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Heavy Ion Collisions

Problems

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1. Coherence: the two-scattering case.

Consider a high-energy massless scalar projectile scattering on some centers located in positions $x_1, x_2, \ldots$ - a nucleus. Use light-cone coordinates $a_\pm = (a_0 \pm a_z)/\sqrt{2}$, $a = (a_0, a_T, a_z) = (a_+, a_-, a_T)$ with $a_T = (a_x, a_y)$ the two-dimensional transverse vector, assume dominance of the +-components for the projectile, and define $q = p' - p$. Employ the optical theorem for purely imaginary amplitudes, $it(q = 0) = it_{\rm forw} = -\sigma$ for projectile-nucleon and $i\mathcal{T}_n(q = 0) = -\sigma_A^n$ for the n-scattering contribution for projectile-nucleus collisions, and the Feynman rules shown in Fig. 1. Then the amplitude with one scattering (Fig. 1 left) reads:

$$\begin{aligned} c(p_+, p'_+) i\mathcal{T}_1(q) &= it_{\rm forw}\ A(p_+ + p'_+) \int d^4x\, \rho_A(x_+, x_T) e^{ix\cdot(p'-p)} \\ &= it_{\rm forw}\ c(p_+, p'_+) A \int d^2x_T\, T_A(x_T) e^{-ix_T\cdot(p'_T - p_T)}. \end{aligned}$$

$\rho_A(x_+, x_T)$ is the nuclear density normalized to 1,

$$T_A(x_T) = \int_{-\infty}^{+\infty} dx_+\, \rho_A(x_+, x_T)$$

the nuclear profile, $|x_T| = b$ the impact parameter and $c(p_+, p'_+) = (2\pi)2p_+\delta(p'_+ - p_+)$ a normalization factor.

a) Show that the corresponding cross section can be written as an incoherent superposition of the contribution from the A scattering centers.

b) Extend this result to two scatterings (Fig. 1 right), performing the integral over k_- using the Cauchy theorem to get

$$\begin{aligned} & c(p_+, p'_+) i\mathcal{T}_2(q) = iA(A-1)(it_{\rm forw})^2 \int \frac{d^4k}{(2\pi)^4} d^4x_1 d^4x_2\, e^{ix_1\cdot(k-p)} \\ & \times e^{ix_2\cdot(p'-k)} \frac{(p_+ + k_+)(k_+ + p'_+)}{k^2 + i\epsilon} \rho_A(x_{1+}, x_{1T}) \rho_A(x_{2+}, x_{2T}) \\ & = c(p_+, p'_+) A(A-1)(it_{\rm forw})^2 \\ & \times \int \frac{d^2k_T}{(2\pi)^2} dx_{1+} dx_{2+} d^2x_{1T} d^2x_{2T}\, e^{-ik_T^2(x_{2+}-x_{1+})/(2p_+)} \\ & \times e^{-i[x_{1T}\cdot(k_T - p_T) + x_{2T}\cdot(p'_T - k_T)]} \rho_A(x_{1+}, x_{1T}) \rho_A(x_{2+}, x_{2T}) \theta(x_{2+} - x_{1+}), \end{aligned} \quad (0.1)$$

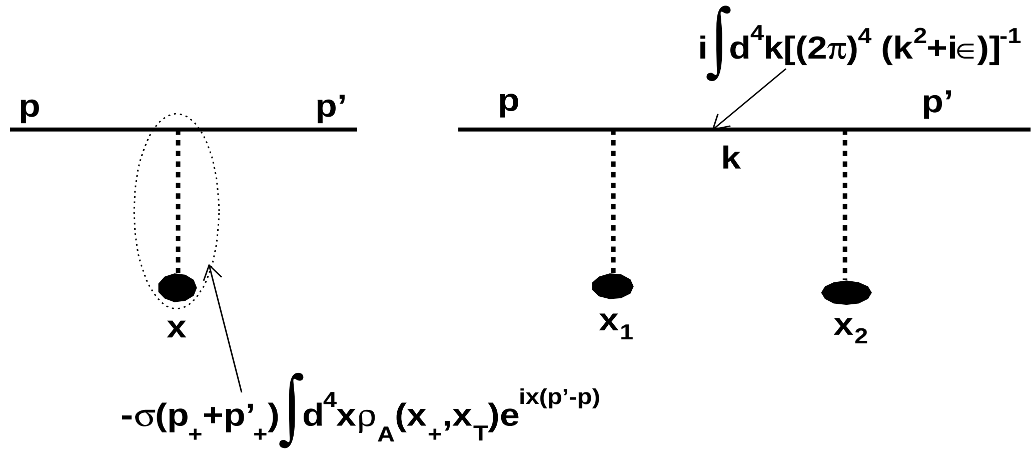

Figure 1: One- (left) and two- (right) scattering diagrams, with the corresponding Feynman rules written on them.

