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Five lectures on

Quantum Field Theory

An introduction to selected topics

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Lecture III

Renormalization

- Basics of renormalization
- Running coupling constants
- Wilsonian renormalization

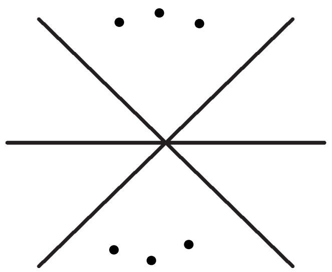
Basics of renormalization

In the perturbative calculation of amplitudes, the expansion in the number of loops is an expansion in powers of $\hbar$.

We see this in a ϕ^n theory. Restoring the powers of $\hbar$ we have

Exercise: restore the powers of $\hbar$

$$\Longrightarrow \quad \frac{i\hbar}{p^2-m^2+i\varepsilon}$$



$$\Longrightarrow \quad -i\frac{\lambda}{\hbar}$$

The power of $\hbar$ of a diagram with E external lines, I internal propagators and V vertices is

$$\#(\hbar) = I - V = L - 1$$

$$\big\downarrow$$

$$L = I - V + 1$$

Hence, a diagram with L loops scales like $\hbar^{L-1}$

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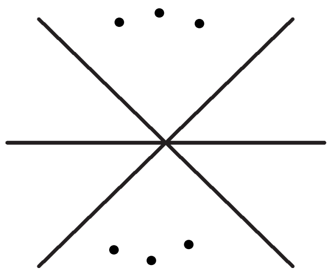

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global momentum conservation

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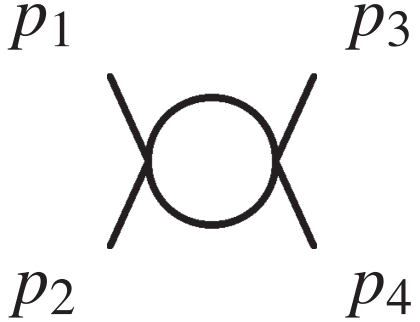
Generically, the calculation of quantum corrections to the tree level (semiclassical) amplitudes leads to divergent integrals. For example, in the case of a ϕ^4 theory the one-loop four-point amplitude diagram,

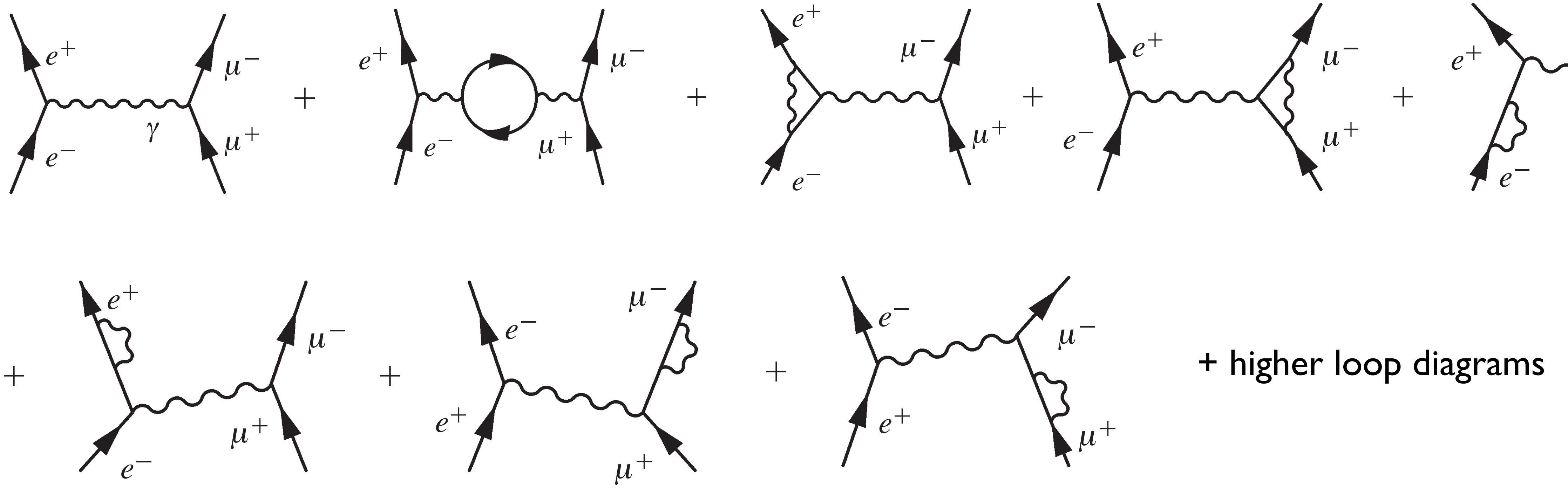


$\sim k^3 dk$

$$= \frac{\lambda^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(k+p_1+p_2)^2-m^2+i\varepsilon} \longrightarrow \infty \quad \Big(\text{logarithmically} \sim \int^{\infty}\frac{dk}{k}\Big)$$

