# On the convexity of relativistic ideal magnetohydrodynamics

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[CQG 30, 057002 (2013)] [CQG 32, 095007 (2015)]

## Outline:

- Introduction: definition of convexity and approach of Lax.

- Relativistic MHD equations.

- Results:

    · Material and Alfvén waves not affected.

    · Generalized fundamental derivative including relativistic effects and magnetic field effects.

    · Recovery of non-relativistic, unmagnetized limits.

- Conclusions: Implications in the GW emission (see N. Sanchis-Gual's talk).

## Introduction:

· Relativistic jets emanating from AGN, explosion in core-collapse supernovae, collapse to black hole or production of gamma ray bursts do need numerical simulations of the RMHD equations obeying a causal EoS (hyperbolic system of conservation laws).

· Influence of exotic states of matter at extreme high densities.

· Definition of the fundamental derivative (FD):

$$\mathcal{G} := -\frac{1}{2}\, V \frac{\left.\frac{\partial^2 p}{\partial V^2}\right|_s}{\left.\frac{\partial p}{\partial V}\right|_s}$$

· The FD measures the convexity of the isentropes in the p-V plan. FD>0 → isentropes are convex, leading to expansive rarefaction waves and compressive shocks. In a VdW-like EoS or in general in a non-convex EoS (FD<0), rarefaction waves can change to compressive and shock waves to expansive, depending on the specific thermodynamical state of the system.

· Observed experimentally and many engineering applications.

· Equivalent definition due to Lax: convex if all its characteristic fields are either genuinely nonlinear or linearly degenerate:

$$\mathcal{P} := \vec{\nabla}_{\mathbf{u}}\lambda \cdot \mathbf{r} \neq 0,$$

$$\mathcal{P} := \vec{\nabla}_{\mathbf{u}}\lambda \cdot \mathbf{r} = 0,$$

· Convexity characterized with the sign of FD which includes dependence of the local speed of sound [Ibáñez, C.-C., Martí, Miralles, CQG, 2013] and magnetic field [Ibáñez, C.-C., Aloy, Martí and Miralles, CGQ, 2015].

# RMHD equations:

Conservation of mass, conservation of energy-momentum and Maxwell equations. EoS closing the system.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = 0 \longleftarrow \mathbf{V} = (\rho,\, v^i,\, \epsilon,\, B^i)^T$$

$$\mathbf{U} = \begin{pmatrix} D \\ S^i \\ \tau \\ B^i \end{pmatrix},$$

$$\mathbf{F}^i = \begin{pmatrix} Dv^i \\ S^j v^i + p^* \delta^{ij} - b^j B^i/W \\ \tau v^i + p^* v^i - b^0 B^i/W \\ v^i B^k - v^k B^i \end{pmatrix}$$

$$b^0 = W v_k B^k,$$

$$b^i = \frac{B^i}{W} + b^0 v^i.$$

$$D = \rho W,$$

$$S^i = \rho h^* W^2 v^i - b^0 b^i,$$

$$\tau = \rho h^* W^2 - p^* - \left(b^0\right)^2 - D.$$

$$W^2 = 1/(1 - v^i v_i)$$

$$p^* = p + b^2/2$$

$$h^* = 1 + \epsilon + p/\rho + b^2/\rho$$

# Results:

· The charactertistic information of the RMHD equations is contained in the Jacobian matrices of the vectors of fluxes with respect to the *conserved* variables along an arbitrary unitary 3-vector (coordinate direction). It can be more easily computed through the Jacobian of the fluxes with respect to the *primitive* variables. Once the characteristic information is known we can study (see details in [Ibáñez, C.-C., Aloy, Martí and Miralles, CGQ, 2015]) the expression

$$\mathcal{P}_\alpha = \vec{\nabla}_{\mathbf{U}} \lambda_\alpha \cdot \mathbf{r}_\alpha$$

which is non-zero if and only if the expression

$$\mathcal{P}^*_\alpha := \vec{\nabla}_{\mathbf{V}} \lambda_\alpha \cdot \mathbf{r}^*_\alpha$$

is also non-zero (we can use again Jacobian with respect to primitive variables).

· Some definitions:

$$a_s := \sqrt{\partial p/\partial e|_s}, \ \phi_\alpha := (-\lambda, \xi_i), \ a := \phi_\alpha u^\alpha, \ G := \phi_\alpha \phi^\alpha, \ \mathcal{B} := \phi_\alpha b^\alpha,$$

$$d = a_s^2 G^2 \mathcal{B}\, \xi_k B^k - (G - \lambda_{m_\pm} a W^{-1}) \rho h W W_s^{-2} a^4.$$

· Trivially zero eigenvalue leads to linearly degenerate characteristic field (LDCF).

