Holography & Hydrodynamics

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MR (2009)

V. Hubeny, S. Minwalla, MR (2011)

Dynamics of conserved currents in CFT

- One consequence of AdS/CFT is that the dynamics of conserved currents maps to dynamics of gauge fields in an asymptotically AdS spacetime.
- Gravitational dynamics in AdS spacetimes captures the dynamics of the conserved energy-momentum-stress tensor of the boundary field theory.
- A natural consequence of the planar limit, coupled with consistent truncations of a wide class of supergravity theories, down to pure gravitational dynamics in AdS spacetimes, results in Einstein's equation with negative cosmological constant, describing the *universal decoupled dynamics* of the stress tensor for an infinite number of gauge theories.
- While the generic story of stress tensor dynamics is as complicated as gravity, in certain sectors it simplifies considerably to allow for exact non-linear treatment.

The fluid/gravity correspondence

- The fluid/gravity correspondence establishes a correspondence between Einstein's equations with a negative cc and those of relativistic conformal fluids.
- Given any solution to the hydrodynamic equations, one can construct, in a gradient expansion, an approximate inhomogeneous, dynamical black hole solution in an asymptotically AdS spacetime.
- The construction heuristically can be viewed as patching together planar AdS black holes of different temperatures with slow variation between patches.
- The fluid in question lives on the timelike boundary of AdS spacetime, and as is familiar, holographically encodes the entire dynamics of the bulk spacetime geometry.

Bhattacharyya, Hubeny, Minwalla, MR (2007)

Outline

- Prelude: AdS/CFT and gravity
- Relativistic hydrodynamics
- The gravitational long-wavelength expansion
- Entropy current in fluids
- Generalizations
- Anomalous transport in fluids
- A viewpoint on the black hole membrane paradigm

Relativistic hydrodynamics

- Hydrodynamics is an IR effective field theory, valid when systems attain local but not global thermal equilibrium.
- We require that deviations away from equilibrium are long-wavelength in nature, i.e., we allow fluctuations that occur at scales larger than the typical mean free path of the theory. $\ell_m \ll L \qquad t_m \ll t$
- This allows for a gradient expansion: higher derivative operators are suppressed by powers of our expansion parameter ℓ_m/L .
- The dynamical content of fluid dynamics is just conservation. The energy momentum tensor and charge currents if any should be covariantly conserved.

$$\nabla_\mu T^{\mu\nu} = 0 \ , \qquad \nabla_\mu J^\mu = 0$$

- Conservation alone does not make for a good dynamical system since there are more dof than equations, but things simplify in the long-wavelength limit.

Relativistic hydrodynamics

- In the long-wavelength limit the dynamical dof are reduced, to local charge densities, local temperature and a (normalized) velocity field which indicates direction of flow of energy flux.

$$T^{\mu\nu}(x) = [P(x) + \rho(x)]\, u^\mu\, u^\nu + P(x)\, g^{\mu\nu} + \Pi^{\mu\nu}(x)$$

$$J^\mu_I = q_I\, u^\mu + J^\mu_{I,\mathrm{diss}}$$

- The definition of the velocity field can be fixed by a choice of fluid frame; typically one chooses the velocity to be the timelike eigenvector of the energy -momentum tensor (defining thus the Landau frame).

- Further specification of the fluid requires constitutive relations which require the operators which characterize the dissipative tensors.

- In addition, a fluid also has an entropy current, which satisfies the 2nd law.

$$\mathcal{J}^\mu_S = s\, u^\mu + \mathcal{J}^\mu_{S,\mathrm{diss}}\ , \qquad \nabla_\mu \mathcal{J}^\mu_S \geq 0$$

Relativistic hydrodynamics

- The dissipative parts of stress-tensor and charge currents can be expanded out in a basis of on-shell inequivalent operators built from the dynamical variables and their derivatives.