$\sim k^4$

The same problem occurs in QED, for example in the process $e^- e^+ \to \mu^- \mu^+$:



+ higher loop diagrams

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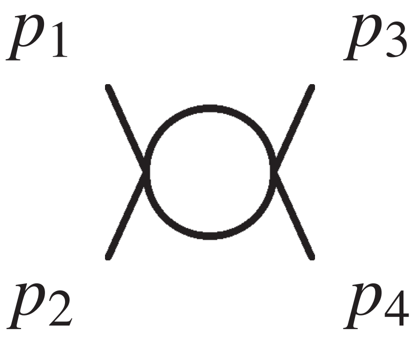

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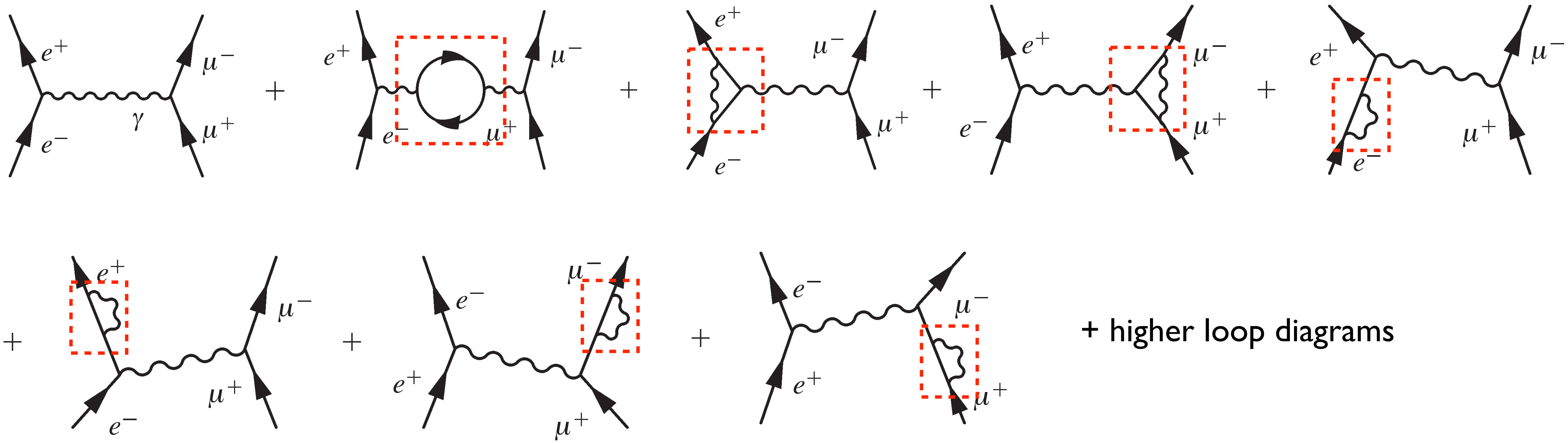


+ higher loop diagrams

All one-loop diagrams are divergent, and these divergences does no cancel between different diagrams.

To handle these divergent expressions and extract physics out of them we need to ***renormalize*** the theory.


Hendrik A. Kramers
(1894-1952)

The underlying idea is simple: the parameters in the Lagrangian (masses, couplings constants...) *are not* physical. The physical quantities, whose values are experimentally fixed, are computed in terms of these parameters.

Divergences in Feynman diagrams are then reabsorbed in the Lagrangian parameters in such a way that the physical quantities remain finite.

The renormalization procedure proceeds in two steps:

- Loop integrals have to be ***regularized*** (so they are finite and we can handle them in a mathematically sound way).
- Physical quantities have to be properly defined, using an appropriate ***renormalization prescription***.

A Word on Regularization

To discuss the different regularization methods, we look at a typical integral like

$$I=\frac{\lambda}{2}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2+i\varepsilon}\qquad\left(\sim\int^{\infty}p\,dp\right)$$

In order to make sense of this integral (i.e., regularize it) we have several alternatives. For example:

- To introduce a cutoff in the integration momentum

$$I(\Lambda)=\frac{\lambda}{2}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2+i\varepsilon}$$

A drawback of this method is its breaking of Lorentz invariance (and gauge invariance in gauge theories as well).

- To introduce a number of fictitious fields with a propagator with the "wrong sign" (Pauli-Villars method)

$$I(M_i) = \frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2-m^2+i\varepsilon} - \sum_{i=1}^{n}\frac{g_i}{p^2-M_i^2+i\varepsilon}\right)$$

Exercise: prove that the regularization of this integral requires two Pauli-Villars fields. Compute g_i.

This method preserves Lorentz and gauge invariance, although its implementation can be rather cumbersome.

- To define the integral in dimension d and continue the result to complex values of the dimension (dimensional regularization).

$$I(d) = \frac{1}{2}\lambda\mu^{4-d}\int \frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m^2+i\varepsilon}$$

μ is introduced to keep the dimensions right while mantaining λ dimensionless.

