BRST-BV treatment of Vasiliev's four-dimensional higher-spin gravity

P. Sundell (Université de Mons, Belgique)

Based on arXiv:1102.2219 with N. Boulanger, arXiv:1103.2360 with E. Sezgin and to-appear with N. B. and N. Colombo.

See also by M. Vasiliev.

Iberian Strings 2012 at Bilbao

Outline

- Motivation
- Abstract
- Generalized Hamiltonian actions *a.k.a.* Poisson sigma models
- BRST quantization using the Alexandrov – Kontsevich – Schwarz – Zaboronsky (AKSZ) implementation of the Batalin – Vilkovisky (BV) field-anti-field formalism for Differential Algebras (DAs) and in particular Cartan – Vasiliev's Unfolded Dynamics (UD)
- Adaptation to Vasiliev's 4D Higher-Spin Gravity (HSGR)
- Conclusions
- Discussions of the key point: non-uniqueness of Poisson structure for HSGR?
  How such a peculiar state-of-affairs seems to fit naturally into the landscape of tensionless string theory and holography.

Motivation

The existence of a class of non-abelian gauge theories tend to be a highly non-trivial fact, not only mathematically but also physically once the dynamics is interpreted properly.

In the case of HSGR, for which the interpretation process is ongoing, a mayor benchmark is set by Vasiliev's fully non-linear models:

- occupy a central position in top-to-bottom approaches to weak/weak-coupling AdS/CFT correspondences (*e.g.* resolution of curvature singularities, tensionless limits of string theory and also various more novel applications, such as for example anyon physics).
- stimulate developments of generally-covariant quantum field theory that may arguably turn out to be crucial for refining String Field Theory (SFT) (*e.g.* unfolded dynamics and twistorization of field theory).
- pending further model building, a new phenomenologically-viable windows to Quantum Gravity?

SFT and HSGR co-exist in harmony — interesting for de Sitter!

Pre-amble

Recently, an action principle for Vasiliev's 4D HSGR has been proposed.

Besides massless higher-spin fields, it contains Stückelberg fields, auxiliaries and Lagrange multipliers.

On-shell, albeit perturbatively, the latter can be eliminated or expressed in terms of the dynamical fields.

Off-shell, however, the plot thickens due to the structure of the very kinetic terms of the "bulk" action and the presence of "boundary" terms (required for non-trivial "amplitudes").

The former requires treating all fields as being independent off-shell and BRST-quantizing the theory using an adaptation of AKSZs specialization of the Batalin and Vilkovisky (BV) field-anti-field formalism.

Abstract

We provide Vasiliev's 4D HSGR with a classical BV master action using the AKSZ formalism.

The construction of the action functional is a straight-forward algebraic procedure:

Essentially, the AKSZ procedure amounts to including a "minimal set "of fields and anti-fields simply by replacing, in the classical generalized Hamiltonian action, each differential form by a corresponding "vectorial superfield" of fixed total degree given by the sum of the form degree plus the ghost number.

Current status

The path-integral quantization, which naively amounts to "choosing boundary conditions", appears to be a formidable story.

Nonetheless, certain classical solution spaces (*e.g.* one-body solitons) as well as certain observables (tree-level Witten diagrams, vacuum bubbles and zero-form charges) are under control.

Moreover, a geometric approach to observables has been proposed, based on the notions of structure groups and topological impurities.

The progress rate is relatively slow, however, and in particular one would like to test it in the case of ordinary relativistic quantum field theories, though mainly due to technical reasons (and sometimes limited skill-sets), so it may very well change as the formalism is exposed to new ideas and insights.

Poisson sigma model

Topological model formulated on open base manifold $\mathcal{M}$ of dimension $\hat{p}+1$.

The degrees of freedom captured by "boundary vertex operators".