- From the velocity field we can for instance define:

$$
\begin{aligned}
\theta &= \nabla_\mu u^\mu = P^{\mu\nu}\,\nabla_\mu u_\nu \\
a^\mu &= u^\nu\,\nabla_\nu u^\mu \equiv \mathscr{D} u^\mu \\
\sigma^{\mu\nu} &= \nabla^{(\mu} u^{\nu)} + u^{(\mu}\,a^{\nu)} - \frac{1}{d-1}\,\theta\,P^{\mu\nu} = P^{\mu\alpha}\,P^{\nu\beta}\,\nabla_{(\alpha} u_{\beta)} - \frac{1}{d-1}\,\theta\,P^{\mu\nu} \\
\omega^{\nu\mu} &= \nabla^{[\mu} u^{\nu]} + u^{[\mu}\,a^{\nu]} = P^{\mu\alpha}\,P^{\nu\beta}\,\nabla_{[\alpha} u_{\beta]}\ .
\end{aligned}
$$

- At first order, upon using the conservation of ideal fluid to eliminate themal gradients, we have

$$
\Pi^{\mu\nu}_{(1)} = -2\,\eta\,\sigma^{\mu\nu} - \zeta\,\theta\,P^{\mu\nu}
$$

Higher order dissipative terms

- # of independent operators for an uncharged fluid at second order: **15**.
- Imposing conformal invariance of the fluid (vanishing of the trace): **5**.
- This is simply the basis of on-shell inequivalent tensors. Further constraints come from positivity of the entropy current.
- While the viscosities are constrained to be positive (fluids dissipate energy flux), the entropy considerations are more powerful at second order.

Romatschke (2009)
Bhattacharyya (2012)

- There are five non-trivial constraints among second-order transport coefficients for an arbitrary fluid (independent of equation of state).
- These constraints are trivially satisfied for a conformal fluid, which has indeed five independent transport coefficients at second order.

Baier, Romatschke, Son, Starinets, Stephanov (2007)

Bhattacharyya, Hubeny, Minwalla, MR (2007)

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Long-wavelength in gravity

- Since fluid dynamics is an effective field theory, we can ask whether we can 'derive' it from AdS/CFT.
- A-priori, the answer would seem to be yes, since the classic works on perturbations of planar AdS black holes reveal long-wavelength quasinormal modes which have a hydrodynamic character.

Policastro, Son, Starinets (2002)

- The quasinormal mode computations indicate that linear hydrodynamics, i.e., small amplitudes coupled with long-wavelength is a good description of perturbed planar AdS black holes.
- The synopsis of fluid/gravity is simply that this continues to hold at the non-linear level and provides a constructive way to demonstrate equivalence of fluid equations and Einstein's equations.

Janik, Peschanski (2006)

