The mechanism of tunneling and formation of bound pairs of electrons

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Abstract.

The classical description of elementary spinning particles shows that the center of mass and center of charge of an elementary particle are different points. This separation is half Compton's wave length and because of this the interaction of two electrons with their spins parallel can produce a bound pair provided the internal phase is opposite and the relative velocity of their centers of mass is below a certain limit. It is also this separation which justifies that an electron under a potential barrier can cross it with an energy below the top of the potential provided the spin is properly oriented and the barrier has a narrow range. This can justify the spin polarized tunneling effect.

1. Two centers

Let us think that the following classical analysis was performed before 1920, i.e., before the emergence of quantum mechanics. The assumption is that the center of mass $\boldsymbol{q}$ and the center of charge $\boldsymbol{r}$, of a charged elementary spinning particle are two different points. If this is the case we can define the angular momentum of the particle with respect to both points. Let us call $\boldsymbol{S}$ the angular momentum w.r.t. the center of charge (CC for short) and $\boldsymbol{S}_{CM}$ the corresponding angular momentum w.r.t. the center of mass (CM for short). They are not independent, because if $\boldsymbol{p}$ is the linear momentum of the particle, then $\boldsymbol{S}_{CM} = (\boldsymbol{r} - \boldsymbol{q}) \times \boldsymbol{p} + \boldsymbol{S}$. But both spins satisfy two different dynamical equations in the free case and under some external electromagnetic interaction.

For any arbitrary inertial observer, the total angular momentum of the particle w.r.t. the origin of observer's frame can be written either as

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{S}, \quad \text{or} \quad \boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{p} + \boldsymbol{S}_{CM}.$$

If the particle is free, $\boldsymbol{J}$ is conserved and thus

$$\frac{d\boldsymbol{J}}{dt} = 0 = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} + \frac{d\boldsymbol{S}}{dt}, \quad \text{or} \quad \frac{d\boldsymbol{J}}{dt} = 0 = \frac{d\boldsymbol{q}}{dt} \times \boldsymbol{p} + \frac{d\boldsymbol{S}_{CM}}{dt},$$

so that

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p} \times \boldsymbol{u}, \quad \text{or} \quad \frac{d\boldsymbol{S}_{CM}}{dt} = 0,$$

because the conserved $\boldsymbol{p}$ is along the CM velocity $\boldsymbol{v} = d\boldsymbol{q}/dt$, but not along the CC velocity $\boldsymbol{u} = d\boldsymbol{r}/dt$. The CM spin is a conserved observable for a free particle while the CC spin is not.

It is moving in an orthogonal direction to the linear momentum, and only its projection on $\boldsymbol{p}$, the helicity $\boldsymbol{S}\cdot\boldsymbol{p}$, is conserved.

Let us assume now that the particle is under some external electromagnetic force $\boldsymbol{F}$ defined at the CC position. In this case, $\boldsymbol{J}$ and $\boldsymbol{p}$ are no longer conserved and thus $d\boldsymbol{J}/dt = \boldsymbol{r}\times\boldsymbol{F}$ and $d\boldsymbol{p}/dt = \boldsymbol{F}$.

$$\frac{d\boldsymbol{J}}{dt} = \boldsymbol{r}\times\boldsymbol{F} = \frac{d\boldsymbol{r}}{dt}\times\boldsymbol{p} + \boldsymbol{r}\times\frac{d\boldsymbol{p}}{dt} + \frac{d\boldsymbol{S}}{dt}, \quad \text{or} \quad \frac{d\boldsymbol{J}}{dt} = \boldsymbol{r}\times\boldsymbol{F} = \frac{d\boldsymbol{q}}{dt}\times\boldsymbol{p} + \boldsymbol{q}\times\frac{d\boldsymbol{p}}{dt} + \frac{d\boldsymbol{S}_{CM}}{dt},$$

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{p}\times\boldsymbol{u}, \quad \text{or} \quad \frac{d\boldsymbol{S}_{CM}}{dt} = (\boldsymbol{r}-\boldsymbol{q})\times\boldsymbol{F}.$$

The CC spin satisfies the same dynamical equation as in the free case, it moves in an orthogonal direction to the linear momentum, although now $\boldsymbol{p}$ is not conserved. The CM spin satisfies the usual torque equation: the torque of the external force w.r.t. the CM is the time variation of this spin.

Both spins can be found in the literature. The Bargmann-Michel-Telegdi spin [1] is the covariant generalization of the CM spin. The CC spin satisfies the same dynamical equation as Dirac's spin operator in the quantum case.

