Lecture II

Symmetries (mostly discrete)

- Quantum mechanical realization of symmetries
- Discrete symmetries: C, P & T
- CP and CPT

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Prelude: Noether's theorem

Emmy Noether
(1882-1935)

In classical mechanics and classical field theory, one of the most important consequences of the existence of symmetries is spelled out by Noether's theorem: associated with each *continuous* symmetry there is a conserved charge.

In classical mechanics:

$$q_i(t) \to q_i'(t,\varepsilon)$$

$$L(q',\dot{q}') = L(q,\dot{q}) + \frac{d}{dt}f(q,\varepsilon) \quad \Longrightarrow \quad \dot{Q} = 0 \quad \text{with} \quad Q \equiv \frac{\partial L}{\partial \dot{q}_i}\delta_\varepsilon q_i - f(q,\delta\varepsilon)$$

In classical field theory:

$$\phi(x) \to \phi(x) + \delta_\varepsilon \phi(x)$$

$$\delta_\varepsilon \mathscr{L} = \partial_\mu K^\mu \quad \Longrightarrow \quad \partial_\mu J^\mu = 0 \quad \text{with} \quad J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta_\varepsilon \phi - K^\mu$$

The associated conserved charge is given by $Q \equiv \int d^3x J^0(t,\mathbf{x})$

Quantum Mechanical Realization of Symmetries

Quantum mechanically, a symmetry acts on states in such a way that both probabilities and transition amplitudes are left invariant, i.e. if $|\alpha\rangle, |\beta\rangle \in \mathscr{H}$

$$|\alpha\rangle \longrightarrow |\alpha'\rangle \qquad |\beta\rangle \longrightarrow |\beta'\rangle \qquad \text{such that} \qquad |\langle\alpha|\beta\rangle| = |\langle\alpha'|\beta'\rangle|$$

Wigner's theorem prescribes that such transformations are implemented by ***unitary*** or ***antiunitary*** operators:

Unitary	Antiunitary
$\mathscr{U}\Big(a\|\alpha\rangle + b\|\beta\rangle\Big) = a\mathscr{U}\|\alpha\rangle + b\mathscr{U}\|\beta\rangle, \qquad a, b \in \mathbb{C}$	$\mathscr{U}\Big(a\|\alpha\rangle + b\|\beta\rangle\Big) = a^*\|\mathscr{U}\alpha\rangle + b^*\|\mathscr{U}\beta\rangle, \qquad a, b \in \mathbb{C}$
$\langle\mathscr{U}\alpha\|\mathscr{U}\beta\rangle = \langle\alpha\|\beta\rangle$	$\langle\mathscr{U}\alpha\|\mathscr{U}\beta\rangle = \langle\alpha\|\beta\rangle^*$

Continuous symmetries are always implemented by *unitary* operators. Exercise

Discrete symmetries be implemented both by *unitary* and *antiunitary* operators.

In the presence of ***continuous symmetries*** the conserved charges Q^a are converted upon quantization in operators such that

But wait until next Monday for important counterexamples!

$$\{Q^a, H\}_{\mathrm{PB}} = 0 \quad \Longrightarrow \quad [Q^a, H] = 0$$

These charges generate the action of the symmetry on the Hilbert space

$$\mathscr{U}(\alpha) = e^{i\alpha^a Q^a} \quad \Longrightarrow \quad \mathscr{U}(\alpha) H \mathscr{U}(\alpha)^\dagger = H$$

There are, however, two ways in which this symmetry can be realized on the spectrum of the theory:

- ***Wigner-Weyl realization.*** The ground state is invariant under the symmetry

$$\mathscr{U}(\alpha)|0\rangle = |0\rangle \quad \Longrightarrow \quad Q^a|0\rangle = 0$$

The spectrum is then classified in multiplets that transform in irreducible representations of the symmetry group, e.g., the hydrogen atom:

$$|\alpha, j, m\rangle \xrightarrow{\mathrm{SO}(3)} |\alpha, j, m'\rangle = \sum_{m=-j}^{j} \mathscr{D}^{(j)}_{m'm}(\theta, \varphi)|\alpha, j, m\rangle$$

total spin (orbital+spin+nuclear)

- ***Nambu-Goldstone realization.*** The ground state is ***not*** invariant under the symmetry:

$$e^{i\alpha^a Q^a}|0\rangle \neq |0\rangle \quad \Longrightarrow \quad Q^a|0\rangle \neq 0 \qquad \text{(at least for some } a\text{'s)}$$

