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BPS states, Wall-crossing and Quivers

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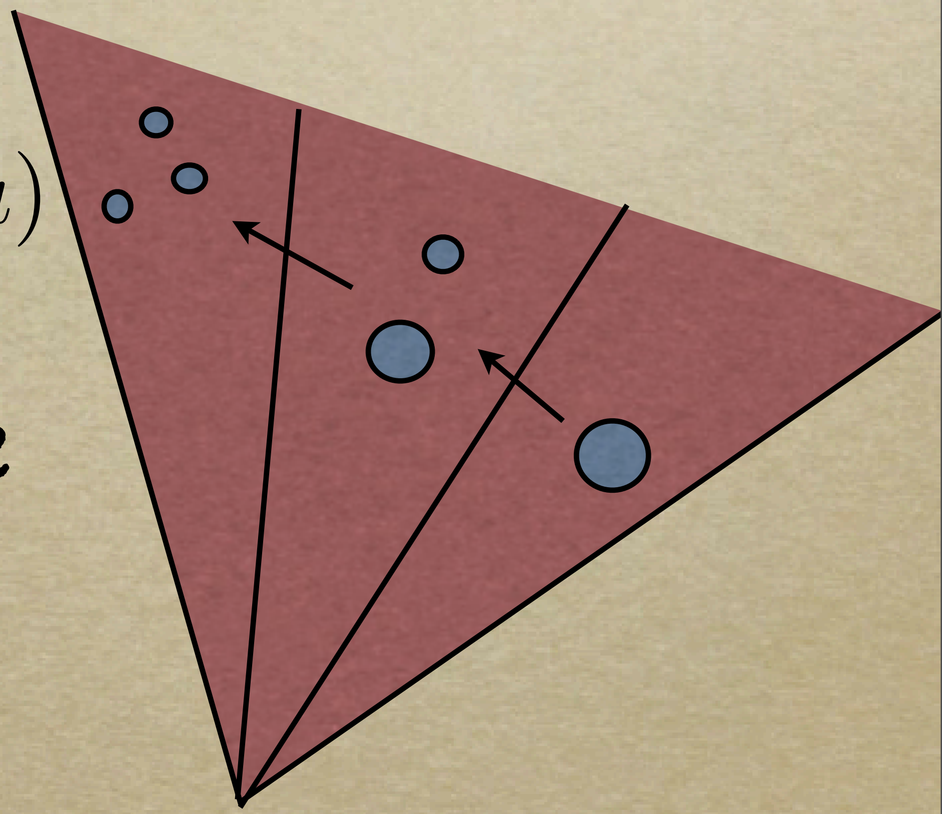
and M.C. to appear

BPS States in String theory

- The problem we would like to address is to compute the spectrum of BPS states in Calabi-Yau compactifications
- In this talk we will focus on type IIA on a local toric Calabi-Yau and consider bound states of D0-D2-D4-D6 branes
- We want to compute the degeneracy $\Omega(\gamma)$ (or Donaldson-Thomas invariant) for a given charge

Chamber Structure

- The vacuum is parametrized by the moduli space of complexified Kähler parameters $u \in \mathcal{B}$
- The degeneracy is really a piecewise constant function $\Omega(\gamma; u)$
- At walls of marginal stability the degeneracy jumps according to a wall-crossing formula
- The moduli space is divided in chambers