2. Classical model of a Dirac particle

If an elementary spinning particle has two separate centers, the free motion implies that the CM is moving at a constant velocity $\boldsymbol{v}$. But, what about the CC motion? If the motion is free it means that we are not able to distinguish, at two different instants, a different dynamical behaviour. But if the trajectory of the CC is a regular curve (i.e. a continuous and differentiable trajectory) it means that the velocity of the CC has to be of a constant modulus, the same at any time, and the trajectory of a constant curvature and torsion. The CC travels along a helix at a constant velocity, and this description must be valid for any inertial observer.

This implies that the CC velocity has to be unreachable for any inertial observer. Otherwise, if some inertial observer is at rest w.r.t. the CC at a certain instant t, because the CC motion is accelerated, it will have for that observer, a velocity different from zero at a subsequent time, and thus contradictory with the assumption that the velocity is of constant absolute value for any inertial observer. The only possibility is that the CC velocity is the speed of light and only a relativistic treatment is allowed.

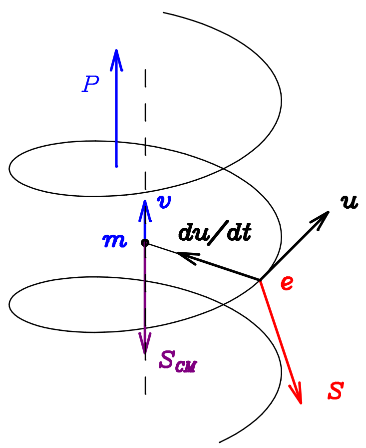

Figure 1. Model of a free classical Dirac particle, with two separate centers, showing the precession of the CC spin $\boldsymbol{S}$ and the conserved CM spin $\boldsymbol{S}_{CM}$. The CC moves along a helix at the speed of light. The CC spin is always orthogonal to the velocity and acceleration of the charge and precesses around $\boldsymbol{p}$. The separation between CC and CM is $\hbar/2mc$, half Compton's wavelength, and the frequency of this internal motion, in the CM frame, is $2mc^2/h$. It is described in [2].

This is precisely the main feature of a classical model of an elementary particle, which satisfies Dirac's equation when quantized, we have developped [2]. The free motion of this model is depicted in Figure 1, where we see the straight motion of the CM and the helical motion at the speed of light of the CC. We also depict the two above mentioned spins, $\boldsymbol{S}$ and $\boldsymbol{S}_{CM}$. The total spin $\boldsymbol{S}$ has two parts $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, one $\boldsymbol{W}$ related tom the rotation of the particle and in the direction of the angular velocity while the *zitterbewegug part* $\boldsymbol{W}$ is due to the separation between CC and CM and has the opposite direction, as depicted in figure 2.

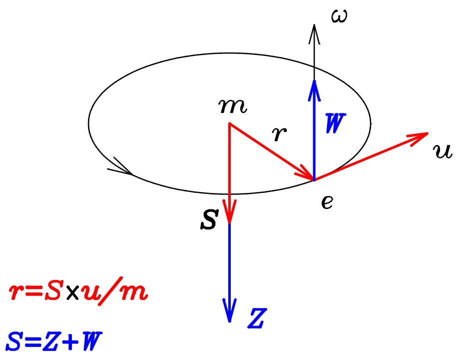

Figure 2. The classical description of a spinning Dirac particle in the CM frame. The CC $\boldsymbol{r}$ moves at the speed of light. The spin has two parts, one $\boldsymbol{W}$ related to the rotation of the particle and another in the opposite direction $\boldsymbol{Z}$ related to the zitterbewegung part of the motion of the CC around the CM.

3. Dirac's analysis of the electron

In his original 1928 papers [3, 4] Dirac describes an electron in terms of a four-component spinor $\psi(t, \boldsymbol{r})$, defined at point $\boldsymbol{r}$, and a Hamiltonian

$$H = c(\boldsymbol{p} - e\boldsymbol{A}(t, \boldsymbol{r})) \cdot \boldsymbol{\alpha} + \beta mc^2 + e\phi(t, \boldsymbol{r})$$

where β and $\boldsymbol{\alpha}$ are Dirac's matrices and ϕ and $\boldsymbol{A}$ the scalar and vector external potentials, also defined at the point $\boldsymbol{r}$.