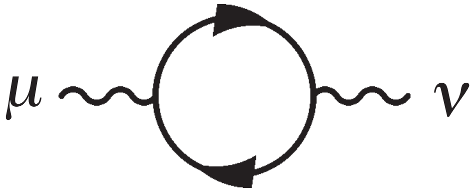
Dimensional regularization preserves Lorentz and gauge invariance. Its implementation in chiral theories requires some care.

- To define the theory on a lattice (see Istvan's lectures).

Renormalization and Running Coupling Constants

To illustrate how renormalization works, we focus on the (amputated) photon self-energy diagram in QED

$$\Pi^{\mu\nu}(q) =$$



Let's have a look at the issue of gauge invariance. Contracting the external legs with the polarization vectors $\varepsilon_\mu(q)$, $\varepsilon'_\mu(q)$, we require that the amplitude is invariant under the gauge transformations

$$\varepsilon_\mu(q) \to \varepsilon_\mu(q) + \lambda q_\mu \qquad \varepsilon'_\mu(q) \to \varepsilon'_\mu(q) + \lambda' q_\mu$$

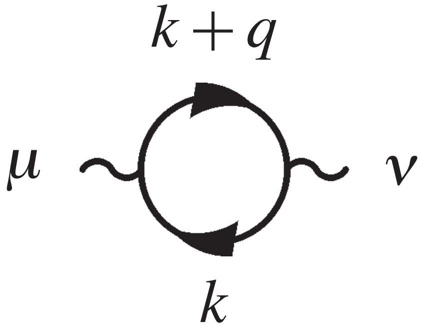
This implies that the polarization tensor has to satisfy

$$q_\mu \Pi^{\mu\nu}(q) = 0 = q_\nu \Pi^{\mu\nu}(q)$$

and therefore it has the transverse form

$$\Pi_{\mu\nu}(q) = \left(q^2 \eta_{\mu\nu} - q_\mu q_\nu\right) \Pi(q^2)$$

We compute the polarization tensor using the QED Feynman rules,



$$\equiv \Pi^{\mu\nu}(q) \;=i^2(-ie)^2(-1)\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{k}+m_e)\gamma^\mu(\not{k}+\not{q}+m_e)\gamma^\nu\right]}{\left[k^2-m_e^2+i\varepsilon\right]\left[(k+q)^2-m_e^2+i\varepsilon\right]}$$

For simplicity we set the electron mass to zero. Regularizing this integral with a momentum cutoff is delicate: it is quadratically divergent and has the structure

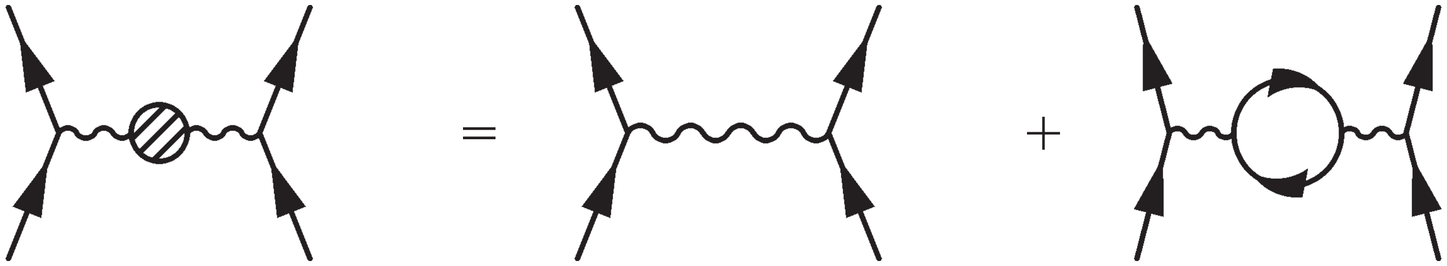
$$\Pi_{\mu\nu}(q)\sim\Lambda^2\eta_{\mu\nu}+\text{transverse part}$$

The term quadratic in Λ spoils gauge invariance but can be subtracted adding a local counterterm. The transverse part is logarithmically divergent and gives

$$\Pi(q^2)\simeq\frac{e^2}{12\pi^2}\log\left(\frac{q^2}{\Lambda^2}\right)+\text{finite terms.}$$

To follow the renormalization program we should reabsorb this divergence in the redefinition of some of the parameters of the theory.

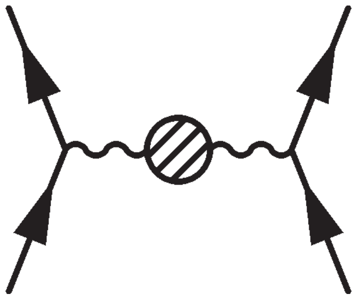
To deal with this logarithmic divergence, we look at the process $e^- e^+ \to \mu^- \mu^+$ proceeding by the interchange of "corrected" photon propagator



$$= \eta_{\alpha\beta} \left(\overline{v}_e \gamma^\alpha u_e\right) \frac{e^2}{4\pi q^2} \left(\overline{v}_\mu \gamma^\beta u_\mu\right) + \eta_{\alpha\beta} \left(\overline{v}_e \gamma^\alpha u_e\right) \frac{e^2}{4\pi q^2} \Pi(q^2) \left(\overline{v}_\mu \gamma^\beta u_\mu\right)$$

$$= \eta_{\alpha\beta} \left(\overline{v}_e \gamma^\alpha u_e\right) \left\{ \frac{e^2}{4\pi q^2} \left[1 + \frac{e^2}{12\pi^2} \log\left(\frac{q^2}{\Lambda^2}\right) \right] \right\} \left(\overline{v}_\mu \gamma^\beta u_\mu\right)$$