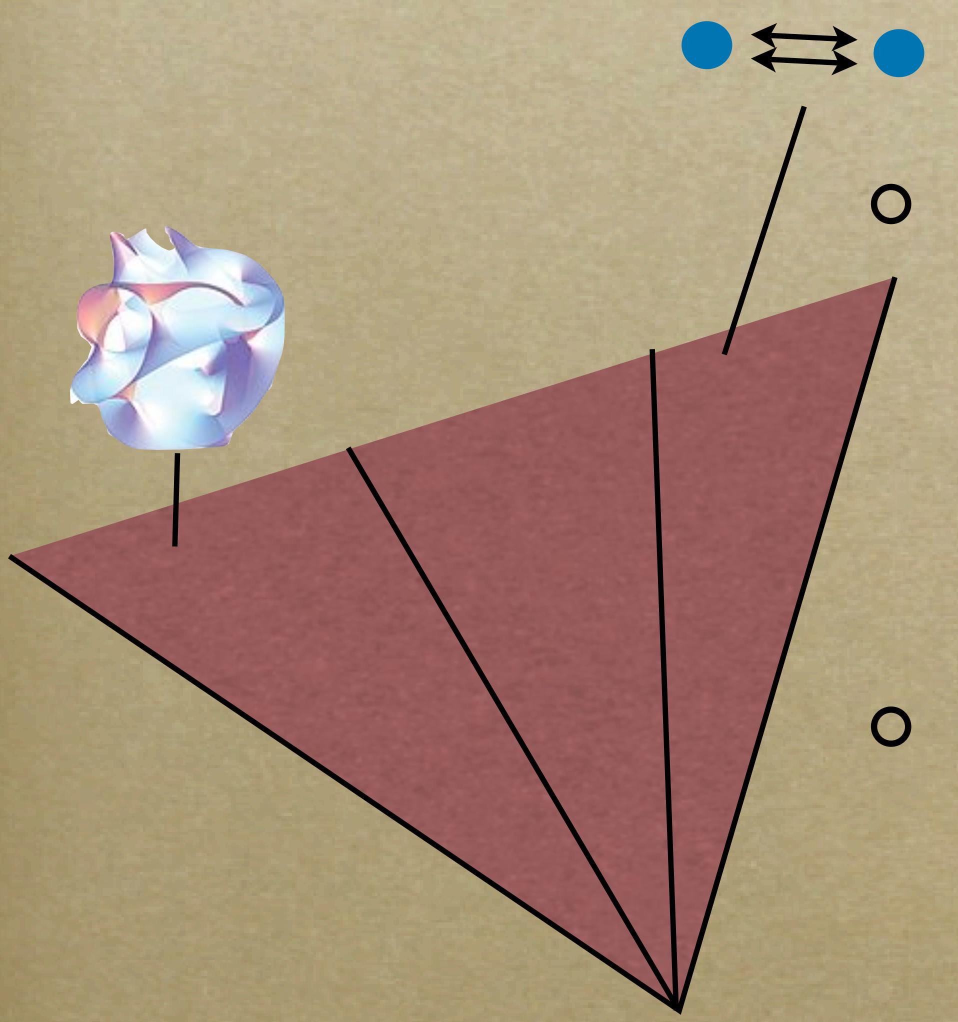


Denef-Moore
Kontsevich-Soibelman

Chamber Structure

- At each point in the moduli space we have a lattice of (electric/magnetic) charges $\Gamma_u = \Gamma_{e,u} \oplus \Gamma_{m,u}$
- Pick a basis $\{\gamma_i\}$ of the lattice. A generic BPS state will be of the form $\gamma = \sum_i n_i \gamma_i$
- For example D-branes (vector bundles) wrapping holomorphic cycles or fractional branes.
- The "wall-crossing problem": is a lattice site occupied by a stable state? what is $\Omega(\gamma; u)$?

Chamber Structure



- In each chamber we have a different counting problem
- Some regions are easier than others: at "large radius" geometrical data are "good" (cycles, bundles...)
- In the noncommutative crepant resolution chamber (NCCR) we use quivers and representation theory

Szendroi
Ooguri&Yamazaki

DT Invariants

- The DT chamber lives in the "large radius"
- Consider bound states of a gas of D2-D0 with a single D6 (just to make life easier).
- If we sit on the D6 brane these bound states will look like "generalized instantons"
- Their spectrum can be computed on any toric CY using a generalization of Nekrasov's instanton counting techniques

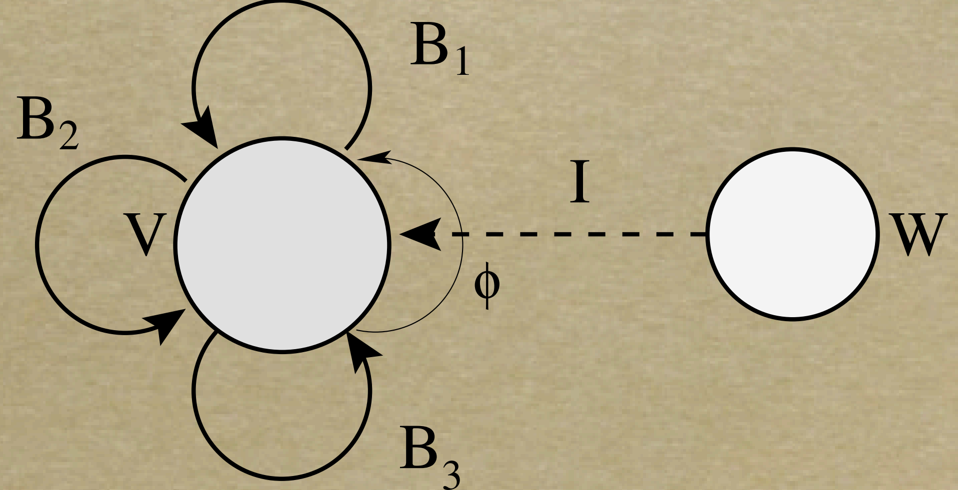
Iqbal Okounkov Nekrasov Vafa
M.C. Sinkovics Szabo