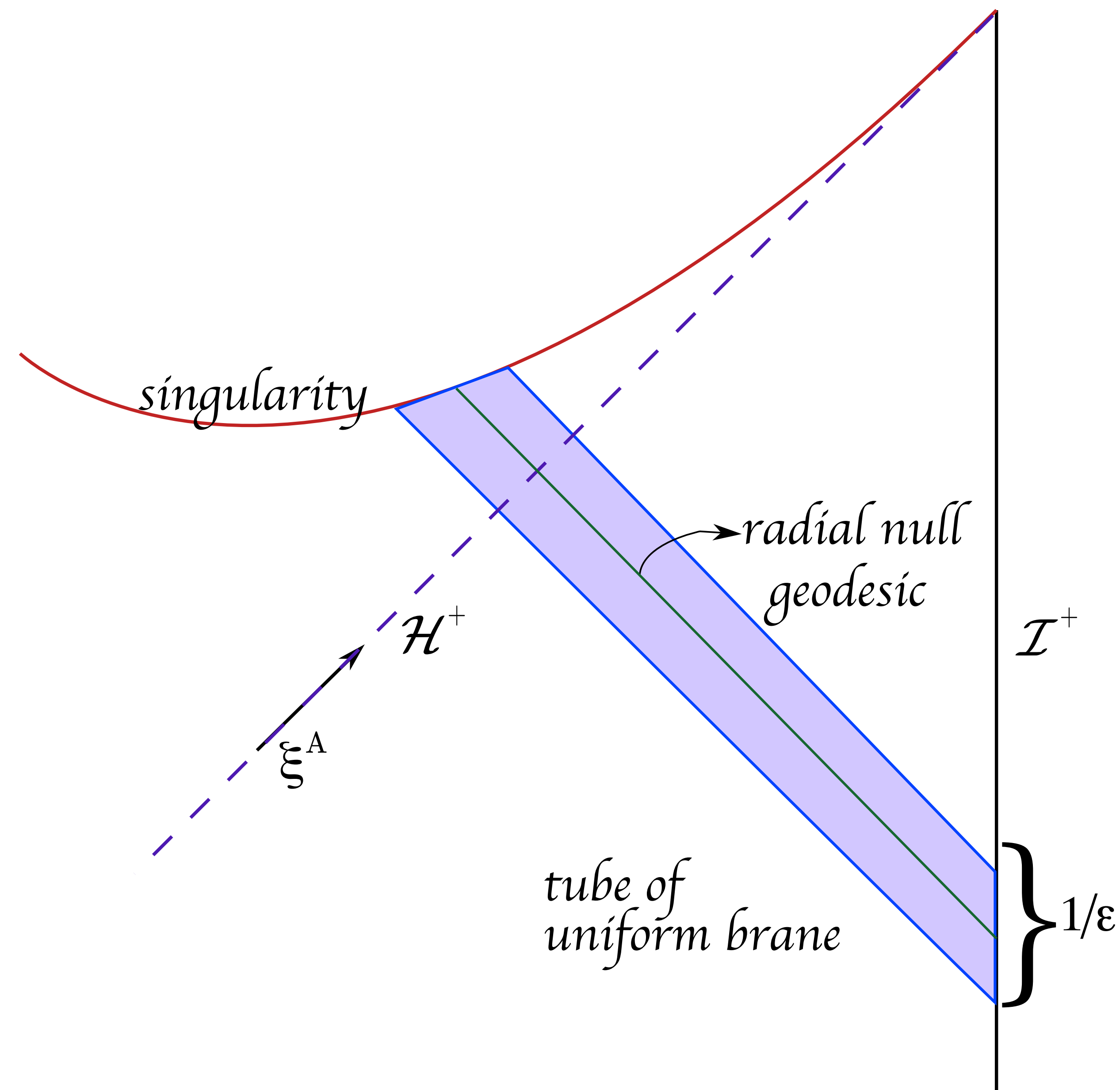
Emparan, Harmark, Niarchos, Obers (2009)
Camps, Emparan (2012)

Nonlinear fluids from gravity

- A basic entry in the AdS/CFT dictionary relates the global thermal equilibrium of a CFT on $\mathbf{R}^{d-1,1}$ in the canonical ensemble to a planar black hole in AdS.
- This equilibrium state is characterized by a temperature T and global boosts (choice of inertial frame) parameterized by a timelike velocity field u^μ.
- The gravitational gradient expansion uses this solution as a starting point for perturbation theory: the perturbation parameter is the scale of fluctuations in $T(x)$ and $u^\mu(x)$ normalized by the local temperature.
- Intuitively, this uses the fact that in local domains the fluid is equilibrated at T(x) and that this domain should holographically be described by a planar AdS black hole.
- Working with regular seed solution (planar black hole in ingoing EF coordinates), the solutions we obtain in perturbation theory exhibit a tubelike structure: local domains on the boundary determine a tubular region in the bulk centered around an ingoing (bulk) null geodesic.

