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Five lectures on

Quantum Field Theory

An introduction to selected topics

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Lecture V (and last)

Effective Field Theories

- Renormalizability vs. nonrenormalizability
- Nonrenormalizable theories as effective field theories
- The scales of the standard model and naturalness

Renormalizability vs. Nonrenormalizability

In Lecture III we learned how the renormalization program consited in absorbing infinities in the Lagrangian parameters and field normalizations. This works in theories like ϕ^4, QED, QCD, and the standard model among others. The question however is:

Does this always work?

The answer is ***no***.

Let us elaborate the answer with the simplest example of ϕ^4 in d dimensions. A general diagram with E external legs, I internal propagators and V vertices gives rise to a contribution with the structure

$$\lambda^V \left[\int \frac{d^d p}{(2\pi)^d}\right]^L \left(\frac{i}{p^2 - m^2 + i\varepsilon}\right)^I$$

The superficial degree of divergence of this integral can be computed from dimensional analysis:

$$\#(\mathrm{div}) = dL - 2I$$

To get a more useful expression, we notice that

$$\left.\begin{aligned} E &= 4V - 2I \\ L &= I - V + 1 \end{aligned}\right\} \Longrightarrow \quad I = 2L + \frac{1}{2}E - 2$$

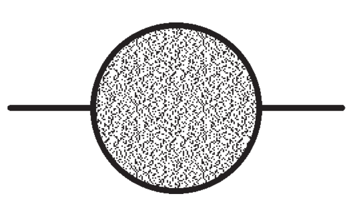
and write:

$$\#(\mathrm{div}) = dL - 2I \quad \Longrightarrow \quad \#(\mathrm{div}) = (d-4)L + (4-E)$$

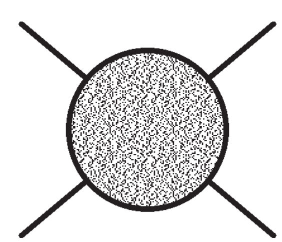
We analyze three different cases

- $d=4$: the degree of divergence is independent of the number of loops. All diagrams with the same number of external legs have the same primitive degree of divergence, and this decreases with the number of external legs

  The only primitively divergent diagrams are the ones for the two- and four-point amplitudes:



(quadratically divergent)

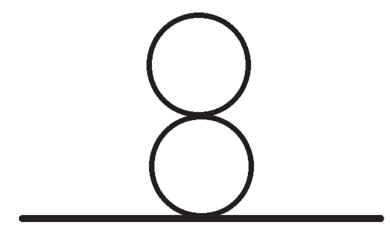


(logarithmically divergent)

The conclusion is that we only need to renormalize the two- and four-point functions, and this can be done absorbing the divergences in the parameters of the Lagrangian (mass, coupling constant and field normalization). This is an example of a ***renormalizable*** theory

- $d < 4$: now, the degree of divergence decreases both with the number of loops and the number of external legs. The number of superficially divergent diagrams is finite.
  For example, in ϕ^4 in three dimensions there are only two divergent diagrams:

(linearly divergent)



(logarithmically divergent)

In this example, the infinities can be disposed of by renormalizing the two-point function at one and two loops. We are dealing then with a ***superrenormalizable*** theory.

Exercise: show that ϕ^3 in $d = 4$ is superrenormalizable.

- $d > 4$: In this case, the superficial degree of divergence increases indefinitely with the number of loops, so there is an infinite number of divergent diagrams with an arbitrary number of external legs.

  These divergences cannot be absorbed in a finite number of parameters, and the theory is ***nonrenormalizable***

In general, the criterion of renormalizability for a QFT boils down to calculating the ***energy dimensions*** of the coupling constants of the theory:

- $[g] = 0$: the theory is renormalizable.

- $[g] > 0$: the theory is superrenormalizable.

- $[g] < 0$: the theory is nonrenormalizable.

To understand this, we remember that the energy dimensions of the correlation function are fixed, so the dimensions of the powers of the coupling constants have to be compensated by the dimension of the integral (i.e., its primitive degree of divergence).