In this case there is a very important result known as Goldstone's theorem:

For every generator broken by the vacuum, the spectrum of the theory contains a massless mode (called the Nambu-Goldstone mode)

Yoichiro Nambu
(b. 1921)

In particle physics, the typical example of Nambu-Goldstone bosons is provided by the pions. Let us consider two-flavor QCD

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \overline{u}\left(i\not{D} - m_u\right)u + \overline{d}\left(i\not{D} - m_d\right)d$$

In the ***chiral limit***, i.e. when $m_u = m_d = 0$, the Lagrangian is invariant under a global $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ acting as

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \longrightarrow M_L \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \longrightarrow M_R \begin{pmatrix} u_R \\ d_R \end{pmatrix} \qquad M_L, M_R \in \mathrm{SU}(2)$$

Exercise: prove it

Jeffrey Goldstone
(b. 1933)

At low energies, the strong interaction between quarks produces quark-antiquark condensates

$$\langle \overline{q}q \rangle = \langle \overline{q}_L q_R + \overline{q}_R q_L \rangle \neq 0$$

These condensates only preserve the "diagonal" SU(2) where $M_L = M_R$. Then the global $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is spontaneously broken

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R = \mathrm{SU}(2)_V \times \mathrm{SU}(2)_A \longrightarrow \mathrm{SU}(2)_V$$

According to Goldstone's theorem, there will be one massless mode for each of the ***three*** broken generators, with the same quantum numbers as the broken currents

$$J_a^\mu = (\overline{u}, \overline{d})\gamma^\mu \frac{\sigma_a}{2}\begin{pmatrix} u \\ d \end{pmatrix} \Longrightarrow \mathrm{SU}(2)_V$$

$$J_{a5}^\mu = (\overline{u}, \overline{d})\gamma^\mu \gamma_5 \frac{\sigma_a}{2}\begin{pmatrix} u \\ d \end{pmatrix} \Longrightarrow \cancel{\mathrm{SU}(2)_A}$$

Exercise: prove that these are the conserved currents associated with $\mathrm{SU}(2)_V$ and $\mathrm{SU}(2)_A$

The broken current interpolates between the vacuum and the one-pion state

$$\langle 0|J_{a5}^\mu(x)|\pi^b(\mathbf{p})\rangle = -if_\pi \delta^{ab} p^\mu e^{-iE_{\mathbf{p}}t + i\mathbf{p}\cdot\mathbf{x}} \neq 0$$

At low energies, the strong interaction between quarks produces quark-antiquark condensates

$$\langle \overline{q}q\rangle = \langle \overline{q}_L q_R + \overline{q}_R q_L\rangle \neq$$

Show that $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ can be decomposed as $\mathrm{SU}(2)_V\times\mathrm{SU}(2)_A$ where

$$\mathrm{SU}(2)_V : \begin{cases} U\psi_L \\ U\psi_R \end{cases} \qquad \mathrm{SU}(2)_A : \begin{cases} U\psi_L \\ U^{-1}\psi_R \end{cases}$$

These condensates only preserve the "diagonal" S ... e global $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ is spontaneously broken

$$\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R = \mathrm{SU}(2)_V\times\mathrm{SU}(2)_A \longrightarrow \mathrm{SU}(2)_V$$

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$$J_a^\mu = (\overline{u},\overline{d})\gamma^\mu\frac{\sigma_a}{2}\begin{pmatrix} u \\ d\end{pmatrix} \quad\Longrightarrow\quad \mathrm{SU}(2)_V$$

$$J_{a5}^\mu = (\overline{u},\overline{d})\gamma^\mu\gamma_5\frac{\sigma_a}{2}\begin{pmatrix} u \\ d\end{pmatrix} \quad\Longrightarrow\quad \cancel{\mathrm{SU}(2)_A}$$

Exercise: prove that these are the conserved currents associated with $\mathrm{SU}(2)_V$ and $\mathrm{SU}(2)_A$

The broken current interpolates between the vacuum and the one-pion state

$$\langle 0|J_{a5}^\mu(x)|\pi^b(\mathbf{p})\rangle = -if_\pi\delta^{ab}p^\mu e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}} \neq 0$$

Thus, the three pions can be identified as the Nambu-Goldstone bosons associated with the spontaneous symmetry breaking of chiral symmetry.

But then, why are they not massless?

The u and d quarks have small but nonzero masses. Therefore, chiral symmetry is only ***approximate*** and the (pseudo)Goldstone bosons are not strictly massless.

