# The problem of gauge fixing in the Newman-Penrose formalism

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# Current problems with wave extraction in NR

- Extraction takes place at finite radius rather than null infinity
- Gauge ambiguity in the determination of the correct tetrad to project the Weyl tensor components.
- Numerical integration of the Weyl scalar $\Psi_4$ to obtain the strain

Systematic errors coming from wave extraction are starting to become the dominant source of errors. The need of a mathematically rigorous wave extraction technique is becoming an urgent topic in numerical relativity.

# Wave extraction with the Newman-Penrose formalism

The Newman-Penrose formalism is used in numerical relativity to obtain gauge invariant information about gravitational waves.

$$\Psi_4 = \frac{\partial^2 h_+^{TT}}{\partial t^2} + i\frac{\partial^2 h_\times^{TT}}{\partial t^2}$$

The Kinnersley tetrad guarantees that $\Psi_4 = 0$ for Kerr.

- The chosen tetrad must converge to the Kinnersley tetrad in the background limit.
- Calculating $\Psi_4$ using tetrads evaluated numerically normally brings unwanted gauge effects into the final waveform.
- Our objective is to eliminate the numerical evaluation of the tetrad (no more Gram-Schmidt!).

# Relevant tetrads in the NP formalism for a general (Petrov type I) space-time

1) Symmetric transverse tetrad (STT)

$$\Psi_1 = \Psi_3 = 0 \qquad\qquad \Psi_0 = \Psi_4$$

2) Quasi-Kinnersley tetrad (QKT)

$$\Psi_1 = \Psi_3 = 0 \qquad\qquad \epsilon = 0$$

- QKT is the "right tetrad", it guarantees the convergence the Kinnersley tetrad in the Petrov type D limit.
- STT in a convenient tetrad because of its symmetric properties, but <u>not good</u> for numerical applications.
- STT and QKT are related by a spin/boost (type III) tetrad transformation (complex parameter $\mathcal{B}$).

# Different approaches to Einstein's equations in vacuum

| Coord. approach | Newman-Penrose | NP in STT |
|---|---|---|
| $g_{\mu\nu}$ | $\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu$ | $\Sigma_{\mu\nu}, \Sigma^+_{\mu\nu}, \Sigma^-_{\mu\nu}$ |
| $\Gamma_{abc}$ | $\rho, \mu, \tau, \pi, \sigma, \lambda, \nu, \kappa, \epsilon, \gamma, \beta, \alpha$ | $A_\mu, B_\mu, C_\mu$ |
| $C_{abcd}$ | $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ | $\Psi_2, \Psi_4$ |

- In STT the only remaining degrees of freedom are $\Psi_2$ and $\Psi_4$

$$
\begin{aligned}
\Psi_2^{STT} &= -\frac{I^{\frac{1}{2}}}{2\sqrt{3}} \left(\Theta + \Theta^{-1}\right), \\
\Psi_4^{STT} &= \frac{I^{\frac{1}{2}}}{2i} \left(\Theta - \Theta^{-1}\right),
\end{aligned}
$$

- $I$ is one of the two curvature invariants and $\Theta = f(I, J)$.

# The Bianchi identities

The Bianchi identities ($\nabla_a C^{*a}{}_{bcd} = 0$) written as functions of the variables introduced within our approach give

$$
\begin{aligned}
A_a &= -\frac{iK}{\sqrt{3}} B_a - \frac{1}{6}\nabla_a \ln I - \frac{K}{3}\nabla_a \ln \Theta \\
C_a &= -\frac{i\left(K + 3K^{-1}\right)}{2\sqrt{3}} B_a + \frac{1}{6}\nabla_a \ln I + \left(\frac{3K^{-1} - K}{6}\right)\nabla_a \ln \Theta.
\end{aligned}
$$

where

$$K = \frac{\Theta - \Theta^{-1}}{\Theta + \Theta^{-1}}$$

It turns out that the Bianchi identities can be used as simple relations to derive the two vectors $A_a$ and $C_a$ once $B_a$ is known. But what about the third vector? Can we find a third potential?

# A quadratic function of the Weyl tensor

We introduce following function of the self-dual Weyl tensor

$$D^*_{abcd} = \nabla_\mu \nabla^\mu C^*_{abcd}.$$

Using the Bianchi identities it is possible to show that $D^*_{abcd}$ is given by

$$D^*_{abcd} = 16\, I\, I_{abcd} - \frac{3}{2}\, C^*_{abef}\, C^{*ef}{}_{cd},$$

where $I_{abcd}$ is the identity tensor: $I_{abcd} = \frac{1}{4}\,(g_{ac}g_{bd} - g_{ad}g_{bc} + i\epsilon_{abcd})$.

The tensor $D^*_{abcd}$ has the same symmetries of the self-dual Weyl tensor, included its trace-free property.