One should not be deceived by the simple arguments presented so far. There are a number of pitfalls to be aware of:

Higher loop diagrams may contain ***overlapping divergences*** and/or ***divergent subdiagrams*** that one should deal with. The message is that a general rigorous proof of renormalizability is very involved.

Naive power counting arguments may fail. For example, the propagator of a massive gauge field

$$G_{\mu\nu}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}\left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)$$

does not go to zero at large momenta. The theory might not be renormalizable even if the coupling constant is dimensionless (in massive QED this works because the "dangerous term" in the bracket does not contribute).

The trouble with nonrenormalizable theories is the computation of observables at arbitrary high energies requires to introduce an infinite number of counterterms in the Lagrangian to absorb the divergences.

In the example of a six-dimensional ϕ^4 theory ($[\lambda]=-2$) we end up with:

$$\mathscr{L}_{d=6}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4\underbrace{-\frac{\lambda_6}{6!}\phi^6-\frac{\lambda_8}{8!}\phi^8-\ldots}_{\text{counterterms}}$$

$$[\lambda_6]=-6$$
$$[\lambda_8]=-10$$
$$\vdots$$

To calculate high energy processes we would need to fix experimentally an arbitrary large number of parameters.

$$\Downarrow$$

Nonrenormalizable theories are not predictive

Or are they?

Nonrenormalizable QFTs as Effective Field Theories

It is clear that if we are interested in the physics at arbitrary high energy, nonrenormalizable theories are no good. The can however be understood as ***effective field theories*** valid below some characteristic energy scale.

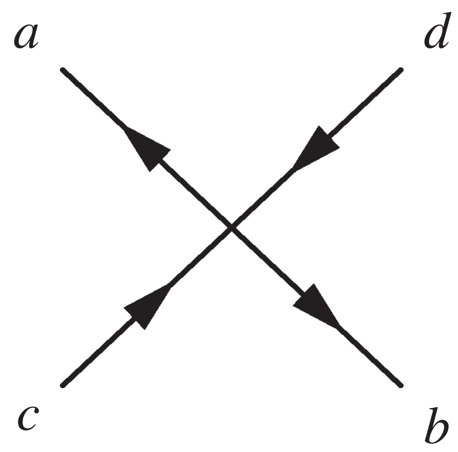
We look at a four-fermion theory with Lagrangian

$$\mathscr{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{a}{\Lambda^2}(\overline{\psi}\psi)^2 \qquad g = \frac{a}{\Lambda^2}, \qquad [g] = -2$$

The theory is not renormalizable, so quantum corrections will induce an infinite number of higher-dimensional operators

$$\mathscr{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{a}{\Lambda^2}(\overline{\psi}\psi)^2 + \sum_n \frac{a_n}{\Lambda^{\dim \mathscr{O}_n - 4}}\mathscr{O}_n[\overline{\psi}, \psi]$$

This theory cannot be used to describe the physics at arbitrary high energies. Apart from being nonrenormalizable it also violates unitarity



$$= -\frac{2ia}{\Lambda^2}\left(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}\right) \quad \Longrightarrow \quad \sigma \sim \left(\frac{a}{\Lambda^2}\right)^2 E^2$$

Typically, the contribution of these higher-dimensional operators to the amplitude of a given process taking place at an energy scale $E \ll \Lambda$ will be suppressed by

$$\left(\frac{E}{\Lambda}\right)^{\dim \mathscr{O}_n - 4}$$

However, ***at a given level of accuracy***, we only need to take into account a finite number of operators, and therefore only a finite number of couplings need to be fixed.

Thus, as effective field theories, nonrenormalizable theories are completely ***consistent*** and ***predictive***.

The effective field theory approximation breaks down at energies $E \sim \Lambda$, when the contribution from all higher-dimensional operators are of the same order. At this scale the theory has to be replaced by some new dynamics.

To find this UV completion, we notice that the four-fermion Lagrangian is classically equivalent to

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{a\Lambda^2}{4}\sigma^2 + a\sigma\overline{\psi}\psi$$

Exercise: prove it.

where σ is a nonpropagating real scalar field. This theory can be regarded as the low energy limit of a Yukawa theory

$$\mathscr{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{a\Lambda^2}{4}\sigma^2 + a\sigma\overline{\psi}\psi$$

in the limit when the typical energies are much below the mass of the scalar field,

$$E \ll M_\sigma = \Lambda\sqrt{\frac{a}{2}}$$

To summarize, we have found that the nonrenormalizable theory

$$\mathscr{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{a}{\Lambda^2}(\overline{\psi}\psi)^2 .$$

can be reliably used to compute observables at energies $E \ll \Lambda$.