In particular, the local degrees of freedom are captured by

- "zero-form invariants", which are boundary vertex operators that can be inserted at points
- localization of "topologically broken" gauge functions to boundaries

Target space and generalized Hamiltonian

The target spaces are phase spaces $T[\hat{p}]\mathcal{N}$ over graded Poisson manifolds $\mathcal{N}$ equipped with:

(i) a nilpotent vector field $Q \equiv \Pi_{(1)} = Q^\alpha(X)\partial_\alpha$ of degree 1, referred to as the Q-structure, governing the classical limit;

(ii) compatible rank-r multi-vector fields $\Pi_{(r)}$ of degree $1 + (1-r)\hat{p}$

$$\{\Pi_{(r)}, \Pi_{(r')}\}_{\text{Schouten}} \; = \; 0 \; .$$

In canonical coordinates, say (X^α, P_α), with $\deg(X^\alpha) + \deg(P_\alpha) = \hat{p}$, the classical Lagrangian is of the generalized Hamiltonian form

$$\mathcal{L}^{\text{cl}}_{\text{bulk}} \; = \; P_\alpha \wedge dX^\alpha - \mathcal{H}(P,X)$$

where $\mathcal{H} = \sum_r \Pi^{\alpha_1 \cdots \alpha_r}(X) P_{\alpha_1} \cdots P_{\alpha_r}$ has vanishing graded Poisson bracket with itself, *viz.* it obeys the structure equation

$$0 \; = \; \{\mathcal{H}, \mathcal{H}\}_{\text{P.B.}} \; \sim \; \partial_\alpha \mathcal{H} \partial^\alpha \mathcal{H} \; .$$

Gauge invariance and structure group

The structure equation implies that under the gauge transformations

$$\delta_{\epsilon,\eta}(X^\alpha, P_\alpha) \;=\; d(\epsilon^\alpha, \eta_\alpha) + (\epsilon^\alpha \partial_\alpha + \eta_\alpha \partial^\alpha)(\partial^\alpha, \partial_\alpha)\mathcal{H} \;,$$

the classical Lagrangian transforms into a total derivative, *viz.*

$$\delta_{\epsilon,\eta}\mathcal{L}^{\rm cl}_{\rm bulk} \;=\; d(\epsilon^\alpha \partial_\alpha (1 - P_\beta \partial^\beta)\mathcal{H} + \eta_\alpha (dX^\alpha + \cdots)) \;,$$

resulting in classically-topological field theory with graded structure group generated by gauge parameters $(t^\alpha, 0)$ obeying

$$t^\alpha \partial_\alpha (1 - P_\beta \partial^\beta)\mathcal{H} \;=\; 0 \;.$$

The remaining gauge symmetries are "topologically broken": the parameters and corresponding fields are sections glued together across chart boundaries by means of transitions from the structure group.

c.f. the treatment of local translations/transvections and local Lorentz rotations in ordinary gravity.

Boundary data and Cartan integrability

The degrees of freedom reside in the boundary data:

- if $\mathcal{H}|_{P=0} = 0$ the variational principle holds with $P_\alpha|_{\partial M} = 0$ which means that P_α can be taken to vanish on-shell.
- integration constants C^α for the X^α of degree zero.
- boundary values of gauge functions λ^α for topologically broken X^α (of strictly positive degree).
- windings in the transition functions and monodromies of flat connections in the structure algebra.

In particular, given C^α and λ^α, the local field configurations can be reconstructed on-shell by means of Cartan's integration formula:

$$X^\alpha_{C,\lambda} \approx \left[\exp((d\lambda^\beta + \lambda^\gamma \partial_\gamma Q^\alpha)\partial_\beta) X^\alpha\right]\Big|_{P=0;X=C} .$$

Topological impurities

The physical degrees of freedom contained in the boundary data are captured by means of classical observables $\mathcal{V}$ that are topological impurities in the sense that

$$\delta_t \mathcal{V} \;=\; 0 \;, \qquad \delta \mathcal{V} \;\approx\; 0 \;,$$

that is, the topologically broken gauge symmetries resurface on-shell, *viz.*

$$\delta_{\epsilon,\eta} \mathcal{V} \;\approx\; 0 \;,$$

which is tantamount to say that $\mathcal{V}$ are intrinsically-defined classical observables (for example, if $\mathcal{V}$ refers to a cycle, then it only depends on the homotopy class of the cycle).

Total action

Thus the PSM is characterized by the total action

$$S^{\rm cl}_{\rm tot} \;=\; S^{\rm cl}_{\rm bulk} + \sum_i t_i \mathcal{V}^i \ ,$$

that is, by the Hamiltonian and the choice of structure group implied by the choice of $\{\mathcal{V}^i\}$.