When computing the velocity of point $\boldsymbol{r}$, Dirac arrives at: $\boldsymbol{u} = i/\hbar[H, \boldsymbol{r}] = c\boldsymbol{\alpha}$, which is expressed in terms of α matrices and writes, '*... a measurement of a component of the velocity of a free electron is certain to lead to the result* $\pm c$. *This conclusion is easily seen to hold also when there is a field present*', because it holds even if the external potentials are not vanishing.

The point $\boldsymbol{r}$ oscillates in a region of order of Compton's wavelength: '*The oscillatory part of* x_1 *is small, ... , which is of order of magnitude* $\hbar/mc$, *...*'. This is the amplitude of the motion of the CC around the CM in our model.

The linear momentum does not have the direction of the velocity $\boldsymbol{u}$, but must be related to some average value of it: ... '*the* x_1 *component of the velocity,* $c\alpha_1$, *consists of two parts, a constant part* $c^2p_1H^{-1}$, *connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least* $2mc^2/h$, *...*', the same as in the above classical model.

The total angular momentum w.r.t. the origin of observer's frame, takes the form

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \frac{\hbar}{2}\boldsymbol{\sigma} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{S}$$

where the orbital part $\boldsymbol{r} \times \boldsymbol{p}$ and the spin part $\boldsymbol{S} = \hbar\boldsymbol{\sigma}/2$, are not separately conserved for a free electron but the spin satisfies,

$$\frac{d\boldsymbol{S}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{S}] = \boldsymbol{p} \times c\boldsymbol{\alpha} = \boldsymbol{p} \times \boldsymbol{u}.$$

even under some external interaction. This is the dynamical equation of the CC spin.

The electron, '... *behaves as though it has a magnetic moment given by*

$$\mu = g\frac{e}{2m}\boldsymbol{S} = \frac{e\hbar}{2m}\boldsymbol{\sigma}, \quad g = 2,$$

an also an instantaneous electric dipole'

$$\boldsymbol{d} = \frac{ie\hbar}{2mc}\boldsymbol{\alpha}.$$

If the previous classical analysis of an elementary particle with two separate centers is taken into account, it is not difficult to conclude that Dirac's electron is an object with two centers, described by a spinor $\psi(t, \boldsymbol{r})$ which is a function of the CC position $\boldsymbol{r}$. The linear momentum is not lying along the velocity of point $\boldsymbol{r}$, but around some average value of it. Dirac spin operator is not the angular momentum w.r.t. the CM, but it represents the angular momentum w.r.t. the CC, even under some external interaction. The magnetic moment is produced by the motion of the charge, and the separation between these two points defines an electric dipole moment $\boldsymbol{d} = e(\boldsymbol{r} - \boldsymbol{q})$.

All these features of Dirac's analysis are contained in the classical description depicted in figure 2 in which the velocity of the CC is always c.

4. Electron dynamical equations

If we call the position of the CM $\boldsymbol{q}$, and its velocity $\boldsymbol{v} = d\boldsymbol{q}/dt$, $v < c$, and for the CC position $\boldsymbol{r}$, and $\boldsymbol{u} = d\boldsymbol{r}/dt$, in the relativistic case always $u = c$. The dynamical equation of the spinning electron in an external electromagnetic field is computed in [5] and are given by the expressions (1) and (2). In the nonrelativistic case, the second equation (2) is replaced by the third (3) showing that the relative motion of the CC around the CM is a kind of harmonic motion with a constant frequency ω while in the relatistic case the internal frequency depends on the velocity of the CM. The internal frequency of a relativistic electron decreases with its velocity, so that a faster electron a slower internal frequency as suggested by the so called twin paradox.

$$\frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E}(t, \boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}(t, \boldsymbol{r})), \quad \boldsymbol{p} = \gamma(v)m\boldsymbol{v} \tag{1}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} + \frac{c^2 - \boldsymbol{v}\cdot\boldsymbol{u}}{(\boldsymbol{r} - \boldsymbol{q})^2}(\boldsymbol{r} - \boldsymbol{q}) = 0, \quad \text{(RELATIVISTIC)} \tag{2}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} + \omega^2(\boldsymbol{r} - \boldsymbol{q}) = 0, \quad \text{(NON REL)} \tag{3}$$

We shall use these dynamical equations to analyze the classical behaviour of a spinning electron in two situations: The analysis of an electron-electron interaction, and the analysis of the interaction of a transversally polarized electron with a triangular potential barrier. In the firs case we shaw that, in addition to the usal sccatering between electrons, it is also possible that two spinning electrons with their spins parallel, can form bound states. This is done in next section. The analysis of tunnelling is defered till section 6.