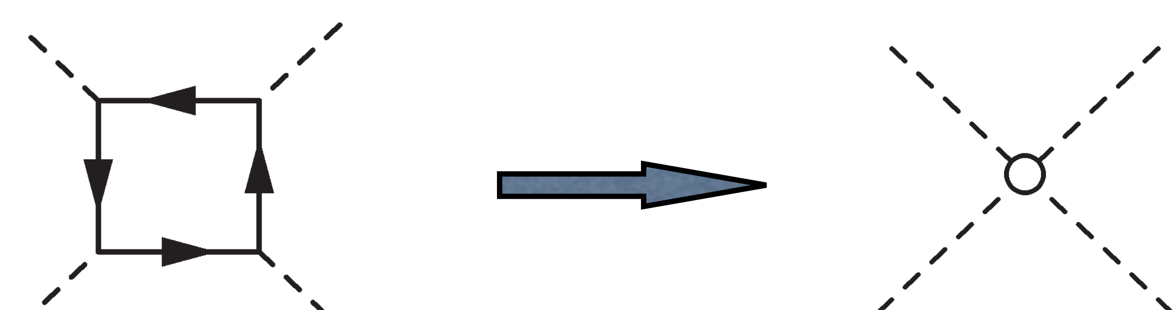
At high energies, the effective field theory approximation breaks down and we have to find an UV completion of the theory that works in this regime.

One possibility is the existence of a massive scalar coupling to the fermion through a Yukawa term

$$\mathscr{L} = \overline{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{a\Lambda^2}{4}\sigma^2 + a\sigma\overline{\psi}\psi - \frac{\lambda_3}{3!}\sigma^3 - \frac{\lambda_4}{4!}\sigma^4$$

This theory is unitary ($\sigma \sim a^4 E^{-2}$ at high energies) and renormalizable. The cubic and quartic terms have to be included to cancel the logarithmically divergence of the one-loop contribution to the three- and four-point scalar amplitude

For example:



$\sim \log \Lambda_{\rm UV}$

counterterm

At low energies it generates a term $\sim \frac{1}{\Lambda^8}(\overline{\psi}\psi)^4$

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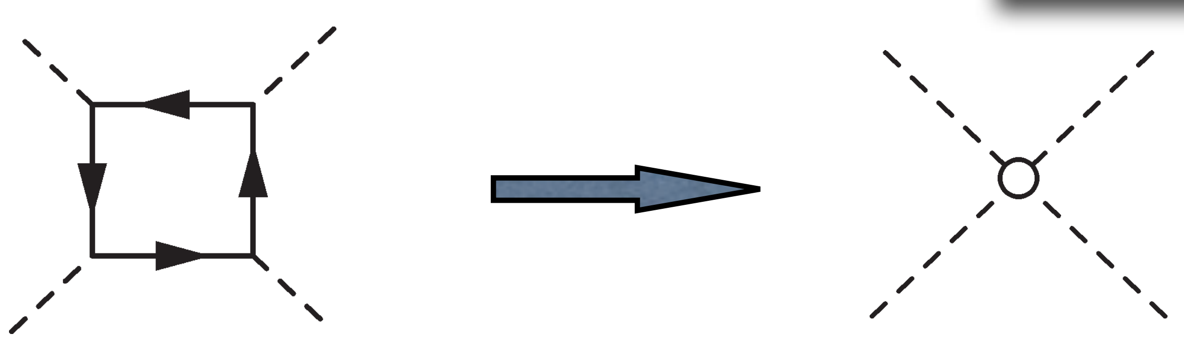
Exercise: replacing the fermion-scalar interaction term by $a\sigma\overline{\psi}\gamma_5\psi$ show that the cubic term is absent. Compute in this case the effective theory at low energies. (use a symmetry argument)

For example:



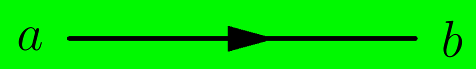
$\sim \log \Lambda_{\rm UV}$ counterterm

At low energies it generates a term

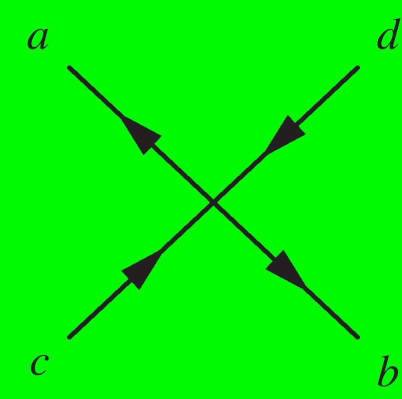
$$\sim \frac{1}{\Lambda^8}(\overline{\psi}\psi)^4$$

Exercise:

Consider the Feynman rules of the four-fermion theory:



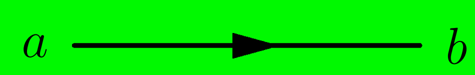
$$= \left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{ba}$$



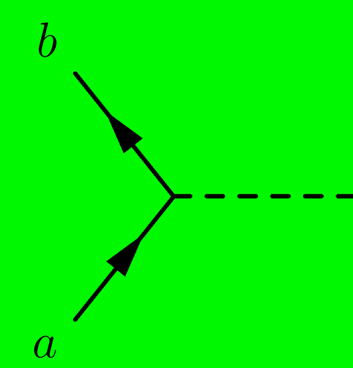
$$= -\frac{2ia}{\Lambda^2}\left(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}\right)$$

and those of the theory with the nondynamical scalar σ

$$= -\frac{2i}{a\Lambda^2}$$



$$= \left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{ba}$$



$$= ia\delta_{ab}$$

Compute the leading order four-fermion amplitude in both cases and show that the results are same. Compute as well the one-loop fermion propagator using both sets of Feynman rules to prove that the two calculations lead to the same integral.

Effective field theories are among the most powerful tools in physics. The very fact that we can make progress in our understanding of the world is based on the fact that the ***physics at different scales decouple***.

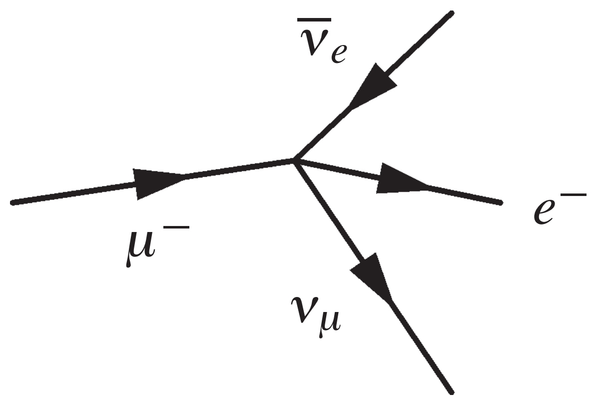
For example, in order to formulate the laws of motion of macroscopic bodies, ***Newton*** didn't have to know anything about atomic physics. Similarly, ***Bohr*** was able to formulate model of the atom only because the influence of subnuclear physics (that were unknown at the time) at the atomic level is very small.

The task of the physical science can be defined as the continuous search for effective descriptions valid at higher energy scales and/or more general setups:

- Aristotelian dynamics can be regarded as the first effective theory: it describes effectively the motion of bodies subjected to (strong) friction.
- Classical (Newtonian) mechanics is an effective description of the motion of bodies valid in the regime of "large" actions and "small" velocities.
- Nonrelativistic quantum mechanics is an effective description of particles moving with "small" velocities (i.e., energies much smaller than their masses)

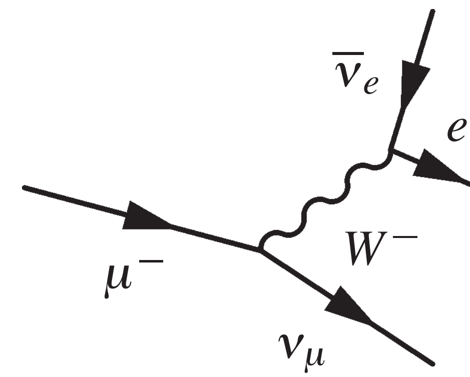
⋮

The same pattern repeats in particle physics:



Fermi's four-fermion interaction

$E \gtrsim m_W$



Glashow-Weinberg-Salam theory

Pion dynamics

$E \gg \Lambda_{\mathrm{QCD}}$

Perturbative QCD

In both examples, the low energy field theory is nonrenormalizable, unlike their UV completion.