Nonlinear fluids from gravity

- The geometry is determined exactly in the radial coordinate away from the boundary, but only perturbatively in the boundary directions.
- The construction intuitively can be thought of as an implementation of the idea of patching together pieces of black holes along the tubes.



$$ds^2 = -2\,u_\mu(x^\alpha)\,dx^\mu\,dr - r^2\,f\left(b(x^\alpha)\,r\right)\,u_\mu(x^\alpha)\,u_\nu(x^\alpha)\,dx^\mu\,dx^\nu + r^2\,P_{\mu\nu}(x^\alpha)\,dx^\mu\,dx^\nu$$

$$G_{AB} = \sum_{k=0}^{\infty} \varepsilon^k\, G^{(k)}_{AB}(T(\varepsilon x), u^\mu(\varepsilon x))\ , \qquad u^\mu = \sum_{k=0}^{\infty} \varepsilon^k\, u^{\mu\,(k)}(\varepsilon x)\ , \qquad T = \frac{1}{b} = \sum_{k=0}^{\infty} \varepsilon^k\, T^{(k)}(\varepsilon x)$$

Nonlinear fluids from gravity

- Einstein's equations with this ansatz splits up into two natural sets
  - dynamical equations (which determine the radial dependence)
  - constraint equations (momentum constraint for radial evolution)

- The former are a set of decoupled ODEs, a consequence of the large symmetry of the background seed spacetime.
- The radial dependence metric can be determined at each order by solving these explicitly imposing regularity an normalizability as boundary conditions.

$$\mathbb{H}\left[g^{(0)}(T^{(0)}, u^{a\,(0)})\right]\, g^{(n)} = s_n$$

- The latter are simply the dynamical equations of fluid dynamics: they enforce conservation of the boundary stress tensor.

$$\nabla_\mu T^{\mu\nu}_{(n-1)} = 0$$

Nonlinear fluids from gravity: Consequences

- Once the bulk solution is known, we can extract the boundary Brown-York stress tensor:

$$T^{\mu\nu} = \lim_{\Lambda_c \to \infty} \frac{\Lambda_c^{d-2}}{16\pi\, G_N^{(d+1)}} \left[K^{\mu\nu} - K\, g^{\mu\nu} - (d-1)\, g^{\mu\nu} - \frac{1}{d-2} \left(R^{\mu\nu} - \frac{1}{2}\, R\, g^{\mu\nu} \right) \right]$$

- This allows us to determine the holographic constitutive relations, by specifying completely the fluid dynamical transport coefficients.

$$\begin{aligned} \text{First order}: &\ \sigma^{\mu\nu} \\ \text{Second order}: &\ \mathfrak{T}_1^{\mu\nu} = 2\, u^\alpha\, \mathcal{D}_\alpha \sigma_{\mu\nu}\ , \quad \mathfrak{T}_2^{\mu\nu} = C_{\mu\alpha\nu\beta}\, u^\alpha\, u^\beta\ , \\ &\ \mathfrak{T}_3^{\mu\nu} = 4\, \sigma^{\alpha\langle\mu}\, \sigma^{\nu\rangle}_{\ \ \alpha}\ , \quad \mathfrak{T}_4^{\mu\nu} = 2\, \sigma^{\alpha\langle\mu}\, \omega^{\nu\rangle}_{\ \ \alpha}\ , \quad \mathfrak{T}_5^{\mu\nu} = \omega^{\alpha\langle\mu}\, \omega^{\nu\rangle}_{\ \ \alpha} \end{aligned}$$

$$\begin{aligned} \Pi^{\mu\nu}_{(1)} &= -2\, \eta\, \sigma^{\mu\nu} \\ \Pi^{\mu\nu}_{(2)} &= \tau_\pi\, \eta\, \mathfrak{T}_1^{\mu\nu} + \kappa\, \mathfrak{T}_2^{\mu\nu} + \lambda_1\, \mathfrak{T}_3^{\mu\nu} + \lambda_2\, \mathfrak{T}_4^{\mu\nu} + \lambda_3\, \mathfrak{T}_5^{\mu\nu} \end{aligned}$$

$$\begin{aligned} &\eta = \frac{N^2}{8\pi}\, (\pi T)^3 \quad \Longrightarrow \quad \frac{\eta}{s} = \frac{1}{4\pi}\ , \\ &\tau_\pi = \frac{2 - \ln 2}{2\pi\, T}\ , \qquad \kappa = \frac{\eta}{\pi\, T} \\ &\lambda_1 = \frac{\eta}{2\pi\, T}\ , \qquad \lambda_2 = \frac{\eta\, \ln 2}{\pi\, T}\ , \qquad \lambda_3 = 0 \end{aligned}$$