This looks pretty much like the tree-level scattering, but with the electric charge replaced by an effective coupling. If μ denotes the typical energy of the process



$$= \eta_{\alpha\beta} \left(\overline{v}_e \gamma^\alpha u_e\right) \left[\frac{e(\mu)^2}{4\pi q^2} \right] \left(\overline{v}_\mu \gamma^\beta u_\mu\right)$$

This effective coupling is observables and therefore cutoff independent.

Since the energy-dependent effective coupling is observable in principle, the logarithmic cutoff dependence has to be compensated by a dependence on the "bare charge" on Λ

$$e(\mu)^2 = e_0(\Lambda)^2\left[1+\frac{e_0(\Lambda)^2}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]$$

This relation can be inverted to obtain the bare coupling

$$e_0(\Lambda)^2 = e(\mu)^2\left[1-\frac{e(\mu)^2}{12\pi^2}\log\left(\frac{\mu^2}{\Lambda^2}\right)\right]+\mathscr{O}[e(\mu)^6]$$

and evaluated at a reference scale μ_0. Substituting back in $e(\mu)^2$

$$e(\mu)^2 = e(\mu_0)^2\left[1+\frac{e(\mu_0)^2}{12\pi^2}\log\left(\frac{\mu^2}{\mu_0^2}\right)\right]$$

Thus, as a consequence of the need to renormalize the theory we find that the effective electric charge depends on the energy scale at which it is measured. For example:

To get this result we have to include in the loop of all SM "light" fermions.

$$\alpha(m_e) = \frac{e(m_e)^2}{4\pi} \simeq \frac{1}{137} \qquad \alpha(m_Z) = \frac{e(m_Z)^2}{4\pi} \simeq \frac{1}{128.9}$$

The running of the coupling constant in a quantum field theory is measured by the ***beta function***

$$\beta(g) = \mu \frac{dg}{d\mu}$$

For QED we have

$$\beta(e)_{\text{QED}} = \frac{e^3}{12\pi^2}$$

whereas for QCD the result is

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)$$

$N_c = \text{number of colors}$

$N_f = \text{number of flavors}$

For “real” QCD with three colors and N_f the number of “active” flavors, the beta function is negative.

This means that the QCD coupling constant tends to zero when the energy scale increases, $\mu \to \infty$ ⟹ ***asymptotic freedom***