Does this mean that the standard model is the "final" fundamental description?

To answer this question we have to take into account that, at low energies, the dynamics of any field theory is dominated by two types of operators:

- ***Relevant operators***, having energy dimension larger than 4 (such as mass terms).

- ***Marginal operators***, whose energy dimension is equal to 4 (usually know as renormalizable interactions terms).

We have learned that theories containing only relevant and marginal operators are renormalizable.

However, experimental evidence might force us to add operators with dimension higher than 4 (called ***irrelevant***) to the action. These are the type of couplings that render a theory nonrenormalizable, and indicate that we are dealing with an effective description.

The discovery of neutrino masses, for example, requires extending the standard model to give masses to the neutrinos without breaking gauge invariance. If neutrinos turn out to be Majorana fermions, the generation of a mass term requires adding a dimension-five operator to the standard model Lagrangian

$$\Delta\mathscr{L}_{\rm SM} = -\frac{1}{M}\sum_{i,j=1}^{3} g_{ij}\left(\overline{\mathbf{L}_i^C}\sigma^2\mathbf{H}\right)\left(\mathbf{H}^T\sigma^2\mathbf{L}_j\right) + \text{h.c.}$$

where $\mathbf{H}$ and $\mathbf{L}_i$ are respectively the Higgs and left-handed lepton doublets. Upon spontaneous symmetry breaking, a Majorana mass term is generated

$$\Delta\mathscr{L}_{\rm SM} = -\frac{\mu^2}{M}\sum_{i,j=1}^{3} g_{ij}\overline{\nu_i^C}\nu_j + \text{h.c.}$$

In order to account for the size of the observed neutrino masses, the scale M signaling the onset of new physics, has to be

$$M \sim 10^{15}\text{GeV} \gg m_W$$

The effects at the electroweak scale are therefore very small.

Naturalness

Neutrino masses are a very strong piece of evidence hinting to new physics at a higher energy scale. Notice that this scale is very close to the one where gauge coupling unification is expected (remember the graph shown in the third lecture).

Being conservative, we can say that high energy physics is then characterized by three energy scales:

- ***The electroweak scale***, set by the vev of the Higgs field $v \approx 246\,\mathrm{GeV}$
- ***The grand unification (GUT) scale***, set by the scale where new physics is expected, $M_{\mathrm{GUT}} \sim 10^{15}$-$10^{16}\,\mathrm{GeV}$
- ***The Planck scale***, where (quantum) gravitational effects cannot be ignored any longer, $M_{\mathrm{Planck}} \sim 10^{19}\,\mathrm{GeV}$

The presence of various energy scales in the theory begs the question of whether it is ***natural*** to have widely separated scales:

In particular, all the masses of the particles in the standard model are much lighter than the GUT or the Planck scale. This includes the Higgs particle, whose mass is expected to be of the order of the electroweak scale.

To address the question, we introduce the following ***naturalness criterion***:

"At a given energy scale μ a set of dimensionless physical parameters $\alpha_i(\mu)$ is allowed to be small only if the replacement $\alpha_i(\mu)$=0 increases the symmetry of the system".