In other words, the rôle of the impurities is to break some Cartan gauge symmetries off-shell in such a way that the latter are restored on-shell as to produce non-trivial "amplitudes" for local degrees of freedom.

In particular, for a relativistic field theory with an "ordinary" action principle $S_{\rm D}$ in D dimensions, one may choose $\mathcal{V} = S_{\rm D}$ inserted on $\partial\mathcal{M}$.

In the case of HSGR, however, the closest candidate to the latter is the cubic Fradkin – Vasiliev action, $S_{\rm FV}$, exhibiting novel features starting from spin 5/2, namely the need to eliminate "extra auxiliary fields" via curvature constraints that do not follow from $\delta S_{\rm FV} \approx 0$.

Gauge fixing and AKSZ procedure

In order to gauge fix, one first needs to exhibit all "deeper" gauge symmetries, which is one of the two main features of the Batalin–Vilkovisky field-anti-field formalism:

the classical fields $(X^\alpha, P_\alpha) \equiv (X^{\alpha,\langle 0\rangle}_{[p_\alpha]}, P^{\langle 0\rangle}_{\alpha,[\hat{p}-p_\alpha]})$ are thus extended by towers of ghosts $(X^{\alpha,\langle q\rangle}_{[p_\alpha-q]}, P^{\langle q'\rangle}_{\alpha,[\hat{p}-p_\alpha-q']})$ with ghost numbers $q = 1, \ldots, p_\alpha$ and $q' = 1, \ldots, \hat{p} - p_\alpha$.

Fields and anti-fields

Moreover, as the Cartan gauge symmetries do not in general close off-shell, the BRST operator and the gauge-fixing terms need to be constructed by going via a "minimal" master action

$$S^{\rm BV}[\phi^i,\phi^*_i]$$

depending on the classical fields and the ghosts, ϕ^i say, as well as all their anti-fields, ϕ^*_i obeying

$${\rm gh}\phi^i+{\rm gh}\phi^*_i \;=\; -1\;,\qquad {\rm deg}\phi^i+{\rm deg}\phi^*_i \;=\; \hat p+1\;.$$

The space of fields and anti-fields is equipped by a natural symplectic structure, corresponding to the BV bracket

$$(A,B) \;=\; \int_{\mathcal M}\delta_i A\delta^i_* B\;,\qquad {\rm gh}(\cdot,\cdot) \;=\; 1\;.$$

Classical and quantum master equation

One can then show that if the anti-fields are eliminated by means of a canonical transformation then the path-integral over the remaining Lagrangian submanifold does not depend on the choice of the former provided that the BV action obeys the quantum master equation

$$(S, S) + \frac{i}{2}\hbar \Delta S \; = \; 0 \; ,$$

where Δ, the BV Laplacian, is the slightly singular operator defined by

$$\Delta \; = \; \int_{\mathcal{M}} \delta_i \delta^i_\star \; , \qquad \mathrm{gh}\Delta \; = \; 1 \; .$$

This operator is nilpotent but it does not act as a differential; rather

$$\Delta(AB) - \Delta(A)B - A\Delta(B) \; = \; (A, B) \; .$$

It follows that the BRST operator s, defined by the differential

$$sA \; := \; (S, A) \; ,$$

is nilpotent and is generated by a BRST current only if $\Delta S = 0$.

The AKSZ minimal quantum master action

For the PSM, the "minimal" quantum master action can be written in a remarkably compact form by extending the classical differential forms into unconstrained vectorial superfields of fixed total degree p given by the sum of form degrees and ghost numbers:

$$\mathbf{X}^\alpha = \underbrace{X^{\alpha\,\langle p_\alpha\rangle}_{[0]} + X^{\alpha\,\langle p_\alpha-1\rangle}_{[1]} + \dots + X^{\alpha\,\langle 0\rangle}_{[p_\alpha]}}_{\textit{fields}} + \underbrace{X^{\alpha\,\langle -1\rangle}_{[p_\alpha+1]} + X^{\alpha\,\langle -2\rangle}_{[p_\alpha+2]} + \dots + X^{\alpha\,\langle p_\alpha-\hat{p}-1\rangle}_{[\hat{p}+1]}}_{\textit{antifields}} ,$$