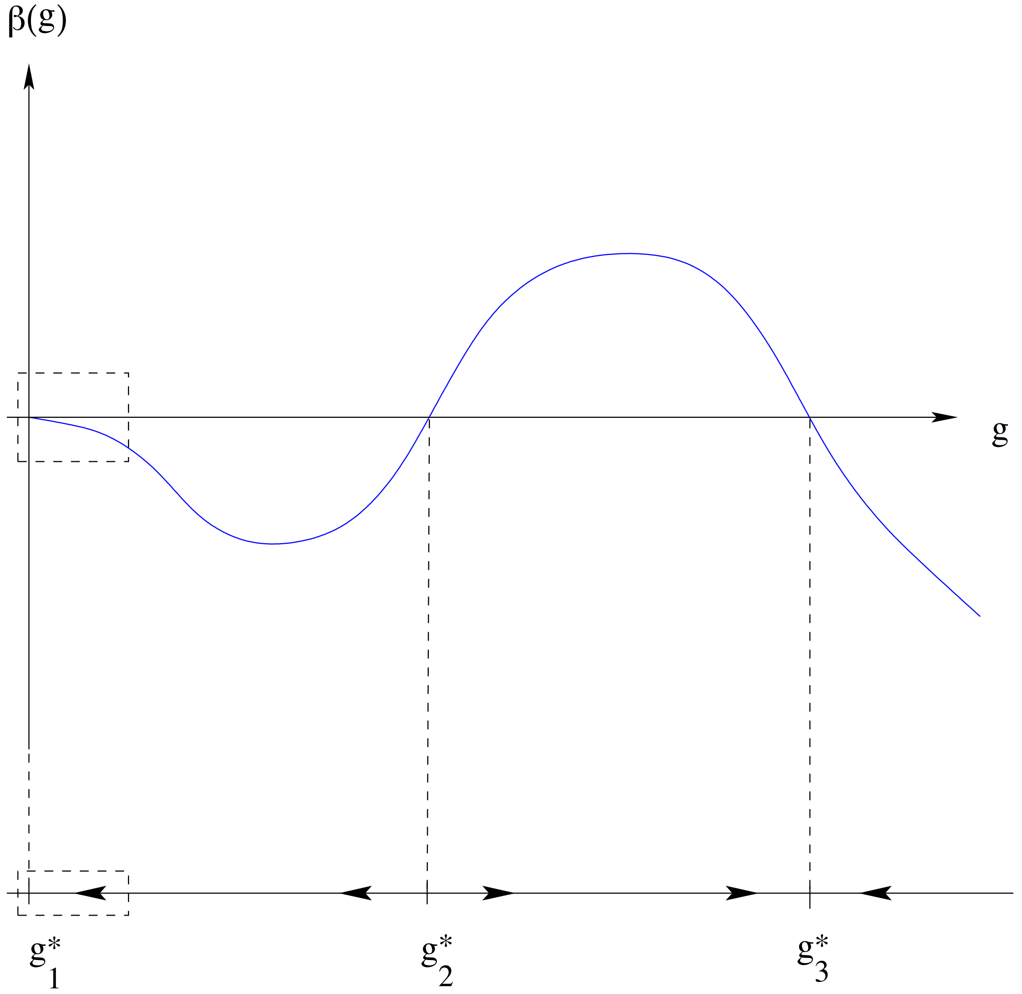
This explains scaling in DIS

QED has a positive beta function and the opposite behavior as QCD. The coupling constant grows with the energy. It becomes of order one at the so-called Landau pole

$$\Lambda_{\text{Landau}} \simeq 10^{34}\,\text{GeV} \qquad \text{(standard model)}$$

$$\Lambda_{\text{Landau}} \simeq 10^{19}\,\text{GeV} \qquad \text{(SUSY)}$$

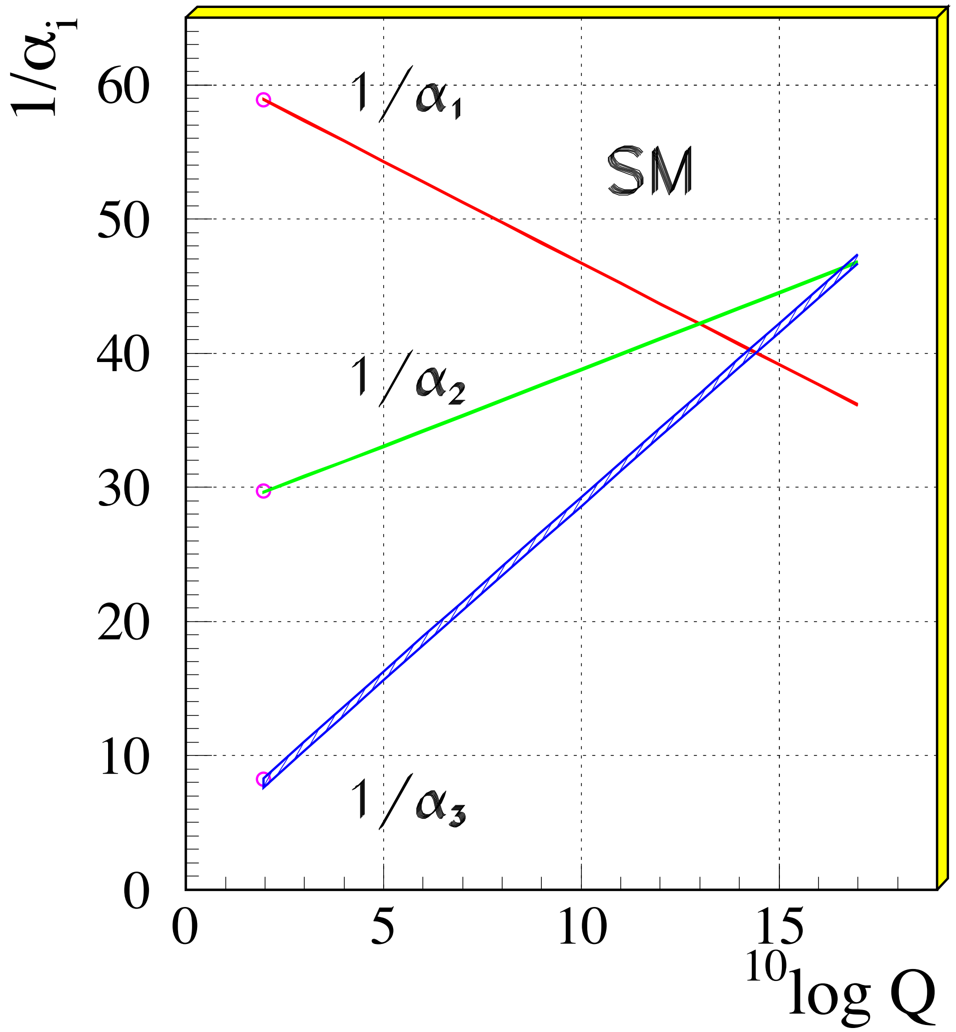
The UV and IR properties of a quantum field theory are determined by the fixed points of the beta function, $\beta(g^*)=0$