Holographic fluids: remarks

- The analysis of conformal fluids is facilitated by the use of the Weyl covariant formalism, which allows easy construction of operators that have definite conformal dimensions.

R. Loganayagam (2008)

- Holographic fluids come with very specific transport coefficients, a consequence of the regularity of the spacetime (outside event horizon).
- For e.g., shear viscosity takes the universal value for all holographic fluids.
- Similarly vanishing of λ_3 is valid for all conformal fluids and moreover

$$2\,\eta\,\tau_\pi = 4\lambda_1 + \lambda_2$$

M. Haack, A. Yarom (2008)

- The construction is also easy to generalize to forced fluids and to fluids on arbitrary curved backgrounds.

Bhattacharyya, Loganayagam, Minwalla, Nampuri, Trivedi, Wadia (2008)
Bhattacharyya, Loganayagam, Mandal, Minwalla, Sharma (2008)

- Incompressible Navier-Stokes fluids can be obtained in a scaling limit.

Bhattacharyya, Minwalla, Wadia (2008)

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Fluids and black holes

- The gravitational solutions dual to fluids are inhomogeneous, dynamical black holes with a regular event horizon.
- The location of the event horizon can be determined at each order in the gradient expansion under the assumption that the fluid settles down at late time and long distances to an equilibrium configuration.

- At each order in perturbation theory the horizon location can be determined algebraically!
- Note that here the teleological nature of the event horizon is being circumvented because we are able to anchor ourselves close to an equilibrium configuration (courtesy the gradient expansion).

Bhattacharyya, Hubeny, Loganayagam, Mandal, Minwalla, Morita, MR, Reall (2008)

Entropy current in fluid dynamics

- Fluids come equipped with entropy currents with non-negative divergence.
- Black hole spacetimes too have a natural notion of entropy a la Bekenstein-Hawking given by the area of the event horizon.
- A natural fluid dynamical entropy current can be obtained by pulling back the area form of the event horizon onto the boundary along the radially ingoing null geodesics.
- The black hole area theorem guarantees that this satisfies the 2nd law & the gravitational entropy current is a special case of fluid entropy current.
- For general conformal fluids we can write down a 4 parameter family of entropy currents that are consistent with 2nd law, while gravity gives only one non-trivial parameter.
- Higher order analyses constrain second order entropy current to reduce the parameter count to agree with gravitational construction. Romatschke (2009)
- Similar constructions can be made using apparent horizons.

Booth, Heller, Spalinski (2011)

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Generalizations I: charged fluids

- Keeping track of other conserved charges involves generalizing the gravitational construction to include gauge fields.
- Using Einstein-Maxwell-Chern-Simons theories in 5d, the charged fluid constitutive relations can be extracted (nb: now we lose universality).

Erdmenger, Haack, Kaminski, Yarom (2008)
Banerjee, Bhattacharya,Bhattacharyya, Dutta, Loganayagam, Surowka (2008)

- The presence of the bulk Chern-Simons term implies that the fluid dynamics has a new parity-violating term in the charge current involving the vorticity:

$$\delta J^\mu \propto \epsilon^{\mu\nu\rho\sigma}\, u_\nu\, \nabla_\rho u_\sigma$$

- The coefficient of proportionality is fixed by the global U(1) anomaly.
- In fact, this term is necessitated by the fact that the current is anomalous and can be ascertained from a 2nd law argument.