Still, they are the lightest hadrons. The reason is that the pion mass goes to zero with the quark masses $m_{u,d}$, whereas the proton mass is determined by $\Lambda_{\rm QCD}$.

Goldstone bosons appear as well in other fields such as condensed matter physics. For example, in the case of a Heisenberg ferromagnet with Hamiltonian

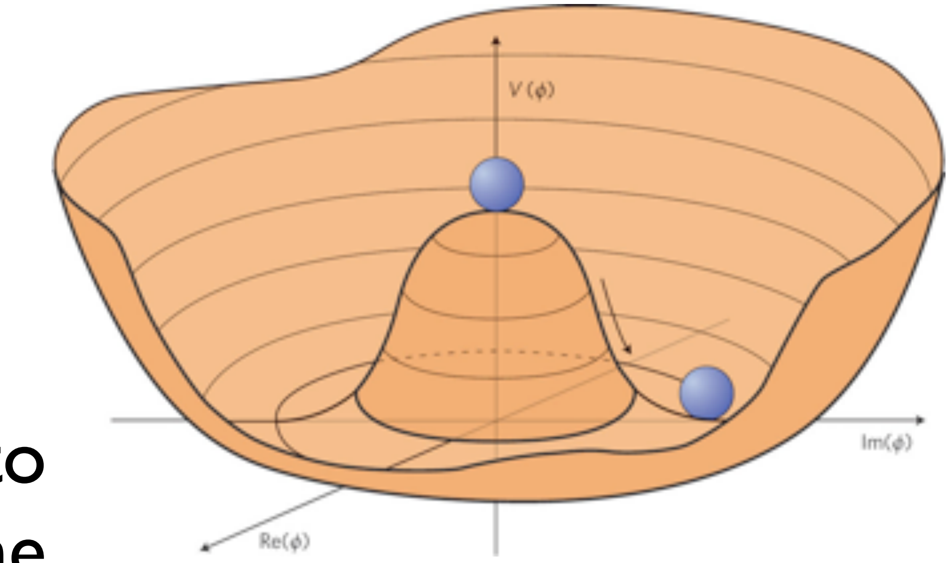
$$H = -J \sum_{\langle \mathbf{x},\mathbf{x}' \rangle} \mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}') \qquad \text{with} \qquad J > 0$$

Although the Hamiltonian is invariant under rotations, the ground state breaks it because of spontaneous magnetization.

The spin waves (magnons) are the associated Goldstone bosons.

Nambu-Goldstone bosons play a very important role in the Brout-Englert-Higgs mechanism (remember the first lecture)

$$V(U^\dagger U) = \frac{\lambda}{4}\left(\frac{M}{g_{\rm YM}}\right)^4\left[\frac{1}{2}\mathrm{Tr}\,(U^\dagger U) - 1\right]^2$$



The "angular" part of the field $U(x)$ corresponds to the massless excitations along the "valley" of the potential

$$U(x) = U_0(x)\left[1 + \frac{g_{\rm YM}}{\sqrt{2}M}h(x)\right] \qquad \text{where} \qquad U_0(x) = \exp\left[i\vartheta^a(x)\frac{\sigma_a}{2}\right]$$

Thus, the three Goldstone bosons associated with the spontaneous symmetry breaking of SU(2) are contained in the Stückelberg field.

When it is gauge away (i.e., in the unitary gauge $U_0(x) = \mathbf{1}$) these Goldstone bosons are transmuted into the longitudinal components of the three massive SU(2) gauge fields.

Discrete Symmetries

We begin with the equations for the electromagnetic field coupled to a number of charged particles

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu, \\ \varepsilon^{\mu\nu\sigma\lambda}\partial_\nu F_{\sigma\lambda} &= 0, \\ m_k \frac{d^2 x_k^\mu}{d\tau^2} &= F^{\mu\nu}(x_k) j_\nu(x_k), \end{aligned}$$

with $$j^\mu(x) = \sum_{k=1}^{N} q_k \int d\tau \frac{dx_k^\mu}{d\tau} \delta^{(4)}\left[x^\mu - x_k^\mu(\tau)\right]$$

These equations are invariant under three discrete symmetries:

- ***Parity***: it reverses the sign of the spatial coordinates leaving time invariant

$$\mathrm{P}: (x^0, \mathbf{x}) \longrightarrow (x^0, -\mathbf{x})$$

On the electromagnetic potential it acts as

$$\mathrm{P}: \begin{cases} A_0(x^0, \mathbf{x}) \longrightarrow A_0(x^0, -\mathbf{x}) \\ \mathbf{A}(x^0, \mathbf{x}) \longrightarrow -\mathbf{A}(x^0, -\mathbf{x}) \end{cases}$$