Warm up: affine space

- In this case the problem reduces to D0-D6 states
- We have constructed an explicit ADHM-like parametrization of the instanton moduli space — Witten
- We end up with a quiver quantum mechanics which compute the index of BPS states

$$[B_i, B_j] + \epsilon^{ijk}[B_k^\dagger, \phi] = 0$$

$$I^\dagger \phi = 0$$



Noncommutative Crepant Resolutions

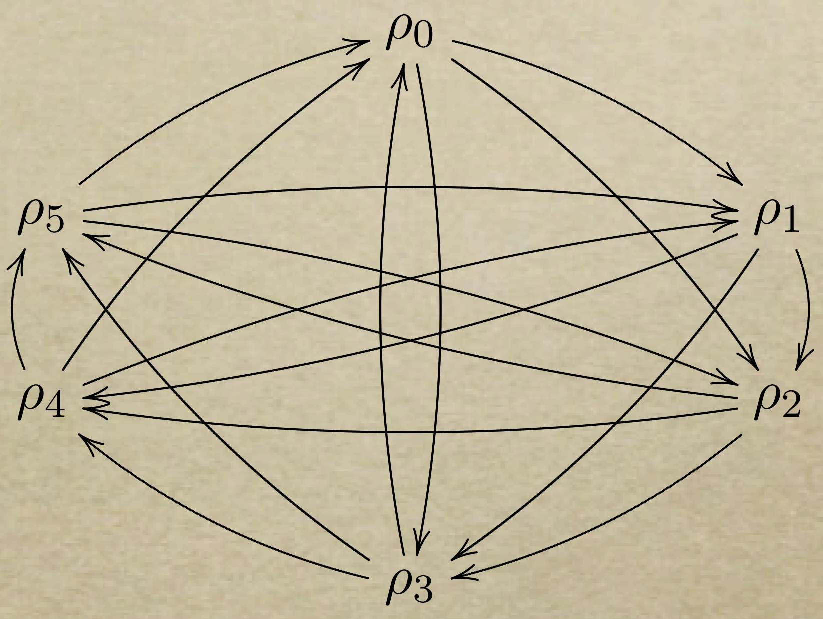
- This chamber corresponds to a singular geometry like $\mathbb{C}^3/\Gamma$ obtained by blowing down a curve or a divisor from the large radius phase
- The cycle has still a non trivial quantum volume measured by the B-field
- Our formalism can be adapted to this situation via a generalization of the Kronheimer-Nakajima construction of instantons on ALE spaces
- The key ingredients are the McKay quiver and the 3D McKay correspondence

The McKay Quiver

- The orbifold action is encoded in the natural representation $Q = (\rho_a, \rho_b, \rho_c)$

- The McKay quiver Q has nodes given by the irreducible representations and arrows determined by the decomposition

$$\rho_k \otimes Q = a^{(1)}_{kl}\, \rho_l$$



$\mathbb{C}^3/\mathbb{Z}_6$

$Q = (\rho_1, \rho_2, \rho_3)$

- The quiver comes equipped with a set of relations:

$$\mathsf{b}^{\rho}_i : \rho \longrightarrow \rho \otimes \rho_{a_i} \qquad \mathsf{r} = \langle \mathsf{b}^{\rho\otimes\rho_{a_i}}_j\ \mathsf{b}^{\rho}_i = \mathsf{b}^{\rho\otimes\rho_{a_j}}_i\ \mathsf{b}^{\rho}_j \rangle$$

The McKay Correspondence

- The 3D McKay correspondence tells us how to extract geometrical data from the McKay quiver

  Ito-Nakajima

- The representation theory of the quiver is encoded all the information about the (canonical, large radius) resolution of the singularity
- But there's more: the path algebra (i.e. the algebra of paths on the quiver) is itself a resolution of the singularity: the NCCR

  van den Bergh
  Ginzburg

- The resolution is "non geometric", in the same sense as noncommutative geometry

Instanton Quivers

- The BPS spectrum in the NCCR chamber can be reformulated as a generalized instanton counting problem
- We start from the resolved geometry and use the McKay correspondence. Our construction uses a "stability" parameter to go to the NCCR chamber
- We give an explicit parametrization of the instanton moduli space. The problem boils down to the study of the representation theory of a certain framed quiver: the instanton quiver