Son, Surowka (2008)

Generalizations II: non-conformal fluids

- While general non-conformal fluids are hard to access holographically, a useful class of examples involves hydrodynamic regime of dilatonic *Dp*-branes, whose duals are supersymmetric gauge theories with 16 supercharges.
- The hydrodynamics of these theories can be obtained using fluid/gravity, and in fact quite simply using a trick. I. Kanitscheider, Skenderis (2009)
- The effective action for holographic dual of *Dp*-branes ($p < 5$) is an Einstein-dilaton system in $p+2$ dimensions.
- This theory can be obtained by a *formal* KK reduction of Einstein-Hilbert action with negative cc living in $p+2+\delta$ on a flat torus T^δ and can be understood for e.g., by using the *M5 → D4 reduction.*
- Likewise the hydrodynamic description can be obtained by reduction of the conformal fluid.
- This continues to hold beyond Einstein-dilaton (e.g., Maxwell) and is a consequence of some curious supergravity consistent truncation relations.

J. David, M. Mahato, S. Wadia (2009) B. Withers, MR (2011)

Generalizations III: superfluids

- It is by now well appreciated that charged scalar hair black holes in AdS correspond to thermal density matrices where a global symmetry is spontaneously broken. Gubser; Hartnoll, Herzog, Horowitz (2009)
- It should be possible to study long-wavelength fluctuations about such density matrices and recover superfluid hydrodynamics.
- In addition to the hydrodynamic mode, there is a new light dof which should be retained: the Goldstone mode that arises from symmetry breaking.
- Analysis of scalar hair black holes + thermodynamics leads to the ideal Tizia-Landau two fluid model of superfluids. Sonner, Withers (2010)
- Note that there is now a new dynamical equation owing to the Goldstone mode: this is encapsulated as the curl-free nature of the superfluid velocity, which in turn can be traced back to the Goldstone mode being the phase of the condensate.
- Using fluid/gravity and positivity of entropy current the complete first order superfluid hydrodynamics (including global anomalies) was worked out recently. Bhattacharya, Bhattacharyya, Minwalla, Yarom (2011)

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Fluids and anomalies

- The most interesting result coming out of the fluid/gravity explorations is the realization that fluid dynamics is cognizant of global anomalies of the microscopic theory.
- Holographic explorations are easy to undertake since the bulk action needs only to be suitably generalized to include Chern-Simons terms, a first principles understanding of these effects is starting to emerge from recent studies.

- Approaches to understanding anomalous transport:
  - adiabaticity arguments involving the entropy current
  - Kubo formulae
  - analysis of free theories with chiral anomalies

Son, Surowka (2008) Landsteiner + Megias + Melgar + Pena-Benitz + Amado (2011)

Loganayagam (2011) Kharzeev, Yee (2011) Loganayagam, Surowka (2012)

Fluids and anomalies

- The anomalous transport coefficients are non-dissipative; they arise from the Hermitian part of the retarded Green's function.
- Kubo formulae for these transport coefficients involve the limiting behaviour of finite momentum, zero frequency correlators.

$$\xi_B = \lim_{k_c \to 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left(\left\langle J^a J^b \right\rangle - \frac{n}{\epsilon + P} \left\langle T^{0a} J^b \right\rangle \right) \Big|_{\omega=0, A_0=0} ,$$

$$\xi_V = \lim_{k_c \to 0} \frac{i}{2k_c} \sum_{a,b} \epsilon_{abc} \left(\left\langle J^a T^{0b} \right\rangle - \frac{n}{\epsilon + P} \left\langle T^{0a} T^{0b} \right\rangle \right) \Big|_{\omega=0} .$$

- Natural way to study the anomalous transport is to couple the fluid to background fields.
- Analysis to date involves coupling hydrodynamic dof to background electromagnetic fields; curiously this already seems to know about gravitational couplings!