5. Formation of bound pairs

We shall use the above dynamical equations to analyze the electron-electron interaction but for particles with spin. Here the fields are the electromagnetic field produced by either particle on each other.

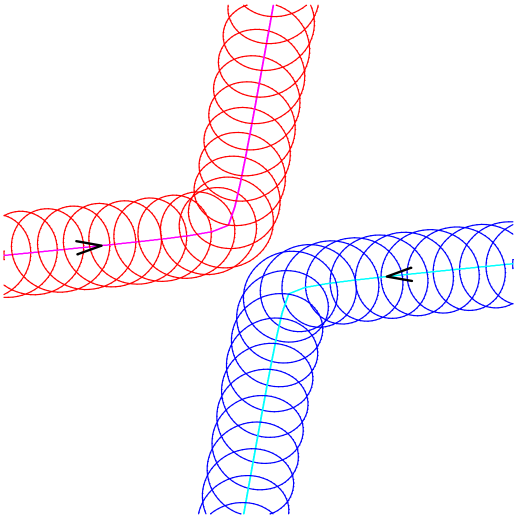

Figure 3. Scattering of two equal charged particles with parallel spins.

See in figure 3 the scattering of two equal charged particles with parallel spins. The centre of mass motion of each particle is depicted with an arrow. If the two particles do not approach each other too much these trajectories correspond basically to the trajectories of two spinless point particles interacting through an instantaneous Coulomb force (see figure 4). By too much we mean that their relative separation between the corresponding centres of mass is always much greater than Compton's wavelength. For high energy interaction the two particles approach each other to very small distances where the interaction term and the exact position of both charges, becomes important. In this case new phenomena appear. We can have, for instance, a forward scattering, which is not described in the classical spinless case, or even the formation of bound pairs for particles of the same charge, which we shall analyse in what follows.

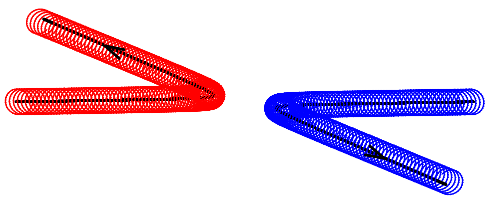

Figure 4. Scattering of two spinning particles with parallel spins. The inner black lines represent the motion of two spinless electrons interacting through a Coulomb force, which have as initial conditions the same positions and velocities as the CM's positions and velocities of the spinning electrons. There is a small difference provided the CC's do not approach each other too much.

In figure 5 we represent an initial situation for two equal charged particles with parallel spins such that the corresponding centres of mass are separated by a distance below Compton's wavelength. Remember that the radius of the internal motion is half Compton's wavelength. We locate the charge labels e_a at the corresponding points $\boldsymbol{r}_a$ and the corresponding mass labels m_a to the respective centre of mass $\boldsymbol{q}_a$. We see that a repulsive force between the charges when both charges have opposite phases implies an atractive force between the corresponding centres of mass. If the initial situation is such that the centres of mass separation is greater than Compton's wavelength, the force is always repulsive irrespective of the internal phases of the particles.

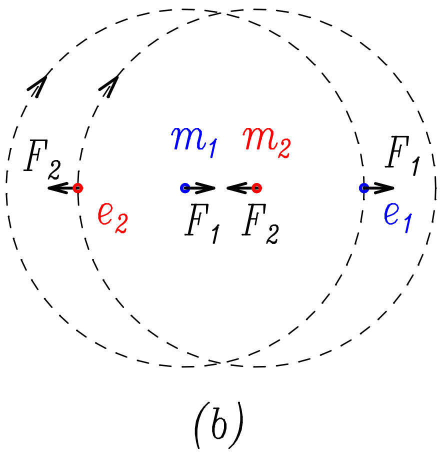

Figure 5. Boundary values for two Dirac particles with parallel spins and with a separation between the centres of mass below Compton's wavelength. The dotted lines represent the previsible clockwise motion of each charge. If the phases are opposite the repulsive force between charges becomes an atractive force between the CM's.

The analysis of this interaction is treated in more detail in [5]. In figure 6 we show the bound motion of both particles when their centres of mass are initially separated $q_{1x} = -q_{2x} =$ 0.2×Compton's wavelength, $\dot{q}_{1x} = -\dot{q}_{2x} = 0.008c$ and $\dot{q}_{1y} = -\dot{q}_{2y} = 0.001c$, and opposite phases.