$$\mathbf{P}_\alpha = \underbrace{P^{\langle \hat{p}-p_\alpha\rangle}_{\alpha\ [0]} + P^{\langle \hat{p}-p_\alpha-1\rangle}_{\alpha\ [1]} + \dots + P^{\langle 0\rangle}_{\alpha\ [\hat{p}-p_\alpha]}}_{\textit{fields}} + \underbrace{P^{\langle -1\rangle}_{\alpha\ [\hat{p}-p_\alpha+1]} + P^{\langle -2\rangle}_{\alpha\ [\hat{p}-p_\alpha+2]} + \dots + P^{\langle -p_\alpha-1\rangle}_{\alpha\ [\hat{p}+1]}}_{\textit{antifields}} ,$$

The action functional

$$S^{\mathrm{AKSZ}}_{\mathrm{bulk}} = \int_{\mathcal{M}} \left[\mathrm{d}\mathbf{X}^\alpha \mathbf{P}_\alpha - \mathcal{H}(\mathbf{X},\mathbf{P}) \right]$$

then obeys

$$(S^{\mathrm{AKSZ}}_{\mathrm{bulk}}, S^{\mathrm{AKSZ}}_{\mathrm{bulk}}) = 0 = \Delta S^{\mathrm{AKSZ}}_{\mathrm{bulk}} .$$

Vasiliev's HSGR and the twistor map

The PSM for Vasiliev's 4D HSGR has a non-commutative base manifold $\mathcal{M}$, referred to as the "correspondence space", containing, as subspaces, a phase-hyperspacetime, say with coordinates $(x^M, p_M) \supset (x^\mu, p_\mu))$, and a twistor space, say with coordinates (y^α, z^α).

Locally, the classical unfolded system theory can be projected down to spacetime or to twistor space; these two formulations are thus related by a "twistor map" made explicit via the "parent formulation" in the correspondence space.

Classical variational principle

Generalized Hamiltonian bulk action

$$S^{\rm cl}_{\rm bulk} \;=\; \int_{\mathcal{M}} \left[U \star DB + V \star \left(F + \mathcal{F}(B;J) + \widetilde{\mathcal{F}}(U;J) \right) \right] \;,$$

where $DB = dB + [A, B]_\star$, $F = dA + A \star A$ and J are d-closed central elements (of even form degrees).

If $\hat{p} = 2n$, the fields decompose under form degree as follows:

$$A \;=\; A_{[1]} + A_{[3]} + \cdots + A_{[2n-1]} \;, \qquad B \;=\; B_{[0]} + B_{[2]} + \cdots + B_{[2n-2]} \;,$$

$$U \;=\; U_{[2]} + U_{[4]} + \cdots + U_{[2n]} \;, \qquad V \;=\; V_{[1]} + V_{[3]} + \cdots + V_{[2n-1]} \;.$$

The variational principle, including boundary conditions on (U, V), implies the duality-extended equations of motion on $\partial\mathcal{M}$:

$$F + \mathcal{F} \;\approx\; 0 \;, \qquad DB \;\approx\; 0 \;.$$

Graded-associative specialization of AKSZ action

Following the adaptation of the AKSZ procedure to the graded-associative case, one can write a minimal BV action by extending the master fields into superfields of fixed total degree, *i.e.* $(A, B, U, V) \to (\mathbf{A}, \mathbf{B}, \mathbf{U}, \mathbf{V})$.

The action

$$S \;=\; \sum_{\xi} \int_{\mathbf{M}_\xi} \mathrm{Tr}\left[\mathbf{U} \star D\mathbf{B} + \mathbf{V} \star \left(\mathbf{F} + \mathcal{F}(\mathbf{B}; J) + \widetilde{\mathcal{F}}(\mathbf{U}; J)\right)\right],$$

obeys the classical master equation

$$(S, S) \;=\; 0\;,$$

modulo boundary terms that vanish provided (A, B) are associated to the structure group and (U, V) are taken to transform as sections.

What's next?