In view of this criterion, light fermions ($m_f \ll M_{\rm GUT}$) are natural. In the limit of massless fermions the system has an additional discrete chiral symmetry

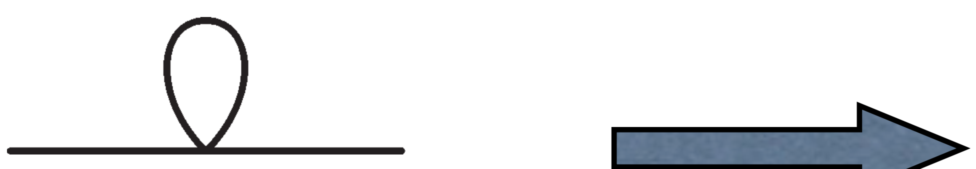
$$\psi \longrightarrow \gamma_5\psi, \qquad \overline{\psi} \longrightarrow -\overline{\psi}\gamma_5$$

In the case of a fundamental scalar, such as the Higgs field, there is no such symmetry enhancement protecting the hierarchy $m_{\rm H} \ll M_{\rm GUT}$

The problem of having light fundamental scalars in the standard model can be seen in a Wilsonian fashion. We look at the simpler model of a ϕ^4 theory that we define at some cutoff scale Λ

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0(\Lambda)^2}{2}\phi_0^2 - \frac{\lambda_0(\Lambda)}{2}\phi_0^4$$

The one-loop correction to the mass comes from the "tadpole" diagram:



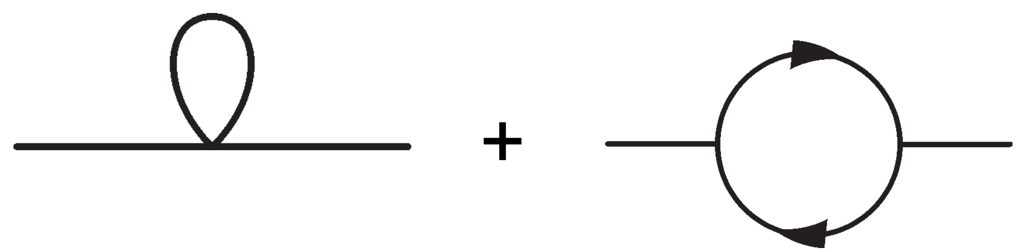
$$m^2 = m_0(\Lambda)^2 + \frac{\lambda_0(\Lambda)}{16\pi^2}\left[\Lambda^2 - \log\frac{\Lambda^2}{m_0(\Lambda)^2}\right]$$

Since the mass depends quadratically on the cutoff, to have a light scalar (i.e., $m \ll \Lambda$) requires a strong fine tuning of $m_0(\Lambda)$.

Since the values of the bare parameters encode the high energy degrees of freedom that we are cutting off at the scale Λ, we find that the mass of the scalar is highly sensitive to the physics at high energies.

This defines the ***hierarchy problem*** in the standard model.

Supersymmetry is the most popular solution to the hierarchy problem. In a supersymmetric theory, the scalar field couples both to itself and to its supersymmetric partner in such a way that the quadratic divergence cancels



$$\sim \log \Lambda_{\rm UV}$$

Now, in order to maintain the hierarchy $m \ll \Lambda$, we require a much milder fine tuning of the bare parameters (more on this in Christophe's lectures).

Another serious naturalness issue is the ***cosmological constant problem***, i.e., why the energy scale of the cosmological constant is so small compared with the only natural scale in the problem, the Planck mass

$$\rho_\Lambda \simeq (10^{-3}\mathrm{eV})^4 = \underbrace{10^{-48}\mathrm{GeV}^4}_{\text{(measured)}} \ll M_{\rm P}^4 \sim \underbrace{10^{76}\,\mathrm{GeV}^4}_{\text{(“expected”)}}$$

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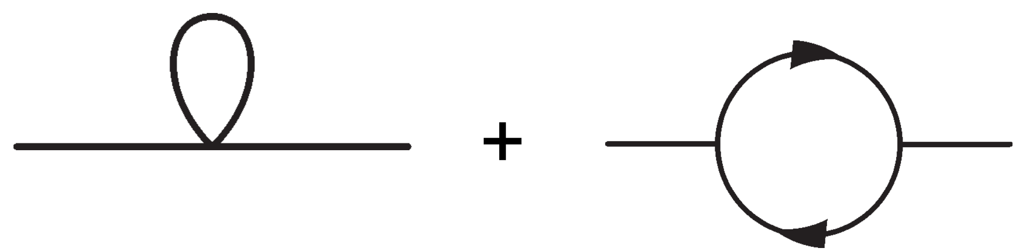

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Exercise: solve the problem (and tell us the answer!)

Thank you