- ***Charge conjugation***: it reverses the sign of all charges

$$\mathrm{C}: q_a \longrightarrow -q_a$$

while reversing as well the sign of the electromagnetic field

$$\mathrm{C}: A_\mu(x) \longrightarrow -A_\mu(x)$$

- ***Time reversal***: it reverses the flow of time, i.e.,

$$\mathrm{T}: (x^0, \mathbf{x}) \longrightarrow (-x^0, \mathbf{x})$$

and

$$\mathrm{T}: \begin{cases} A_0(x^0, \mathbf{x}) \longrightarrow A_0(-x^0, \mathbf{x}) \\ \mathbf{A}(x^0, \mathbf{x}) \longrightarrow -\mathbf{A}(-x^0, \mathbf{x}) \end{cases}$$

The electric field $\mathbf{E}$ is reversed by P and C, and left invariant by T, while the magnetic field $\mathbf{B}$ changes sign under C and T but is left invariant by P.

Exercise: show that the equations shown in the previous transparency are indeed invariant under P, C and T

Specially interesting are the action of P, C and T on a classical Dirac field. On general grounds the transformation has to be of the form

$$\psi_\alpha(x) \longrightarrow \Gamma_{\alpha\beta}\psi_\beta(x') \qquad \text{or} \qquad \psi_\alpha(x) \longrightarrow \Gamma_{\alpha\beta}\psi_\beta(x')^*$$

where

$$x'^{\mu} = (x^0, -\mathbf{x}) \qquad \text{for P}$$

$$x'^{\mu} = x^\mu \qquad \text{for C}$$

$$x'^{\mu} = (-x^0, \mathbf{x}) \qquad \text{for T}$$

Now, the transformed fields have to satisfy the Dirac equation with respect to the coordinates x^μ

$$\Big(i\gamma^\mu\partial_\mu - m\Big)\psi(x) = 0 \longrightarrow \begin{cases} \Big(i\gamma^\mu\partial_\mu - m\Big)\Gamma\psi(x') = 0 \\ \Big(i\gamma^\mu\partial_\mu - m\Big)\Gamma\psi(x')^* = 0 \end{cases}$$

In the case of ***parity***, assuming $\psi_\alpha(x) \longrightarrow \Gamma_{\alpha\beta}\psi_\beta(x')$ we find that

$$[\Gamma,\gamma^0]=0 \qquad \{\Gamma,\gamma^i\}=0$$

Hence, we can choose

$$\mathrm{P}: \psi(x^0,\mathbf{x}) \longrightarrow \eta_P\gamma^0\psi(x^0,-\mathbf{x})$$

with η_P an arbitrary phase.

For ***charge conjugation***, we assume $\psi_\alpha(x) \longrightarrow \Gamma_{\alpha\beta}\psi_\beta(x')^*$. Then

$$\Gamma^{-1}\gamma^\mu\Gamma = -\gamma^{\mu *}$$

The solution depends on the representation of the Dirac algebra. Taking

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

we find

$$\mathrm{C}: \psi(x) \longrightarrow \eta_C(-i\gamma^2)\psi(x)^*$$

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Exercise:

Consider a complex scalar field coupled to electromagnetism

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$$S=\int d^4x\left[(D_\mu\phi)^*D^\mu\phi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \qquad \text{with} \qquad D_\mu=\partial_\mu+iqA_\mu$$

and show that

$$\mathrm{C}:\phi(x)\longrightarrow\eta_C\phi(x)^* \qquad \mathrm{T}:\phi(t,\mathbf{x})\longrightarrow\eta_T\phi(-t,\mathbf{x})^*$$

with η_C,η_T phases.

with

For ***charge conjugation***, we assume $\psi_\alpha(x)\longrightarrow\Gamma_{\alpha\beta}\psi_\beta(x')^*$. Then

$$\Gamma^{-1}\gamma^\mu\Gamma=-\gamma^{\mu *}$$

The solution depends on the representation of the Dirac algebra. Taking

$$\gamma^0=\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}, \qquad \gamma^i=\begin{pmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{pmatrix}$$

we find

$$\mathrm{C}:\psi(x)\longrightarrow\eta_C(-i\gamma^2)\psi(x)^*$$

Finally, in the case of ***time reversal***, assuming $\psi_\alpha(x) \longrightarrow \Gamma_{\alpha\beta}\psi_\beta(x')^*$,

$$\Gamma^{-1}\gamma^0\Gamma = \gamma^{0*}, \qquad \Gamma^{-1}\gamma^i\Gamma = -\gamma^{i*}$$