- Whether the bulk action also obeys the quantum master equation is under investigation.
- The classical theory admits a number of topological impurities $\mathcal{V}$, all of which require the topological breaking of some gauge symmetries; these obey $(S, \mathcal{V}) = 0$ making it an interesting problem to study the properties of $(\mathcal{V}, \mathcal{V})$ and $\Delta\mathcal{V}$.
- In particular, there exists such perturbations based on four-forms that are reminiscent of on-shell actions; whether some of these actually correspond to the free 3D CFT is an open problem.
- An interesting feature of the AKSZ formalism is the existence of "non-minimal" gauge-fixing schemes — some of which are based on including large canonical BV transformations that exchange fields and anti-fields, — that yield cancellation of all vacuum bubbles up to topological terms implying that the partition function can be summed over topologies; in particular, there exists a "maximal" gauge-fixing scheme with this property that we claim generalizes to HSGR.

Conclusions

- Simplicity of the unfolded approach HSGR encourages excursions into new domains of field theory.
- While progress is being made, it is still to early to tell whether there are ways leading back to the "camp", *e.g.* Witten diagrams with external massless fields (and possibly only very mild quantum corrections).
- So far, several positive results have been accumulated, though one cannot say yet that there exists a bridge between standard and unfolded relativistic field theory.

Thanks for your attention!

Progression of structures

- Standard BV: "metric formalism" on commutative manifolds; all objects assigned a degree = second-quantized ghost number; the Grassmann parity governed by the ghost number.
- AKSZ: "frame formalism" on graded-commutative manifolds; all objects assigned a bi-degree = form degree, *i.e.* first-quantized ghost number, and second-quantized ghost number; the Grassmann parity is now governed by the sum of the two ghost numbers.
- Vasiliev's HSGRs: "frame formalism" on graded-associative base manifolds with first-quantized $\star$-products; all objects are still assigned bi-degree and Grassmann parity as in the graded-commutative case.

Certain key features of AKSZ's formalism carries over to Vasiliev's HSGRs, which consist perturbatively of symmetric tensors. Going beyond these, to mixed-symmetry fields, first-quantized, homotopy-associative DAs may play a rôle: preliminary results show that certain features of AKSZ indeed carry over also to this case as well.

Key point

A classical system may admit several PS, giving rise to distinct quantum algebras, each of which, in its turn, may admit many distinct representations.

The notions of "algebras" and "representations", blurred already within the associative context where the latter emerge as (non-polynomial) functions in enveloping algebras, become more intimately connected as associativity is deformed by generalized PS.

Moreover, in the path-integral, the aforementioned algebraic structures are linked to super-selection sectors: various boundary conditions and corresponding classical solution subspaces are paired with different classes of observables.

Use AKSZ formalism to explore interplay between PSs and super-selection in generally-covariant field theories, and, in particular, (tensionless) string (field) theory.

Rough picture

Classical field theories, containing local degrees of freedom, arise, unfolded, as classical limits of **topological "bulk" actions** of the AKSZ type; the latter can be **deformed in three sectors**:

- First-quantization: non-commutative base-manifolds including phase-spacetime and twistor spaces
- Second-quantization: "boundary" actions and other topological impurities
- Third-quantization: topological summation including proper (generalized) PS

“Matrioshka” and Quantum Gauge Principle

Non-commutative manifolds, associative as well as homotopy-associative, arise from path integrals Poisson sigma models (PSMs) of the AKSZ-type.

Deforming the latter by topological impurities, grouped into a master field, and assuming that the PS contains a bi-vector field, yields master-field equations that are again of the unfolded type but now in the target space of the PSM: their Cartan gauge-invariances subsume target-space diffeomorphism invariance and other gauge symmetries.

Influenced also by String Field Theory (SFT) and desires to improve its (second-quantized!) BRST operations, one is led to propose a refined Gauge Principle:

Nature is described by layers — or a web — of PSMs such that radiative corrections at one level corresponds to the topological summation at the level below and the DA structure at the level above.

What's special about Vasiliev's 4D HSGR 4D in all this?

- Examples of **fully non-linear** unfolded descriptions of self-interacting, generally-covariant *classical* field theories — currently, they and their lower-D avatars (including 3D CS HSGR) are the only known such models, while (super)gravities remain to be unfolded in the curvature sector *i.e.* normal-coordinate expanded on-shell.
- Exhibit a remarkable combination of **integrable structures** based on Maurer – Cartan forms and Wigner-deformed oscillators, facilitating constructions of complete spaces of algebraically special solutions and corresponding classical observables, opening up not only for super-selection studies at the classical level but, potentially, also for geometric quantization (on the phase space of the field theory).
- **"First-quantized" descriptions** introducing "germs" of extended objects realizing Flato – Fronsdal's massless particle=(singleton)^2, either via 3D conformal fields (CFT3), or via 2D PSMs of the Cattaneo – Felder - type, opening up for studying *choices of Poisson structures*.