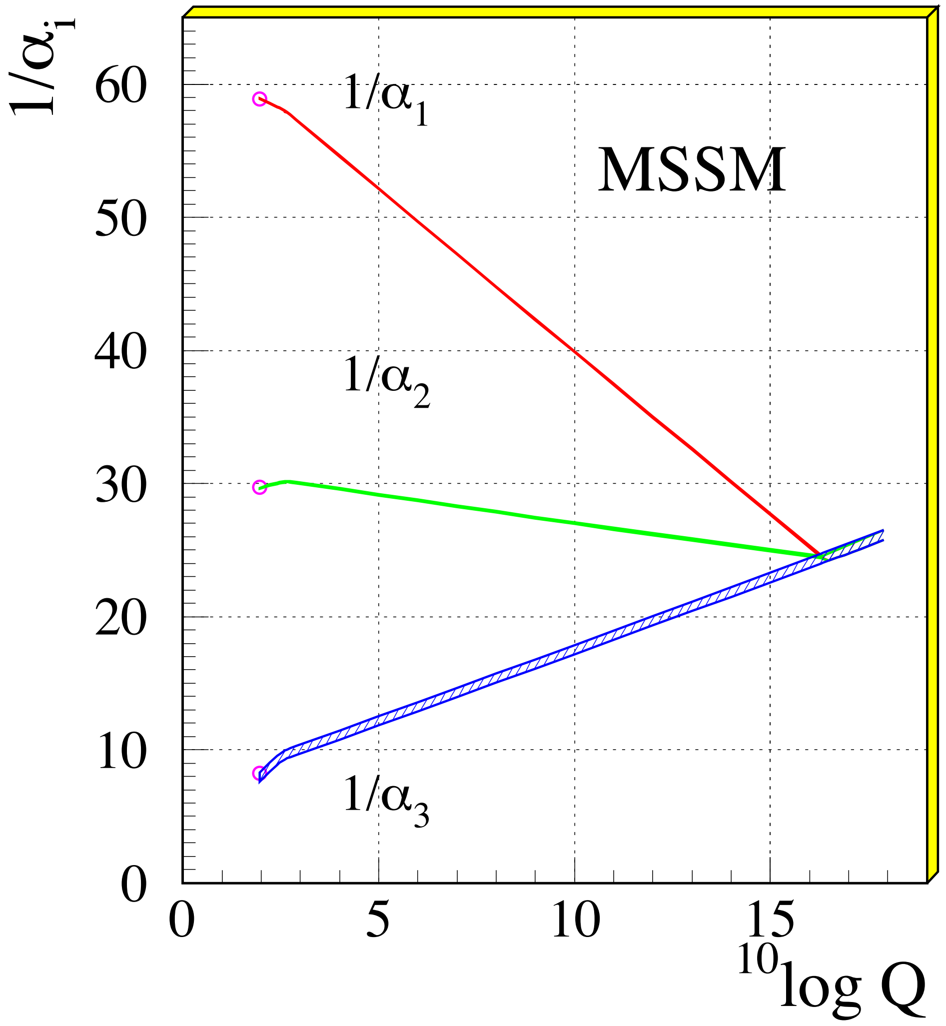


- g_1^* :"trivial", UV stable
- g_2^* :"nontrivial", IR unstable
- g_3^* :"nontrivial", UV stable

e.g., g=0 is a UV stable fixed point for QCD and IR stable fixed point for QED

In the standard model, the three coupling constants get close but fail to meet at a point:





From hep-ph/0012288

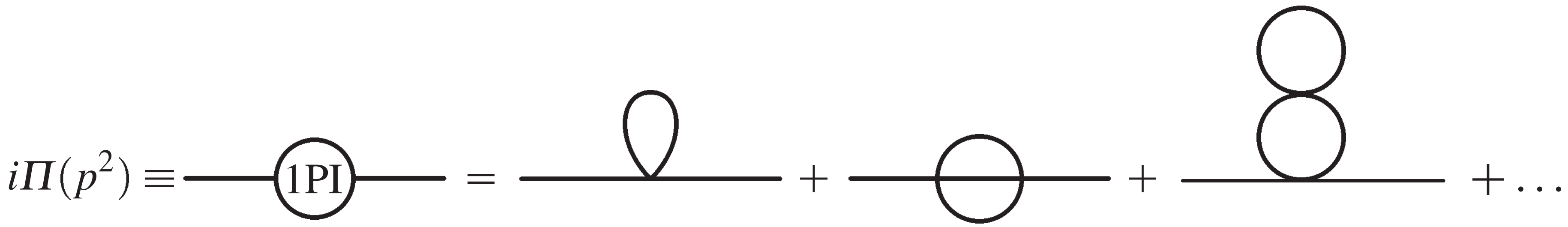
In the minimal supersymmetric standard model (MSSM), on the other hand, the three couplings meet at an energy around 10^{15}-10^{16} GeV

(More about this in Christophe's lectures)

The renormalization program can be applied ***systematically*** to other QFTs by reabsorbing the cutoff dependence of the amplitudes in the mass and coupling constant parameters of the Lagrangian, as well as in the field normalizations.