In the previous representation the solution is

$$\mathrm{T}: \psi(t,\mathbf{x}) \longrightarrow \eta_T(-\gamma^1\gamma^3)\psi(-t,\mathbf{x})^*$$

The fields transformed under P, C and T actually transform as Dirac spinors under the Lorentz group. This follows from the fact that the matrices

$$\begin{aligned} \gamma^{\mu\dagger} &= \gamma^0\gamma^\mu\gamma^0, \\ -\gamma^{\mu *} &= (-i\gamma^2)\gamma^\mu(-i\gamma^2)^{-1}, \\ \gamma^{\mu T} &= (-\gamma^1\gamma^3)\gamma^\mu(-\gamma^1\gamma^3)^{-1} \end{aligned}$$

are representations of the Dirac algebra. Exercise: prove it

With this results we can now construct the P, C and T transformation of quantum fields.

According to Wigner's theorem, these symmetries are implemented by unitary or antiunitary operators. In the case of parity and charge conjugation they are unitary:

$$\mathscr{P}\psi(x^0,\mathbf{x})\mathscr{P}^{-1} = \eta_P\gamma^0\psi(x^0,-\mathbf{x})$$

$$\mathscr{C}\psi(x^0,\mathbf{x})\mathscr{C}^{-1} = \eta_C(-i\gamma^2)\psi(x^0,\mathbf{x})^* = \eta_C(i\gamma^0\gamma^2)\overline{\psi}(x)^T$$

In terms of the creation-annihilation operators

$$\mathscr{P}b(\mathbf{k},s)\mathscr{P}^{-1} = \eta_P b(-\mathbf{k},s)$$

$$\mathscr{P}d(\mathbf{k},s)\mathscr{P}^{-1} = -\eta_P^* d(-\mathbf{k},s)$$

$$\mathscr{P}|\mathbf{k},s;0\rangle = \eta_P|-\mathbf{k},s;0\rangle$$

$\downarrow$

intrinsic parity

$$\mathscr{C}b(\mathbf{k},s)\mathscr{C}^{-1} = \eta_C d(\mathbf{k},s)$$

$$\mathscr{C}d(\mathbf{k},s)\mathscr{C}^{-1} = \eta_C^* b(\mathbf{k},s)$$

C interchanges particles and antiparticles

Time reversal, on the other hand, is implemented by an antiunitary operator. To see this, we notice that, by definition, it inverts time evolution

$$\mathscr{T} e^{-itH} \mathscr{T}^{-1} = e^{itH} \quad \Longrightarrow \quad \mathscr{T}(iH)\mathscr{T}^{-1} = -iH$$

If $\mathscr{T}$ is unitary, then

$$\mathscr{T} H \mathscr{T}^{-1} = -H$$

and the transformed system is unbounded from below. Hence, the time reversal operator is *antiunitary* (and therefore it leaves the Hamiltonian invariant).

The action on the spinor field operator is given by

$$\mathscr{T} \psi(x^0, \mathbf{x}) \mathscr{T}^{-1} = \eta_T (-\gamma^1 \gamma^3) \psi(-x^0, \mathbf{x})$$

and the transformation of the creation-annihilation operators is

$$\mathscr{T} b(\mathbf{k}, s) \mathscr{T}^{-1} = (-1)^{\frac{1}{2}-s} \eta_T b(-\mathbf{k}, -s) \qquad \mathscr{T} d(\mathbf{k}, s) \mathscr{T}^{-1} = -(-1)^{\frac{1}{2}-s} \eta_T^* d(-\mathbf{k}, -s)$$

It reverses the spin and the momentum.