Instanton Quivers

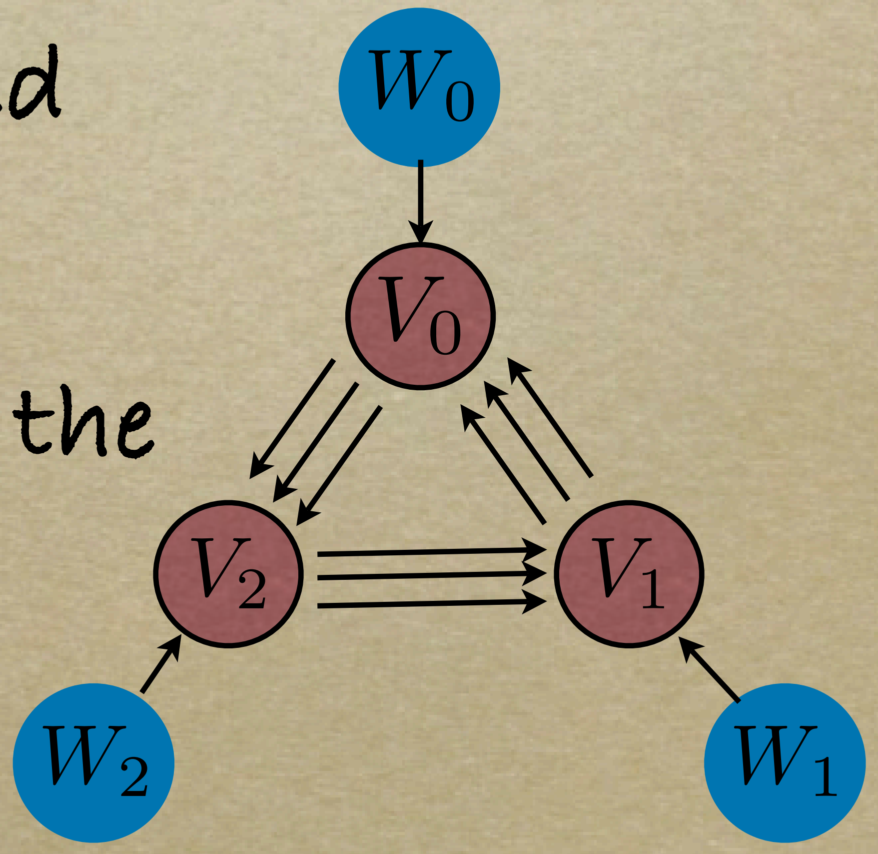
- The ingredients are two vector spaces.

$$V = \sum_k V_k \otimes \rho_k^\vee$$

$$W = \sum_k W_k \otimes \rho_k^\vee$$

- The vector spaces W_k label boundary conditions for the instanton. Each instanton at infinity is associated with the representation $\bigoplus_k \rho_k^{w_k}$

- The dimensions $v_k = \mathrm{dim} V_k$ count the number of fractional branes in a certain representation.