The dependence of these parameters on the cutoff is fixed by the renormalization prescriptions. We see how this works in a simple example:

In a ϕ^4 theory we consider the sum of all 2-point, 1PI (one particle irreducible) diagrams



The *full* two point function (propagator) can be formally written as

Substituting the contribution of each diagram we find

$$G(p^2) = \frac{i}{p^2 - m^2 + i\varepsilon} + \frac{i}{p^2 - m^2 + i\varepsilon} i\Pi(p^2) \frac{i}{p^2 - m^2 + i\varepsilon} + \ldots = \frac{i}{p^2 - m^2 + i\varepsilon} \sum_{n=0}^{\infty} \left[i\Pi(p^2) \frac{i}{p^2 - m^2 + i\varepsilon} \right]^n$$

We have arrived at a geometric series whose sum is

$$G(p^2) = \frac{i}{p^2 - m^2 - \Pi(p^2) + i\varepsilon}$$

All this is purely formal. Now we apply it to a computation at a given order in perturbation theory. Regularizing the theory using, say, a cutoff Λ we find

$$G(p^2) = \frac{i}{p^2 - m^2 - \Pi(p^2, \Lambda) + i\varepsilon}$$

Naively, the limit $\Lambda \to \infty$ gives a meaningless result. The question, however, is:

How do we define the mass of the scalar particle?

In a free theory we have

$$G(p^2)_{\text{free}} = \frac{i}{p^2 - m^2 + i\varepsilon} \qquad \Longrightarrow \qquad G(p^2 = m^2)^{-1}_{\text{free}} = 0$$

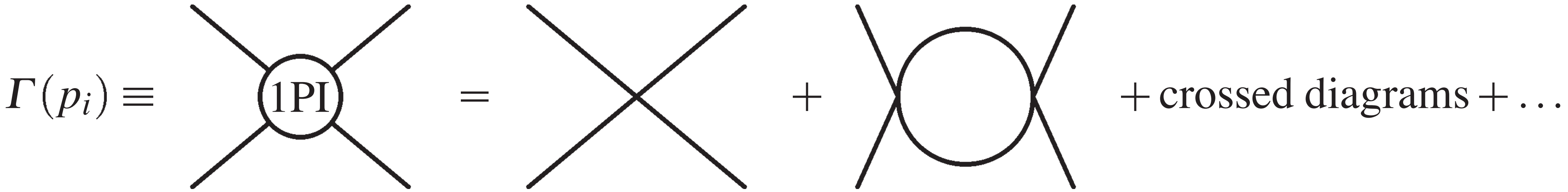
so the mass of the particle is identified as the pole of the free propagator.

We extend this definition to the regularized interacting theory and define the renormalized mass parameter m as the pole of the propagator:

$$m^2 - m_0(\Lambda)^2 - \Pi(m^2, \Lambda) = 0$$

Since the physical mass is cutoff-independent, we have to assume that the mass parameter in the Lagrangian depends on the cutoff (not a problem, it's just a unphysical parameter).

The renormalized coupling constant can be defined using the four-point, 1PI diagrams



The final result is that all observable quantities can be written solely in terms of the renormalized mass an coupling.

Different renormalization prescriptions give different definition of the renormalized quantities, but *observables* are independent of the renormalization scheme.

Wilsonian renormalization

Kenneth Wilson
(b. 1936)

A very profound physical interpretation of the renormalization program in QFT can be extracted from statistical mechanics.

Let us look at a “simple” statistical system: the Ising model with Hamiltonian

$$H = -J\sum_{\langle i,j\rangle} s_i s_j$$

$$s_i = \pm\frac{1}{2}$$

$$J > 0$$

and compute its thermal partition function

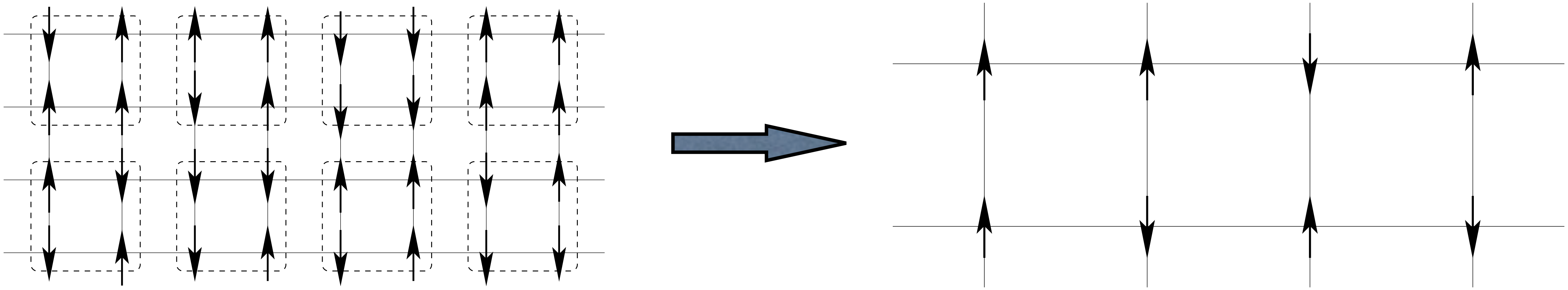
$$\mathscr{Z} = \sum_{\{s_i\}} e^{-\beta H}$$