# On the divergence of $D^*_{abcd}$

Analogously to $\nabla_a C^{*a}{}_{bcd} = 0$, the divergence of $D^*_{abcd}$ must satisfy

$$\nabla_a D^{*a}{}_{bcd} = \mathcal{S}_a \, C^{*a}{}_{bcd} + \mathcal{T}_a \, D^{*a}{}_{bcd}.$$

$\mathcal{T}_a$ and $\mathcal{S}_a$ are tetrad invariant vectors given by

$$\mathcal{T}_a = \nabla_a \ln\left[I^{\frac{1}{2}} \, \left(\Theta^3 + \Theta^{-3}\right)^{\frac{1}{3}}\right] - \frac{I^{-\frac{1}{2}}}{\sqrt{3}\,(\Theta^3 + \Theta^{-3})}\,\mathcal{S}_a.$$

$$\mathcal{S}_a = f(\nabla_a I, \nabla_a \Theta) + D^{*\,bcd}_a \nabla_e D^{*e}{}_{bcd}.$$

These two vectors naturally introduce a third tetrad invariant quantity that cannot be expressed as a function of $I$ and $J$. Gradient of a third scalar?

# The Petrov type D limit

$A_a$, $B_a$ and $C_a$ can be expressed as functions of $\nabla_a I$, $\nabla_a \Theta$ and $\mathcal{S}_a$. In the limit of Petrov type D they are given by

$$\begin{aligned} A_a &= \frac{1}{6}\nabla_a \ln\, I, \\ B_a &= 0, \\ C_a &= -\frac{1}{6}\nabla_a \ln\, I - \frac{I^{-\frac{1}{2}}}{6\sqrt{3}}\mathcal{S}_a. \end{aligned}$$

$\rho, \mu, \tau, \pi$

$\lambda, \sigma, \nu, \kappa$

$\epsilon, \gamma, \beta, \alpha$

- These values are consistent with the known expressions for the spin coefficients in Kerr.
- The value of $C_a$ calculated in the Kerr space-time confirms that $\mathcal{S}_a = \nabla_a \Phi$! (at least in this limit)
- Knowing $C_a$ in STT and in QKT allows to calculate the spin/boost parameter $\mathcal{B}$ between STT and QKT.

# Final expressions

Knowing the spin-boost parameter $\mathcal{B}$ between STT and QKT we find that the values of $\Psi_2$ and $\Psi_4$ in QKT are given by

$$\begin{aligned} \Psi_2^{QKT} &= -\frac{I^{\frac{1}{2}}}{2\sqrt{3}} \left(\Theta + \Theta^{-1}\right), \\ \Psi_4^{QKT} &= \frac{\mathcal{B}^2 I^{\frac{1}{2}}}{2i} \left(\Theta - \Theta^{-1}\right). \end{aligned}$$

Moreover: the spin coefficient $\sigma$ in QKT vanishes in the Kerr limit and is naturally related to

$$\sigma^{QKT} = \frac{\partial h_+^{TT}}{\partial t} + i\frac{\partial h_\times^{TT}}{\partial t}.$$

No need for numerical integration!

# The Ricci identities

It is known that the Ricci identities in STT simplify to

$$
\begin{aligned}
\nabla_a A^a &= A_a A^a - B_a B^a - \frac{2I^{\frac{1}{2}}}{\sqrt{3}}\left(\Theta + \Theta^{-1}\right) \\
\nabla_a B^a &= -2B_a C^a + 2i\Psi_- \\
\nabla_a C^a &= A_a A^a - B_a B^a + 2A_a C^a - \frac{4I^{\frac{1}{2}}}{\sqrt{3}}\left(\Theta + \Theta^{-1}\right)
\end{aligned}
$$

Our result of obtaining $A_a$, $B_a$ and $C_a$ as functions of $\nabla_a I$, $\nabla_a \Theta$ and maybe $\nabla_a \Phi$ would then lead to equations of the type

$$
\begin{aligned}
\nabla_a \nabla^a I &= \ldots \\
\nabla_a \nabla^a \Theta &= \ldots \\
\nabla_a \nabla^a \Phi &= \ldots
\end{aligned}
$$

A set of three non-linear wave-like equations for $I$, $\Theta$ and $\Phi$.

# Conclusions

- Transverse tetrads ($\Psi_1 = \Psi_3 = 0$) are an elegant form of fixing the gauge in the Newman-Penrose formalism for wave extraction.
- Using STT as a starting point, we only need to calculate the spin-boost parameter $\mathcal{B}$ to obtain the scalars in QKT.
- Bianchi identities and the divergence of $D^*_{abcd}$ allow to find the expression for the spin coefficients in STT, and consequently $\mathcal{B}$.
- Work in progress to determine whether the additional degree of freedom ($\mathcal{S}_a$) can in general be expressed as gradient of a third potential (it can in the Petrov type D limit).
- This procedure allows to study alternative quantities to $\Psi_4$, like $\sigma$, related to the first time derivative of the strain.