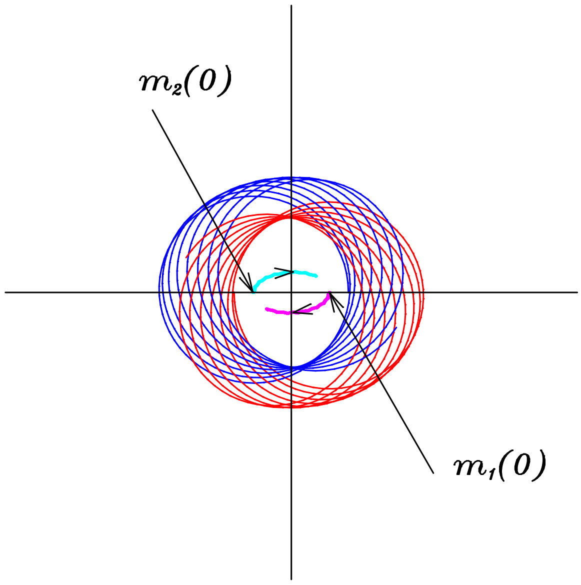

Figure 6. Bound motion of the CC's and CM's of two spinning particles with parallel spins, and with a centre of mass velocity $v \simeq 0.008c$, for an initial separation between the centres of mass of 0.2×Compton's wavelength.

We have found bound motions provided the velocity of each electron, in the CM frame, will be below $0.01c$. If the phases of the two particles are the same (or almost the same) there is no possibility of formation of a bound state. The two fermions of the bound state have the same spin and energy. They differ that their phases and linear momenta are opposite to each other. Is this difference in the phase a way to overcome at the classical level, the Pauli exclusion principle?

6. Tunneling

As a consequence of the zitterbewegung and therefore of the separation between the center of mass and center of charge, we shall see that spinning particles can have a non-vanishing crossing of potential barriers.

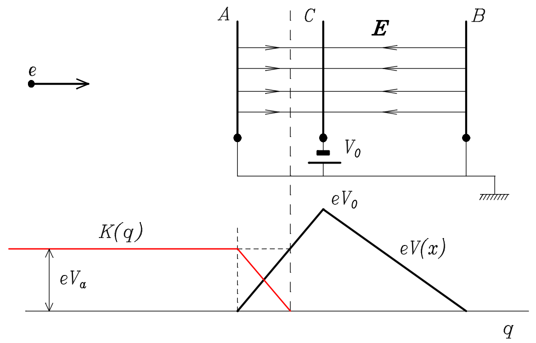

Figure 7. Uniform field triangular potential barrier. We show in red, the variation of the kinetic energy $K(q)$ of a spinless electron. It goes to zero, when the eelctron penetrates into the barrier. Then the electron stops and is rejected backwards. A spinless electron never crosses the barrier.

Let us consider the potential barrier depicted in figure 7. On the left side AC the electrostatic field produces a force on the electrons to the left while on the right side region CB, the force is to the right. A spinless electron, accelerated with a potential V_a has a kinetic energy $K(q)$ such that when the electron enters into the field decreases till zero, stops and is finally rejected. A spinless electron with kinetic energy below eV_0, never crosses that barrier. However, a tranversally polarized spinning electron can tunnel the above device.

We solve the electron dynamical equations (1) and (2) in that potential of left width a and right width b, respectively, and depict in figure 8 the variation of the kinetic energy during the crossing. In this figure, the width of the potential is $a = b = 1$, i.e., in units of the separation between the CC and CM. In figure 9 we depict the variation of the kinetic energy when the right side of the potential has a width of $b = 10$ times this separation. The whole classical analysis is dimensionless so that the crossing is independent of the absolute value of the potential V_0. A more detailed analysis is done in [6]

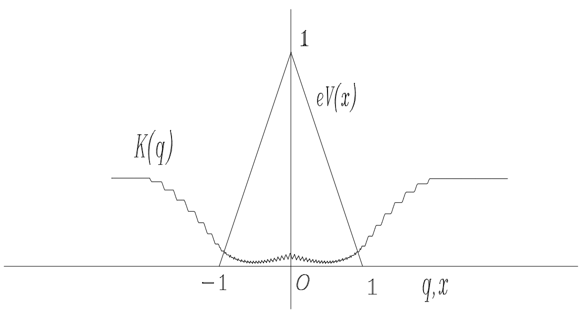

Figure 8. Evolution of the kinetic energy of a transversally polarized electron in a triangular potential barrier of left width $a = 1$ and right width $b = 1$.

