Proposed problems on the Standard Model

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Problem 1

The scalar sector of the Standard Model Lagranagian has the form:

$$\mathcal{L}_S = (D_\mu \phi)^\dagger D^\mu \phi - V(\phi), \qquad D^\mu \phi = \left[\partial^\mu + ig\frac{\vec{\sigma}}{2}\vec{W}^\mu + ig' y_\phi B^\mu\right]\phi,$$

$$V(\phi) = \mu^2 \phi^\dagger \phi + h(\phi^\dagger \phi)^2 \qquad (h > 0, \mu^2 < 0) \ ,$$

where $\phi(x)$ is an $SU(2)_L$ doublet of complex scalar fields. The potential, which only depends on the modulus of the scalar field doublet, has its minimumm at $|\phi|_{\rm min} = \sqrt{\frac{-\mu^2}{2h}} \equiv \frac{v}{\sqrt{2}}$. The scalar doublet can then be parametrized as:

$$\phi(x) \equiv \begin{pmatrix} \phi_a(x) \\ \phi_b(x) \end{pmatrix} = \exp\left\{i\frac{\vec{\sigma}}{2}\vec{\theta}\right\} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \ .$$

The (arbitrary) choice of vacuum configuration, $\phi_0^T = \frac{1}{\sqrt{2}}(0, v)$, breaks the $SU(2)_L \times U(1)_Y$ gauge symmetry spontaneously, leaving one generator unbroken:

$$Q \equiv (T_3 + Y) \equiv \frac{1}{2}(\sigma_3 + I_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \qquad Q\phi_0 = 0, \qquad e^{iQ\gamma}\phi_0 = \phi_0 \ .$$

We would like to identify Q with the electric charge and the associated unbroken symmetry with $U(1)_{\rm em}$. Thus, $\phi_b(x)$ and $H(x)$ are neutral fields, $\phi_a(x)$ has $Q = +1$ and $y_\phi = 1/2$.

a) Under a local $U(1)_{\rm em}$ transformation, $\phi'(x) = e^{iQ\gamma(x)}\phi(x)$, the electromagnetic field should transform as $A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \gamma(x)$, while the Z field should remain invariant. Show that this requirement implies $e = g \sin\theta_W = g' \cos\theta_W$, where θ_W is the electroweak mixing angle,

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_m u \\ A_m u \end{pmatrix} \ .$$

b) Find the explicit expression of the Lagrangian $\mathcal{L}_S$ in the unitarity gauge $\vec{\theta}(x) = \vec{0}$. Show that $M_Z \cos\theta_W = M_W = \frac{1}{2}gv$.

c) Working in a general gauge where charged scalars are present, show that one gets the correct $U(1)_{\rm em}$ covariant derivative, $D^\mu = \partial^\mu + ieQA^\mu$.

Problem 2

The Goldstone boson equivalence principle states that at high energy, the amplitude for emission or absortion of a longitudinally polarized massive gauge boson becomes equal to the amplitude for emission or absortion of the Goldstone boson that was eaten by the gauge boson. Consider a heavy Majorana neutrino (SM singlet), which has a Yukawa coupling only with one generation $SU(2)_L$ doublet lepton, (ν_e, e^-). Show that before spontaneous symmetry breaking (SSB) the branching ratio $\text{BR}(N \to \phi^+ e^-) = 1/4$ and, as a consequence of the Goldstone boson equivalence principle, after symmetry breaking $\text{BR}(N \to W^+ e^-) = 1/4$, in the limit $m_W \ll M_N$.

Problem 3

Have a look at the PDG. Why the charged pions ($\pi^\pm$) decay to $\mu^\pm \nu_\mu$ with probability 0.999877 and to $e^\pm \nu_e$ with probability 1.23 $\times 10^{-4}$?

Problem 4

Consider the process $e^+e^- \to W^+W^-$. In the Standard Model there are three amplitudes contributing to lowest order: t-channel neutrino exchange and s-channel photon and Z exchange.

a) Show that the t-channel amplitude leads to a cross section which increases with energy, violating unitarity.

b) Show that the s-channel Z exchange amplitude cancels exactly the bad high-energy behaviour.

c) The process $e^+e^- \to ZZ$ does not receive any s-channel contribution (the vertices $Z^3, \gamma Z^2$ are not present in the Standard Model). Show that the t-channel contribution is well-behaved in this case.

Problem 5

Calculate the muon decay width, $\mu \to e\bar{\nu}_e\nu_\mu$, neglecting the mass of the electron and neutrinos.