· Eigenvalue associated with the material waves leads to LDCF.

· Eigenvalues associated with the Alfvén waves (2) leads to LDCF.

· Eigenvalues associated with the fast and slow magnetosonic wavespeeds (4):

$$\mathcal{P}^*_{m_\pm} = \frac{W^3 a^4 G^2}{2 a_s^2 d} \mathcal{P}^*_1 \mathcal{P}^*_2$$

$$\mathcal{P}_1^* = b^2 G - \rho h a^2,$$

$$\mathcal{P}_2^* = \left( \rho \left. \frac{\partial a_s^2}{\partial \rho} \right|_\epsilon + \frac{p}{\rho} \left. \frac{\partial a_s^2}{\partial \epsilon} \right|_\rho \right) W_s^2 \left( \frac{\mathcal{B}^2}{a^2} - \mathcal{E} \right)$$
$$- b^2 \left( 3 - a_s^2 \right) - 2\rho h a_s^2 + \frac{a_s^2 \left( 5 - 3a_s^2 \right) \mathcal{B}^2}{a^2}.$$

· First term is zero iff the corresponding magnetosonic eigenvalue is also an Alfvén eigenvalue → LDCF.

· Other cases: to study loss of convexity we focus on second term.

· We recap the relativistic FD derived in [Ibáñez, C.-C., Martí, Miralles, CQG, 2013]:

$$\tilde{\mathcal{G}} = 1 + \frac{\rho}{2a_s^2} \left. \frac{\partial a_s^2}{\partial \rho} \right|_s - a_s^2$$

· We define the FD for relativistic, magnetized fluids [Ibáñez, C.-C., Aloy, Martí and Miralles, CGQ, 2015]:

$$\tilde{\mathcal{G}}_{\mathrm{M}} := \tilde{\mathcal{G}} + F,$$

$$F := \frac{3}{2} W_s^{-4} \left( \frac{c_a^2 / a_s^2 - R}{1 - R} \right).$$

Using the comoving frame and some algebra it can be checked that F>0 in any reference frame.

· Previous expression can be rewritten as:

$$\mathcal{P}_2^* = -2 a_s^2 W_s^2 \mathcal{E} (1 - R)\, \tilde{\mathcal{G}}_{\mathrm{M}}$$

$$\mathcal{E} := \rho h + b^2, \; R := \frac{\mathcal{B}^2}{\mathcal{E} a^2}, \; c_a^2 := \frac{b^2}{\mathcal{E}}$$

## Recovery of limits:

· Zero magnetic field implies $\tilde{\mathcal{G}}_{M,b^2=0} = \tilde{\mathcal{G}}$ , so we recover the relativistic unmagnetized limit [Ibáñez, C.-C., Martí, Miralles, CQG, 2013].

· Imposing non relativistic effects but taking into account magnetic fields (MHD equations) in previous FD, we recover the classical magnetized limit [Serna, Marquina, Phys Fluids, 2014].

· More details about all calculations and in particular the analysis of the degenerate cases can be found in [Ibáñez, C.-C., Aloy, Martí and Miralles, CGQ, 2015].

# Conclusions:

· Analysis of the influence of special relativistic effects and magnetic field in the convexity properties of the RMHD equations using approach of Lax (study of LDCF).

· Material and Alfvén waves are LDCF and, then, not affected by the convexity issue.

· Analysis of the characteristic fields associated with the magnetosonic waves: dependence of the convexity condition through the sign of the *generalized FD*.

· Generalized FD can be written as the sum of two terms: the first one is the FD in the case of purely hydrodynamical relativistic flow; the second one contains the effects of the magnetic field and is always positive (reduction of the domain of thermodynamical states for which the EoS is non-convex).

## Conclusions:

· FD in the case of purely hydrodynamical relativistic flow includes a negative correction to the non-relativistic case (increasing of the domain of thermodynamical states for which the EoS is non-convex).

· Recovery of non-relativistic, unmagnetized limits.

· This result can be relevant in the context of massive stellar core collapse. Needed of numerical simulations: several EoS, relativistic negative contribution, magnetic positive contribution.

· Relativistic effects can induce a non-convex thermodynamics, and, as a consequence, a non-convex dynamics. A non-convex dynamics have implications in the emission of GWs: see N. Sanchis-Gual's talk.

Thank you for your attention!!