with $\theta(x)$ the step function.
c) Analyze the low- and high-energy behavior of the exponential involving the difference in longitudinal position of the scattering centers, an interference term which makes this two-scattering contribution negligible at low energies and coherent at high energies. Show that in the totally coherent limit becomes negative - shadowing.
Note: In the coherent limit, the contribution of an arbitrary number of scatterings can be resummed, resulting in a path-ordered exponential for the $\mathcal{S}$-matrix. For QCD, this is the Wilson line.
References: N. Armesto, hep-ph/0604108, Section 2; A. Hebecker, hep-ph/9905226, Section 3.1 and Appendix A.

2. The Balitsky-Kovchegov equation.

The Balitsky-Kovchegov (BK) equation gives the evolution with rapidity $Y = \ln(E_{cm}^2/s_0) = \ln(x_0/x)$ of the scattering probability $N(r,Y)$ of a $\bar{q}q$ dipole of transverse size r on a hadronic target. This probability has the limiting behaviors $N(r,Y) \propto r^\delta$ ($\delta > 0$) for $r \to 0$ and $N(r,Y) \to 1$ for $r \to \infty$. Its Fourier transform is proportional to the unintegrated gluon density of the hadron, $\phi(x,k_\perp) \propto \int dk_\perp e^{i\vec{k}_\perp\cdot\vec{r}} N(r,Y)/r^2$. The equation reads

$$\frac{\partial N(r,Y)}{\partial Y} = \int \frac{d^2 r_1}{2\pi} K(\vec{r},\vec{r}_1,\vec{r}_2)\left[N(r_1,Y) + N(r_2,Y) - N(r,Y) - N(r_1,Y)N(r_2,Y)\right],$$

with the vectors in two dimensions and the kernel

$$K(\vec{r},\vec{r}_1,\vec{r}_2) = \frac{\alpha_s N_c}{\pi}\frac{r^2}{r_1^2 r_2^2} \quad , \quad \vec{r} = \vec{r}_1 - \vec{r}_2.$$

a) Assuming scaling, $N(r,Y) \equiv N(\tau)$, $\tau = r\,Q_s(Y)$, show that the saturation momentum $Q_s(Y)$ which separates the high occupacy, dense soft modes from the dilute hard ones, verifies the equation

$$\frac{\partial N(\tau)}{\partial Y} = \frac{\partial \ln[Q_s^2(Y)/\Lambda^2]}{\partial Y}\, r^2 \frac{\partial N}{\partial r^2}\,.$$

b) Using the scale invariance of the kernel K and the previous result, integrate both the right-hand and left-hand sides of the BK equation over d^2r/r^2 to get that, for the coupling constant α_s fixed, $Q_s^2(Y) = Q_0^2 \exp(d\alpha_s N_c Y/\pi)$ with $d = \int \frac{d^2r}{\pi r^2}\frac{\partial N(\tau)}{\partial Y}$ and $Q_0^2 = Q_s^2(Y=0)$.
c) In the running coupling case, the kernel in no longer scale-invariant. Taking the running coupling

$$\alpha_s(r) = \frac{12\pi}{\beta_0 \ln \frac{4}{r^2\Lambda^2}}, \quad \beta_0 = 11N_c - 2N_f,$$

evaluated at $r = 2/Q_s(Y)$, show in the same way as done previously that $Q_s^2(Y) = \Lambda^2 \exp\left[\Delta'\sqrt{Y+X}\right]$ with $(\Delta')^2 = 24N_c d/\beta_0$ and $X = (\Delta')^{-2}\ln(Q_0^2/\Lambda^2)$.
Note: Taking $\alpha_s(r)$ inside the integrand of the integral over d^2r/r^2 previously mentioned and using the behavior of the kernel and the limiting behaviors of the scattering probability $N(r,Y)$ given at the beginning of the problem, it can be argued the bulk of the contribution to the integral comes from $\tau \sim 1$. This justifies the choice of scale $r \sim 1/Q_s(Y)$ made in c).
References: J. L. Albacete *et al.*, hep-ph/0408216, Sections 2, 3 and 5.3.