Majorana spinors

Having introduced charge conjugation, we can define spinors that are self-conjugate under C, i.e.,

$$\psi(x) = \eta_C (i\gamma^0\gamma^2)\overline{\psi}(x)^T \quad \Longrightarrow \quad \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \eta_C \begin{pmatrix} i\sigma^2 u_-^* \\ -i\sigma^2 u_+^* \end{pmatrix}$$

Spinors satisfying this condition are called ***Majorana spinors***. The equation is solved by

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} u_+ \\ -i\eta_C \sigma^2 u_+^* \end{pmatrix}$$

Hence, a Majorana spinor has as many degrees of freedom as a Weyl spinor. In fact,

$$\psi(x) = \frac{1}{\sqrt{2}}\left[\psi_+(x) + \psi_+^C(x)\right] \qquad \text{where} \qquad \psi_+ = \begin{pmatrix} u_+ \\ 0 \end{pmatrix}$$

Remember that we are using a representation where

$$\gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

We can write now the Dirac Lagrangian for a Majorana spinor $[\sigma^\mu_\pm = (\mathbf{1}, \pm\sigma_i)]$

$$\mathscr{L}_{\text{Dirac}} \equiv \overline{\psi}(i\partial\!\!\!/ - m)\psi = \frac{i}{2} u_+^\dagger \sigma_+^\mu \overleftrightarrow{\partial}_\mu \, u_+ - \frac{m}{2} \underbrace{\left[\eta_C^* u_+^T (i\sigma^2) u_+ + \text{h.c.}\right]}_{\text{Majorana mass term}}$$

This mass term violates the global U(1) phase symmetry

$$u_+ \longrightarrow e^{i\alpha} u_+$$

A Majorana spinor represents a "neutral" fermion that is its own antiparticle,

$$b(\mathbf{k}, s) = \eta_C d(\mathbf{k}, s)$$

Exercise: prove it

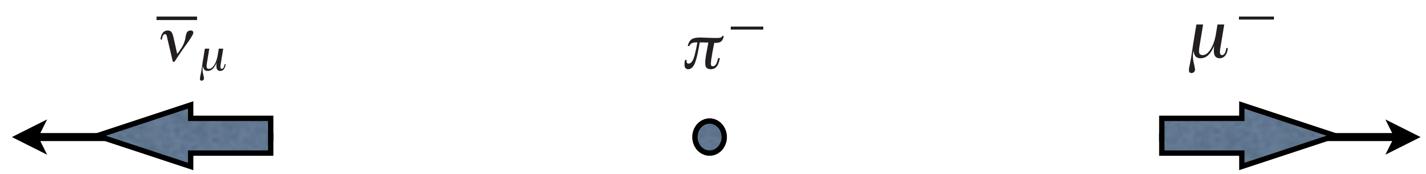
It is possible to find a representation of the Dirac matrices in which the Majorana fermion is real (this is called the Majorana representation).

CP and CPT

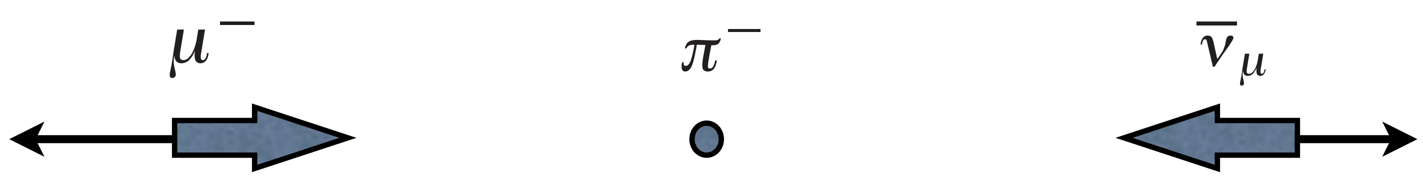
Parity is violated by weak interactions. This can be seen in the pion decay

$$\pi^- \longrightarrow \overline{\nu}_\mu + \mu^-$$

The muon is always emitted with positive helicity



whereas the parity transformed process

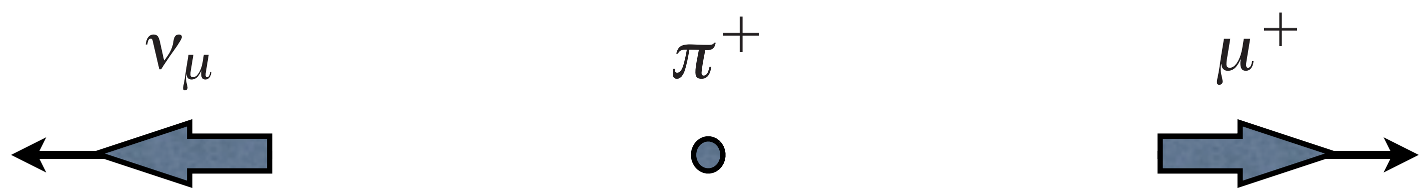


does not happens. The violation is therefore maximal.