In dimension larger than one, the system undergoes spontaneous magnetization at a critical temperature T_c . Away from this, the correlation between spins decreases exponentially,

$$\langle s_i s_j\rangle \sim e^{-\frac{|x_{ij}|}{\xi}}$$

(ξ = correlation length)

If we are interested in the long distance properties of the system we can "integrate out" short distance physics. This can be done using Kadanoff's decimation method:



We replace each spin-block by an effective spin calculated using some rule (e.g., a majority rule supplemented by some prescription in case of a draw)

$$\{s_i : i \in B_a\} \quad \longrightarrow \quad s_a^{(1)} \qquad\qquad s_a^{(1)} = \frac{1}{2}\mathrm{sign}\left(\sum_{i \in B_a} s_i\right) \quad \text{with} \quad \mathrm{sign}(0) = 1$$

The partition function can be written now as

$$\mathscr{Z} = \sum_{\{\mathbf{s}\}} e^{-\beta H[s_i]} = \sum_{\{\mathbf{s}^{(1)}\}} \sum_{\{\mathbf{s} \in B_a\}} \delta\left[s_a^{(1)} - \mathrm{sign}\left(\sum_{i \in B_a} s_i\right)\right] e^{-\beta H[s_i]}$$

The basic point now is that the sum over the spins inside each block can be written itself as

$$\sum_{\{s\in B_a\}} \delta\left[s_a^{(1)} - \text{sign}\left(\sum_{i\in B_a} s_i\right)\right] e^{-\beta H[s_i]} = e^{-\beta H^{(1)}[s_a^{(1)}]}$$

The new Hamiltonian will be very complicated, but it has the form

$$H^{(1)} = -J^{(1)} \sum_{\langle i,j\rangle} s_i^{(1)} s_j^{(1)} + \ldots$$

other interaction terms

and only depends on the new variables. The partition function is now written as

$$\mathscr{Z} = \sum_{\{s^{(1)}\}} e^{-\beta H^{(1)}[s_a^{(1)}]}$$

This decimation operation can be understood as a map in the space of all Hamiltonians,

$$\mathscr{R} : H \to H^{(1)}$$

$$a \to 2a$$

$$\xi \to \frac{\xi}{2}$$

This operation can be written an arbitrary number of times, until a fixed point is reached

$$H \xrightarrow{\mathscr{R}} H^{(1)} \xrightarrow{\mathscr{R}} H^{(2)} \xrightarrow{\mathscr{R}} \ldots \xrightarrow{\mathscr{R}} H_\star$$

The fixed-point Hamiltonian is scale invariant, that we take to have $\xi = \infty$

These transformations can be understood in terms of couplings by writing the Hamiltonian as a combination of all possible operators

$$H[s_i] = \sum_{a=1}^{\infty} \lambda_a \mathscr{O}_a[s_i] \qquad\qquad \lambda_a \in \mathbb{R}$$

Hence, integrating out short distance physics results in a ***renormalization*** of the couplings

$$\mathscr{R} : \lambda_a \longrightarrow \lambda_a^{(1)}$$

Our ignorance about the physics at short distances is parametrized in the values of the coupling constants that characterize the Hamiltonian at long distances (i.e, the couplings *run* with the scale).

After this excursion in statistical mechanics, we proceed now back to QFT. Suppose we have a quantum field theory defined at a scale Λ by an action

$$S[\phi_a,\Lambda]=\int d^4x\left\{\mathscr{L}_0[\phi_a]+\sum_i g_i(\Lambda)\mathscr{O}_i[\phi_a]\right\}$$

To know how the theory looks like at an energy $\mu<\Lambda$ we compute

$$e^{iS[\phi_a',\mu]}=\int_{\mu<p<\Lambda}\prod_a\mathscr{D}\phi_a\, e^{iS[\phi_a,\Lambda]}$$

After integrating out the physics between the scales Λ and μ we get an action for the renormalized fields ϕ_a' of the form

$$S[\phi_a',\mu]=\int d^4x\left\{\mathscr{L}_0[\phi_a']+\sum_i g_i(\mu)\mathscr{O}_i[\phi_a']\right\}$$

The couplings are different from their values at the cutoff scale. What happens is that the running of the coupling constants encodes the physics at energies above the scale μ.

The Wilsonian approach shows that renormalization cannot be seen as a simple trick to “sweep infinities under the rug”. The running of the coupling constants is the way in which the physics at high energies shows up when going to lower energies.

It also offers a radically different view on the concept of renormalizability. Nonrenormalizable theories are safe to use, provided we are only interested in physics at low energies.

We will further ellaborate on all this in next Wednesday’s lecture (on effective field theories).