“Massless” Poisson structure

Massless HS fields, Fronsdal tensors, arise in Vasiliev’s HSGR upon classical perturbative expansion around four-dimensional spacetimes. In asymptotically AdS spacetimes, they correspond holographically to bilinear HS currents in CFT3.

Conjecture: the weakly-coupled 1/N-expansions of the CFT3s correspond to choosing a PS for Vasiliev’s 4D HSGR that is trivial in its Fronsdal sector.

In other words, there exists an off-shell formulation such that upon imposing asAdS boundary conditions, the effective bulk action is given on-shell by the classical action plus, possibly, vacuum bubbles — not as an effect of remarkable cancellations of but rather by construction.

Other PS? Hints from HS symmetry breaking

4D HSGR contains composite Goldstone particles built from two massless particles.

Conjecture: the HS gauge symmetries are broken via “radiative” corrections whereby massless propagators mix with cubic and quartic tree-level diagrams with one and two external Goldstone particles, respectively, and the resulting reshuffling of the classical action brings about (sufficient) $\hbar$-corrections corresponding to the 1/N-corrections to the critical CFT3.

In other words, broken phases include massive particles arising as multi-singleton states that are not totally symmetric.

"Stringy" PS: 1P-states as cyclic multi-singletons

Following the Regge trajectories of Type IIB string theory in 10D flat spacetime into $AdS(5) \times S(5)$ with decreasing radius, focusing on fixed spins, assuming Polyakov's "non-intersection principle" and counting modulo Kaluza-Klein modes and Goldstone modes, one is led to "le Grande Buffe" ($P > 1$):

$(P-1)^{\mathrm{st}}$ Regge trajectory $\leftrightarrow$ cyclic P-singleton states $\leftrightarrow$ single-trace operators in free $\mathcal{N} = 4_4$ super Yang – Mills theory (SYM4).

For finite string and gauge couplings, g_{s} and g_{YM} say, the quantum corrections from the stringy PS, for cyclic multi-singletons, amounts, pictorially, to genus expansion of worldsheets/t'Hooft "fishnets".

Fitting HSGR into classical tensionless SFT

In the zero-radius, or tensionless, limit, keeping 10D Planck length fixed, one has g_{s}, $g_{\mathrm{YM}} \to 0$, *i.e.* the stringy PS is scaled away and, correspondingly, the YM connection, say $A \in \mathfrak{su}(N)$, can be re-scaled such that $dA + g_{\mathrm{YM}} A^2$ becomes abelian.

The limiting SFT is thus classical; it contains massless HS particles in AdS5, corresponding to bi-linear single-trace currents, which happen to be $S_2 \equiv Z_2$- and $O(N^2-1)$-invariant and close among themselves; it also contains Goldstone particles (and Kaluza – Klein modes on $S(5)$). One is led to:

Conjecture: To leading order in $1/N$, the single-trace sector of free SYM4 corresponds to classical limit of Type IIB SFT on $AdS(5) \times S(5)$ containing Vasiliev-type $\mathcal{N} = 4_5$ HSGR as a consistent truncation.

HS symmetry breaking mediated in classical theory via fundamental Goldstone fields.

Mild sub-leading 1/N-corrections arise via "course fishnets" plus mixing

"Membrany" PS: 1P-states as totally symmetric multi-singletons

However, in the high-energy limit of mechanical strings in AdS, the motion of relativistic "cusps", realized as solitons of the Nambu – Goto action, exhibit a loss of cyclicity in favor of total symmetry.

The cusps carry energy-momenta that are essentially the same as those for singletons, at the classical as well as the semi-classical level.

The arising of totally-symmetric multi-singleton states can be manifest by discretization followed by taking the tensionless limit $\rightsquigarrow$ two-dimensional chiral Wess – Zumino – Witten models
Conjecture: Vasiliev-type HSGR can be embedded consistently in theories