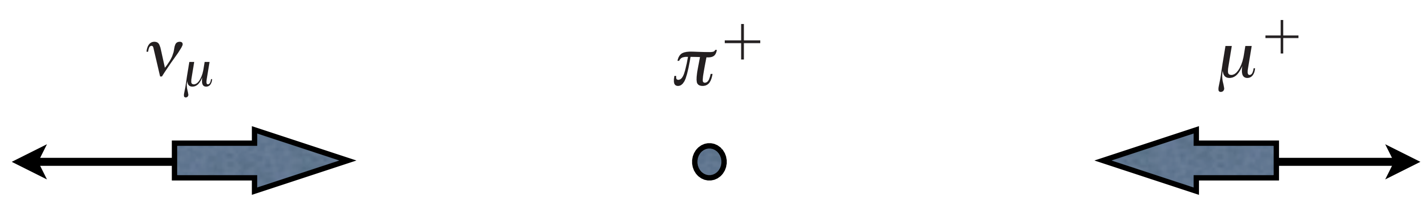
We can now look at the charge conjugate decay

$$\pi^+ \longrightarrow \nu_\mu + \mu^+$$

Since C does not change the momentum or the helicity one would expect



However, the emitted antimuon *always* has negative helicity, the process that takes place is:

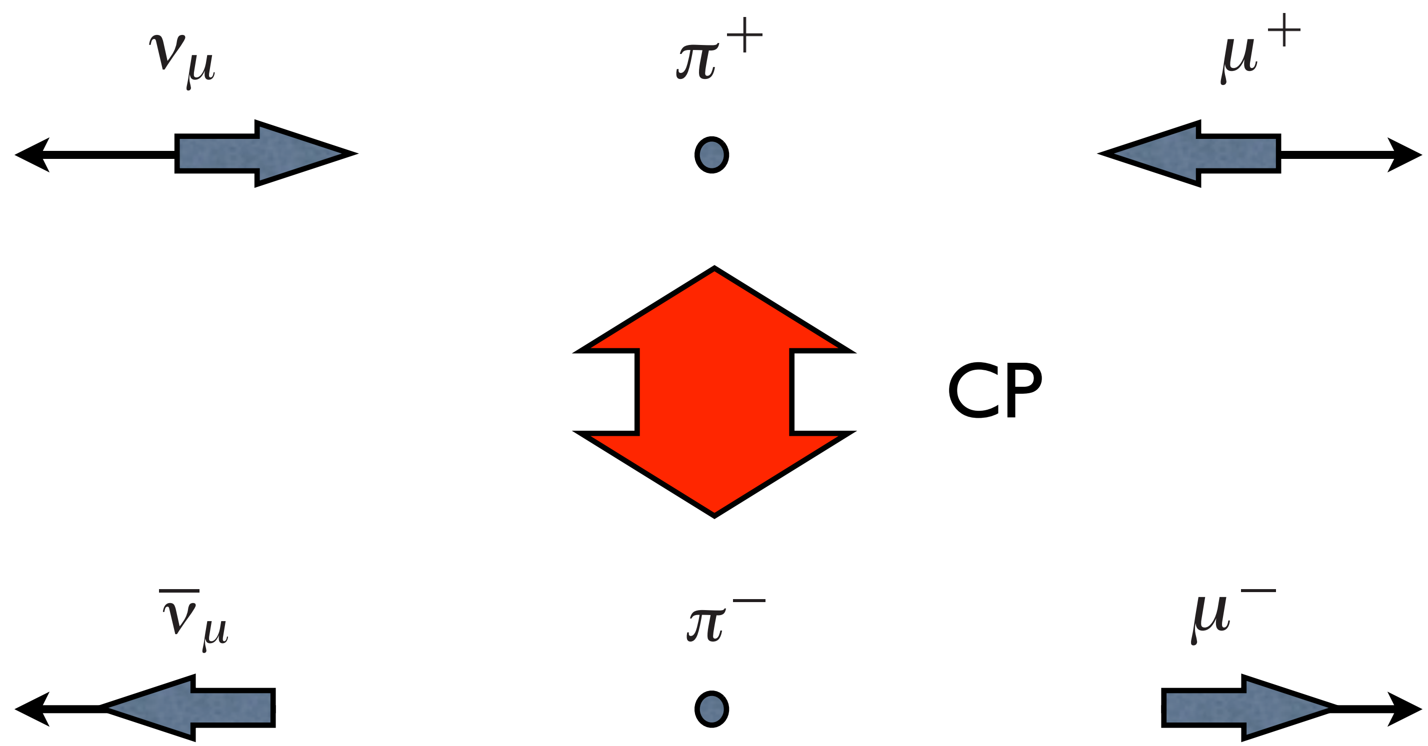


This shows that C is also maximally violated.