To compare the classical crossing with the quantum one we use the solution of this quantum mechanical problem as solved by Landau [7]. The quantum probability depends on the potential V_0 and for different V_0 values is depicted in figure 10 as a function of the right width b of the barrier, with a fixed value for the left width a. This quantum probability has been obtained by assuming that we have electrons of a uniform distribution in energy, below the top of the potential. Simmilarly, in the same figure we shaw the classical probability of tunneling $P(b)$ computed from the previous solved equations for different values of b. If we consider for the classical spinning particle the same uniform distribution of particles, then, the function $P(b) = 1 - K_c(b)$, where $K_c(b)$ is the minimum dimensionless kinetic energy for crossing

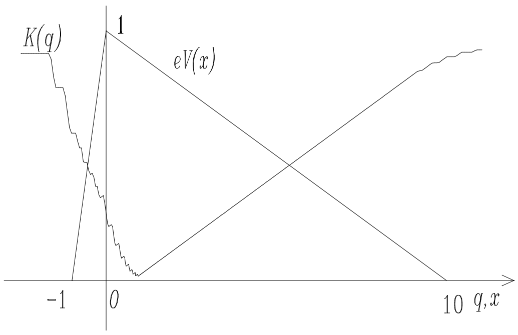

Figure 9. Evolution of the kinetic energy of a transversally polarized electron in a triangular potential barrier of left width $a = 1$ and right width $a = 10$.

computed before, represents the ratio of the particles that with kinetic energy below the top of the potential cross the barrier because of the spin contribution.

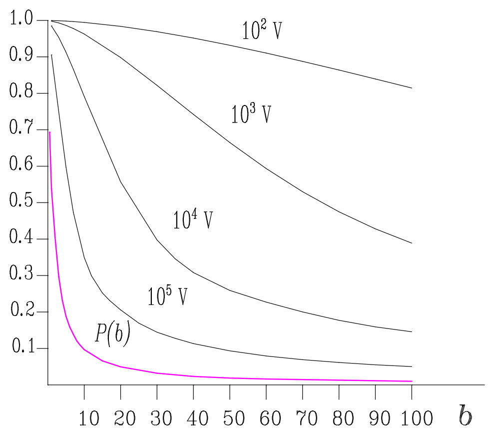

Figure 10. Classical probability $P(b)$ and Quantum probability of tunneling for various potentials V_0. Classical Probability is independent of the potential and quantum probability decreases with V_0. The Quantum Probability is greater than the Classical Probability for crossing because of the Uncertainty Principle.

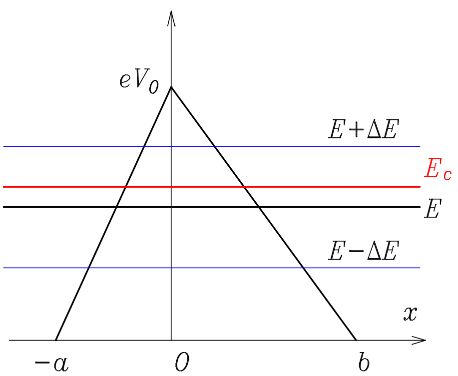

Figure 11. Potential barrier where E_c represents the minimum kinetic energy for crossing for the classical spinning particle. For a quantum particle of kinetic energy E, the value of the crossing energy is between $E + \Delta E$ and $E - \Delta E$, with $\Delta E \Delta t_c \geq \hbar$, being Δt_c the uncertainty in the time of crossing. It thus implies that a quantum particle with an energy below the crossing energy, has a probability of having a greater energy than E_c and thus it crosses the barrier.

The classical probability of crossing is smaller then the quantum one because of the uncertainty principle. But when the potential V_0 rises, the quantum probability approaches the classical one. The reason is described in figure 11. If Δt_c is the uncertainty in the time of crossing the uncertainty principle implies that the uncertainty in the energy of the electron satisfies $\Delta E\ \Delta t_c \geq \hbar$. It therefore implies that a quantum particle with an energy below the crossing energy can have a nonvanishing probability of having a greater energy than E_c if this value lies in the range $E_c \in (E - \Delta E, E + \Delta E)$. This ΔE decreases when the uncertainty Δt_c increases which is the case for faster particles. This means that a quantum particle with an average energy below the crossing energy E_c, and an uncertainty $\pm\Delta E$ can cross the barrier while the classical spinning particle does not. When the potential V_0 increases the quantum probability of crossing approaches the probability computed for the classical spinning particle.

Acknowledgments

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