3. The AGK cutting rules and the Glauber model.

Consider hadron-nucleus (hA) scattering of a hadron h on a nucleus A with mass number A at fixed impact parameter b, whose dependence will be usually implicit in the following. Denoting with uppercase letters the quantities corresponding to hA and with lowercase letters those corresponding to hN (N for nucleon), we have

$$\mathcal{S} = 1 + i\mathcal{A}, \quad s = 1 + ia, \quad 2\,\mathrm{Img}a = \sigma,$$

$$\sigma_{tot}^{\mathrm{hA}}(b) = 2\,\mathrm{Re}(1-\mathcal{S}), \quad \sigma_{elastic}^{\mathrm{hA}}(b) = |1-\mathcal{S}|^2.$$

In the Glauber-Gribov theory, the amplitude is given by a superposition of scatterings,

$$\mathcal{A} = \sum_{k=1}^{A} \mathcal{A}_k, \quad i\mathcal{A}_k = \binom{A}{k}(iaT_A)^k \Longrightarrow \mathcal{S} = (1+iaT_A)^A,$$

with $T_A \equiv T_A(b) = \int dz \rho_A(z, \mathbf{b})$ the longitudinal integral of the nuclear density normalized to one, called the profile function, and k the number of scatterings (exchanged hadron-nucleon amplitudes). At high energies, the scatterings take place simultaneously. We want to compute the contribution to the hA cross sections from cuts of these amplitudes $\mathcal{A}_k$. The cuts are arbitrarily ordered and given by a single cutting plane which may cut an arbitrary number of amplitudes (or none of them), but cuts only can take place on or after (on or before) the first (last) scattering. The AGK cutting rules provide the way to do this: $ia \to ia$ for amplitudes located to the left of the cut, $ia \to (ia)^*$ for amplitudes located to the right of the cut, and $ia \to \sigma$ for cut amplitudes, taking into account the combinatorial factors.
a) Obtain that $D_{k,m}$, the contribution from k exchanged amplitudes with $m \leq k$ cut ones, is

$$D_{k,m} = (-1)^{k-m}\binom{A}{k}\binom{k}{m}(\sigma T_A)^k.$$

b) Using the identity

$$\binom{A}{k}\binom{k}{m} = \binom{A}{m}\binom{A-m}{k-m},$$

obtain the inelastic (non-diffractive) cross section

$$\sigma^{\rm hA}_{inel} = \sum_{m=1}^{A} \sigma_m = 1-(1-\sigma T_A)^A, \quad \sigma_m = \sum_{k=m}^{A} D_{k,m} \ .$$

c) Obtain

$$D_{k,0} = (-\sigma T_A)^k - [iaT_A]^k - [(ia)^* T_A]^k$$

to get $\sigma_0 = \sum_{k=1}^{A} D_{k,0}$, the inelastic plus diffractive cross section, and verify that $\sigma^{\rm hA}_{inel} + \sigma_0 = \sigma^{\rm hA}_{tot}$.

d) Compute the mean number of participant nucleons (or nucleon-nucleon collisions)

$$\langle m\rangle(b)\, \sigma^{\rm hA}_{inel}(b) = \sum_{m=1}^{A} m\, \sigma_m = A\, \sigma\, T_A(b).$$

From here you get, integrating both sides over d^2b, $\langle m\rangle = A\,\sigma/\sigma^{\rm hA}_{inel}$. Considering that each cut gives the same contribution to the multiplicities, this shows that in hA the inclusive cross section is proportional the A times the inclusive cross section in hN, the same result we get in collinear factorization for a large scale. It means that rescatterings, which make $\sigma^{\rm hA} < A\,\sigma$, have no effect on the one-particle inclusive cross section – the so-called AGK cancellation.

Notes:

- Sometimes the binomials are exponentiated, so $(1-\sigma T_A)^A \simeq \exp(-\sigma A T_A)$.
- Many times the amplitude a is taken as purely imaginary, so $ia = -\sigma/2$.
- Under certain approximations, the above expression for σ_m can be used in AB collisions replacing $A \to AB$, $T_A(b) \to T_{AB}(b) = \int d^2s\, T_A(s)\, T_B(\mathbf{b}-\mathbf{s})$.
- We have taken the cut of the exchanged amplitudes as purely inelastic (no diffraction), and the cut between the amplitudes with the same structure as the hadron or nucleon. This may not be so, which forbids the identification of σ with the total hadron-nucleon cross section. In practice, to compute σ_m the inelastic or inelastic plus diffractive hadron-nucleon cross section is employed.

References: V. A. Abramovsky, V. N. Gribov and O. V. Kancheli, Yad. Fiz. 18 (1973) 595 [Sov. J. Nucl. Phys. 18 (1974) 308], Sections 1 and 2; C.-Y. Wong, *Introduction to High-Energy Heavy-Ion Collisions*, World Scientific 1994, Chapter 12; M. L. Miller et al., nucl-ex/0701025.