Although the P- and C-transformed of the processes

$$\pi^- \longrightarrow \overline{\nu}_\mu + \mu^- \qquad\qquad \pi^+ \longrightarrow \nu_\mu + \mu^+$$

do not occur in Nature, the ones transformed under the combination of C and P (called CP) exists



It was believed for some time that CP was a good symmetry of the weak interactions.

In the case of a complex scalar field, the transformation under CP is given by

$$(\mathscr{C}\mathscr{P})\phi(x^0,\mathbf{x})(\mathscr{C}\mathscr{P})^{-1} = \eta_{CP}\phi(x^0,-\mathbf{x})^\dagger$$

Exercise: prove it

It interchanges particles with antiparticles while reversing their helicities.

As it was the case with P and C, CP is also violated by weak interactions. To look for the possible sources of these violation, let us look at a theory with an interaction Hamiltonian of the form

$$H_{\text{int}} = \int d^3x \left[\sum_i g_i \mathscr{O}_i(x) + \sum_i g_i^* \mathscr{O}_i(x)^\dagger\right]$$

CP is unitary and acts on the operators by hermitian conjugation. Hence we find

$$(\mathscr{C}\mathscr{P})H_{\text{int}}(\mathscr{C}\mathscr{P})^{-1} = \int d^3x \left[\sum_i g_i \mathscr{O}_i(x)^\dagger + \sum_i g_i^* \mathscr{O}_i(x)\right]$$

The conclusion is that CP is violated unless all couplings are real.

The standard model Lagrangian contains the Cabibbo-Kobayashi-Maskawa mixing matrix:

3x3 unitary matrix $\Longrightarrow$ 3 mixing angles + 6 complex phases

Since there are six quarks, some phases could be absorbed in the quark fields. The simultaneous phase shift of all quarks is a global symmetry of the standard model Lagrangian, hence *only five phases can be absorbed*.

1 remaining complex phase $\Longrightarrow$ CP violation

There is another source of CP violation in the strong interaction sector of the standard model. There is no reason to exclude from the QCD action a term of the form

$$S_\theta = -\frac{\theta g_{\rm YM}^2}{32\pi^2}\int d^4x F_{\mu\nu}^A \widetilde{F}^{\mu\nu A} \qquad \text{where} \qquad \widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}F^{\sigma\lambda}$$

Under CP this new term changes sign, whereas the QCD action remains invariant

$$S_{\mathrm{QCD}} + S_\theta \xrightarrow{\mathrm{CP}} S_{\mathrm{QCD}} - S_\theta$$

Exercise: prove this. You need first to find the C and P transformations of a nonabelian gauge field.

The coupling θ can be bounded experimentally to be

$$|\theta| < 10^{-9}$$

It is not understood why this is so small (this is called the strong CP problem).

Additional sources of CP violation are present in various extensions of the standard model. For example, if the neutrinos have ***Dirac masses*** there is an additional phase in the leptonic mixing matrix. For ***Majorana neutrinos*** the number of phases is bigger (no phase redefinition possible).

CP violation is needed for explaining baryon asymmetry (Sakharov condition). It is also a promising window to new physics.

CPT is the combination of the three discrete symmetries we studied. For example, on a complex scalar and a Dirac field it acts respectively as

$$\Theta\phi(x)\Theta^{-1} = \phi(-x)^{\dagger} \qquad\qquad \Theta\psi(x)\Theta^{-1} = -i\gamma_5\psi(-x)^{*}$$

In the case of the fermion, we find

$$\Theta|\mathbf{p},s;0\rangle = i(-1)^{\frac{1}{2}+s}|0;\mathbf{p},-s\rangle \qquad\qquad \Theta|0;\mathbf{p},s\rangle = -i(-1)^{\frac{1}{2}+s}|\mathbf{p},-s;0\rangle$$

i.e., it interchanges particles with antiparticles and reverse the sign of the spin.

Unlike C, P, T and CP, there are very strong reasons to believe that CPT is a preserved symmetry of Nature. It can be proved that any local field theory that is Poincaré invariant is also invariant under CPT (this is known as the ***CPT theorem***)

One of the consequences of the CPT theorem is that the lifetimes and widths of a particle and its antiparticle are equal (Lüders-Zumino theorem).