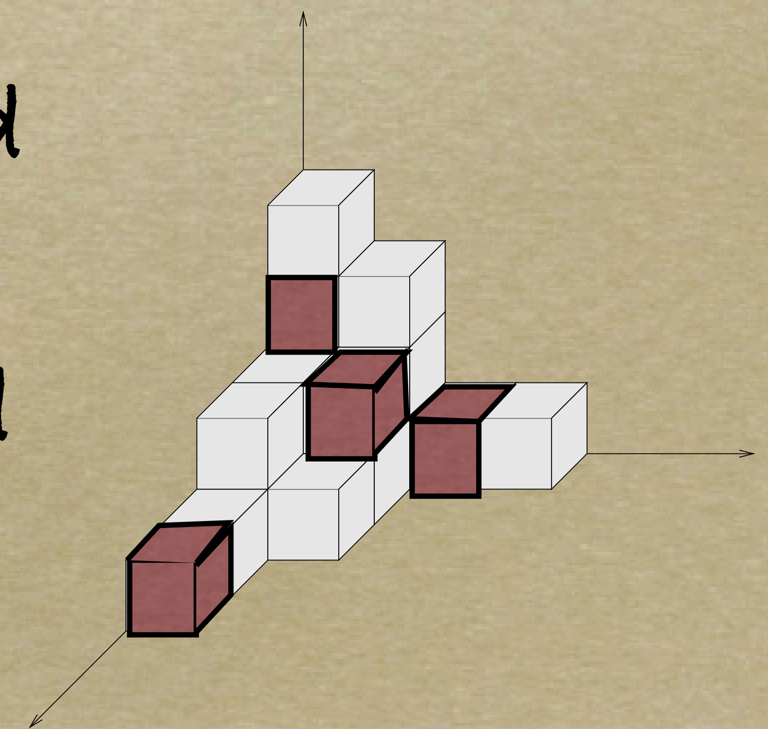


Instanton Quivers

- Between the nodes there are maps obeying certain relations

$$B_j^{\rho\otimes\rho_{a_i}} \; B_i^{\rho} = B_i^{\rho\otimes\rho_{a_j}} \; B_j^{\rho} \qquad\qquad B_i^{\rho} : V_\rho \longrightarrow V_{\rho\otimes\rho_{a_i}}$$

- For a fixed configuration, the Witten index of the instanton quantum mechanics compute the spectrum of BPS states
- Instanton configurations are labelled by coloured partitions where the "color" degree of freedom is associated with the irreps $V = \sum_k V_k \otimes \rho_k^{\vee}$



Example

- As an example the $\mathbb{C}^3/\mathbb{Z}_3$ partition function is

$$\mathcal{Z}_{DT}(\mathbb{C}^3/\mathbb{Z}_3) = \sum_{\pi} (-1)^{(-|\pi_1|-|\pi_2|+|\pi_0||\pi_1|+|\pi_0||\pi_2|+|\pi_1||\pi_2|)} q^{\frac{1}{3}|\pi|-\frac{1}{6}(7|\pi_0|-8|\pi_1|+|\pi_2|)} v^{\frac{1}{2}(3|\pi_0|-4|\pi_1|+|\pi_2|)}$$

- The instanton action is computed again via the McKay correspondence from the anomalous couplings
- The generating function only depends on two parameters as in the large radius limit.

Wall-Crossing Formula

- Having in principle solved the BPS spectrum in one chamber (the NCCR) we can move in nearby chambers with a wall-crossing formula
- The wall-crossing formula is written in terms of McKay data
- To each irrep we associate an operator X_r with commutation relations

$$\mathsf{X}_r \, \mathsf{X}_s = \lambda^{2a^{(2)}_{rs} - 2a^{(1)}_{rs}} \, \mathsf{X}_s \, \mathsf{X}_r \qquad \bigwedge^i Q \otimes \rho_r = \bigoplus_s a^{(i)}_{sr} \, \rho_s$$

Wall-Crossing Formula

- To a charge vector $\gamma = \sum_{r \in irrep} g_r \gamma_r$ we associate

$$\mathsf{X}_\gamma = \lambda^{-\sum_{r<s} g_r g_s (a^{(2)}_{rs} - a^{(1)}_{rs})} \prod_r^{\curvearrowright} \mathsf{X}_r^{g_r}$$

- We construct the quantum monodromy invariant

$$\mathsf{M}(\lambda) = \prod_{\theta_\rho}^{\curvearrowright} \Psi\left(\lambda^{2s_\rho} \mathsf{X}_\rho; \lambda\right)^{\Omega^{\mathrm{ref}}_{2s_\rho}(\rho)}$$

Cecotti-Neitzke-Vafa
Kontsevich-Soibelman

- The ordering is determined by the central charges (fixed by the McKay correspondence). As this changes crossing walls of marginal stability, the degeneracies change to keep $\mathsf{M}(\lambda)$ invariant

Motivic Invariants

- We can use our formalism to study motivic DT
- Roughly speaking the motivic invariant represents the BPS Hilbert space itself
- They can be computed via our instanton quivers

$$\left[\mathrm{NDT}_{\mu=0}(\mathbf{k})\right] = \mathbb{L}^{\frac{1}{2}\,\chi_{\mathsf{Q}}(\mathbf{k},\mathbf{k})}\ \frac{\left[f_{\mathbf{k}}^{-1}(0)\right] - \left[f_{\mathbf{k}}^{-1}(1)\right]}{\left[G_{\mathbf{k}}\right]}$$

- With the motivic wall-crossing formula one can study directly the Hilbert space across the moduli space and the algebra of BPS states

Conclusions

- We have developed a formalism to study the spectrum of BPS states on toric CY
- Our formalism is very efficient in certain chambers (such as at large radius or the NCCR)
- It can be generalized to study wall-crossing, motivic invariants, cluster algebra structures, noncommutative mirror symmetry and defects (which I haven't mentioned)
- yet, still a small piece of the puzzle...