Fluids and anomalies

- All available data points to fluid dynamics being aware of the anomaly polynomial.

$$\mathfrak{F}^{\omega}_{anom} = \mathcal{P}_{anom}\left(F \mapsto \mu,\ p_1(\mathfrak{R}) \mapsto -T^2,\ p_{k>1}(\mathfrak{R}) \mapsto 0\right)$$

- The anomalous transport coefficients can be extracted from an 'anomalous Gibbs current' which is built from the anomaly polynomial and its derivatives:

$$\begin{aligned}\bar{\mathcal{G}}_{anom} = \frac{1}{(2\omega)^2}\Big\{&\mathfrak{F}^{\omega}_{anom}[T(2\omega), B+\mu(2\omega)] - \Big[\mathfrak{F}^{\omega}_{anom}[T(2\omega), B+\mu(2\omega)]\Big]_{\omega=0} \\ &- \quad \omega\left[\frac{\delta}{\delta\omega}\mathfrak{F}^{\omega}_{anom}[T(2\omega), B+\mu(2\omega)]\right]_{\omega=0}\Big\} \wedge u\end{aligned}$$

  - anomalous charge current $\qquad \bar{J}_{anom} = -\dfrac{\partial \mathcal{G}_{anom}}{\partial \mu}$
  - anomalous entropy current $\qquad \bar{J}_{S,anom} = -\dfrac{\partial \bar{\mathcal{G}}_{anom}}{\partial T}$
  - anomalous heat current $\qquad \bar{q}_{anom} = \bar{\mathcal{G}}_{anom} + T\bar{J}_{S,anom} + \mu \bar{J}_{anom}$

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The black hole membrane paradigm

- Connections between gravity and fluids originated in the *black hole membrane paradigm.*

T. Damour; K. Thorne, R. Price (1970s)

- The membrane paradigm associates a dynamical membrane with electro-mechanical properties to the black hole.
- In particular, it does away with the interior of the black hole; matter falling into the black hole instead interacts with the membrane.
- Membrane dynamics, obtained by projecting Einstein's equations onto a null hypersurface, has formal similarities with the non-relativistic Navier-Stokes dynamics.

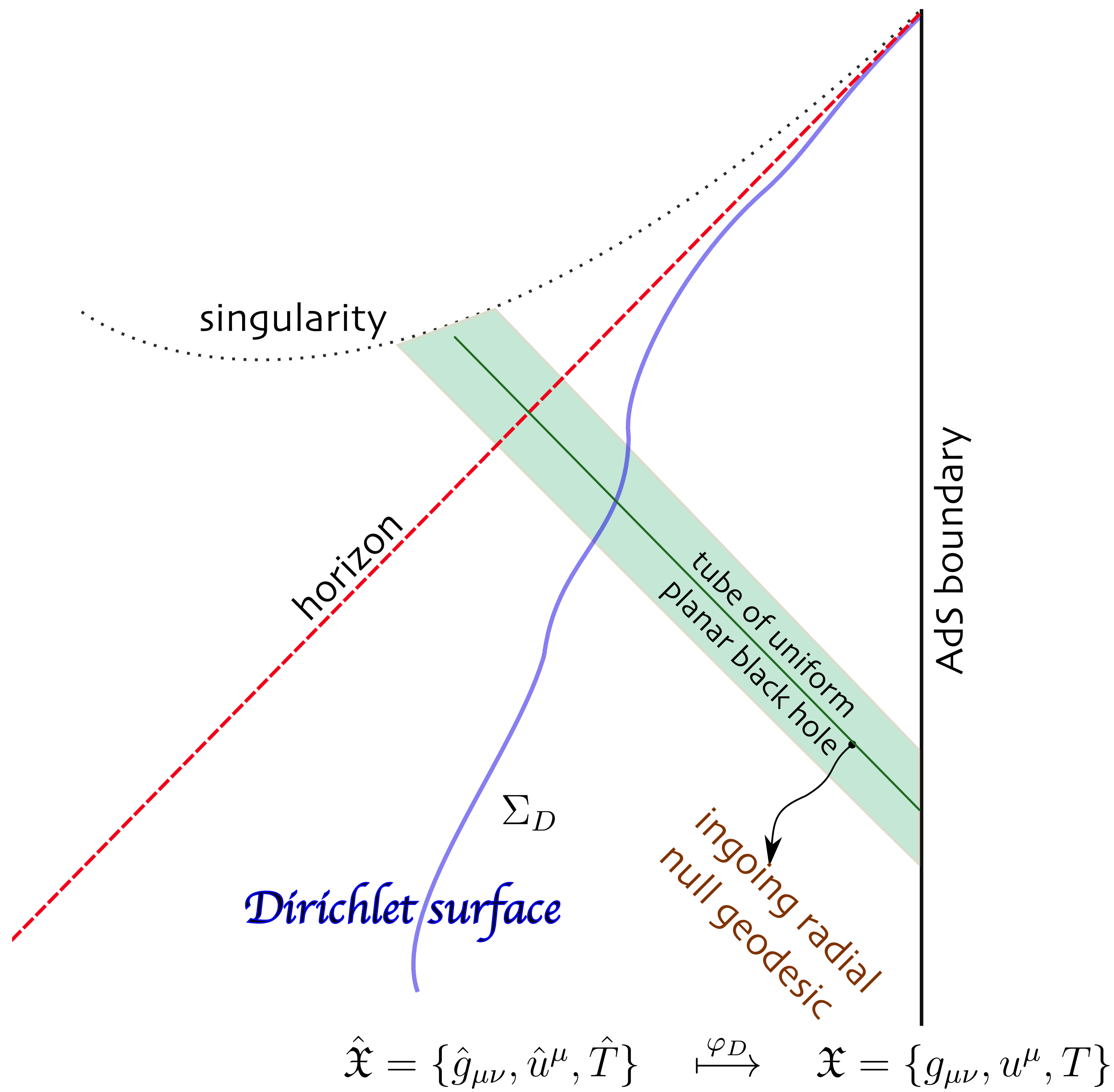
C. Eling, Y. Oz (2008)

- More recently, using a gradient expansion in the near-horizon Rindler region, the membrane dynamics has been shown to correspond to an incompressible Navier-Stokes system.

Bredberg, Lysov, Keeler, Strominger (2010-11)
Compere, McFadden, Skenderis, Taylor (2011)

Dirichlet problem for gravity

- ❖ The general non-linear gravity problem is hard to solve.
- ❖ Can make progress in the long-wavelength regime, where we can bring the results of the fluid/gravity correspondence to bear.
- ❖ The tubewise structure of the spacetime allows one to implement the gradient expansion with Dirichlet bc.



Brattan, Camps, Loganayagam, MR (2011)

Dressed up the boundary fluid

- The long-wavelength solution to the Dirichlet problem results in a Dirichlet fluid whose constitutive relations are explicitly known.
- In particular, this construction allows for a non-linear proof of the statement that shear viscosity does not run under radial evolution. Iqbal, Liu (2008)
- The Dirichlet fluid can be viewed in terms of a conformal fluid (the usual boundary fluid) which happens to reside on a 'dynamical background'.
- The geometry on which the conformal fluid propagates depends on the local fluid dof and is determined only upon solving fluid equations.
- One can view the fluid as carrying a local gravitational cloud with it; a field redefinition allows one to absorb the dynamical variables into the fluid and leaving behind an inert background.
- Pushing the Dirichlet surface to the horizon, gives an AdS embedding of the membrane paradigm: the boundary fluid lives on a degenrate manifold with Newton-Cartan structure.

Five questions (I'd like to know the answer to)

- Fluids coupled to gravity: gravitational anomalies from first principles.
- Extremal fluids: finite density locally critical dynamics as an effective field theory.
- Entropy current away from local equilibrium: what geometric construct does the job?
- Insight into turbulence: can knowledge of gravitational dynamics shed light on fluid dynamics?
- Einstein's equations from a Boltzmannian perspective: does incorporation of higher quasinormal modes lead to a Boltzmann type equation?