On the equilibrium configuration of point charges placed on an ellipse

J. M. Aguirregabiria, A. Hernández, M. Rivas, and M. A. Valle
Física Teórica, Facultad de Ciencias, Universidad del País Vasco, Apdo. 644, 48080 Bilbao, Spain

(Received 28 October 1988; accepted 25 September 1989)

The equilibrium configurations of several different systems of pointlike charges placed on an ellipse are considered. The suitability of these discrete systems as models for finding the equilibrium charge distribution in a conducting wire of negligible cross section is also discussed. This example might prove useful in a course on computational physics, especially in order to illustrate the dangers of naive discretizations of continuous physical problems.

INTRODUCTION

The problem of finding the equilibrium charge distribution in a conducting wire with negligible cross section has recently been considered by Ross.[1] In his paper, a discrete system of point charges is substituted for the continuous problem. This, in turn, suggests analyzing and comparing other discrete systems that could also be considered as naive models for the same continuous problem.

It should be noted that we want to consider the discrete models as physical systems by themselves. Analogous discrete problems, which refer to charges or magnets placed instead on a spherical surface or in bounded regions of the plane or the space, have received quite a lot of attention[2–7] in the last years.

Nevertheless we shall also discuss the suitability of the discrete physical systems as models for the continuous problem, in particular in relation to the method of moments.[8–10]

We think that this problem could be a useful example for a course on computational physics. Indeed, several different numerical techniques can be tried, compared, and tested in its analysis. Moreover, it can be useful for discussing the most fundamental concept of computational physics: discretization. As we shall see, different discrete models that seem, at first sight, to be plausible models for the continuous problem will give different results for the linear charge density. Furthermore, all the numerical computations described below can be done rather easily on a microcomputer and the necessary numerical techniques and codes are well described in textbooks and widely available in different computer languages and formats.[11]

I. THE CONTINUOUS PROBLEM

Ross[1] considers an elliptic shaped conducting wire in equilibrium with total charge Q. The dimensions of the cross section are assumed to be negligible compared to the ellipse axes, and the unknown to be found is the charge distribution per unit length.

The method of Ref. 1 is based on an equilibrium condition, the wire must be equipotential, and on a discrete model for the wire: It is divided in N pieces subtending the same polar angle from the center and each piece is replaced by a point charge q_i located in the middle and whose polar angle is $2\pi i/N$, $\mathbf{r}_i$ being its corresponding position vector. The problem is thus reduced to the solution of the linear system formed by the N equations, which state that the potential V is the same in the N sites of the charges and the condition that the total charge is Q:

$$\frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\ j\neq 1}}^{N}\frac{q_j}{|\mathbf{r}_i-\mathbf{r}_j|}=V \quad (i=1,2,\ldots,N), \tag{1a}$$

$$\sum_{j=1}^{N} q_j = Q. \tag{1b}$$

This system can be solved for the $N+1$ unknowns (the N charges q_i and the potential V) and the linear density at each site is defined as $\lambda_i \equiv q_i/l_i$, where l_i is the length of the corresponding piece of wire. The results obtained by Ross are rather surprising and, we think, incorrect. The linear charge density is far more discontinuous (in the figure of Ref. 1, values that seem to be strictly null appear close to rather high values) than what can be expected by elementary physical intuition. Moreover, we think that these unacceptable results occur not only because the number of points N used is not sufficient for the cases analyzed but also, and more important, because the method is not correct, as we shall discuss now, and later in Secs. II and IV.

In fact, we can interpret system (1) in two apparently similar yet very different forms. First, we can think that it is a discrete mathematical approximation to the equations for the continuous case, which would be something like

$$\frac{1}{4\pi\epsilon_0}\oint_C \frac{\lambda(s')ds'}{|\mathbf{r}(s)-\mathbf{r}(s')|}=V \quad (\forall s\in C), \tag{2a}$$

$$\oint_C \lambda(s')ds' = Q, \tag{2b}$$

where s is the curvilinear abscissa along the ellipse C. The difficulty is that the first integral in (2) has a logarithmic singularity when $s=s'$. So Eqs. (2) are not valid for the wire and system (1) should be an approximation to incorrect equations. On the other hand, the method can be seen as a variant of the *point-matching approximation* in Ref. 8, but the important—in fact, infinite—contribution to the potential of the division of the wire where each point is located is simply ignored.

The second possible interpretation for system (1) is to consider the discrete model as a physical system by itself, i.e., like N real point charges restricted to move on an ellipse. In the realm of this second interpretation, the problem consists of finding for what values of the N charges these stay at rest in their initial positions. From this point of view, we might perhaps expect that in the limit $N \to \infty$ the result for the linear charge density in the equilibrium configuration of this discrete system will be similar to the one corresponding to the continuous case. As we shall see, however, this cannot always be the case.

Furthermore, though in this discrete system there is no divergence—because, as usual with point charges, the self-contributions are ignored—the right equilibrium condition is not that the potential must be the same on all charges. In fact, the necessary and sufficient condition for the point charges to be at rest is that they are under the action of an electric field that must be normal to the ellipse. In the continuous case, the fact that the potential is constant on the surface of a conductor is strictly equivalent to the vanishing of the tangent component of the electric field in all points of that surface. But this is no longer true in the case of the discrete physical system, as we shall show in Sec. III.

In the following, we shall deal mainly with the discrete systems and their possible connection with the continuous conducting wire will only occasionally be considered until Sec. IV.

II. A FAMILY OF DISCRETE PHYSICAL SYSTEMS

We shall consider a class of discrete physical systems in which N point charges are located initially at certain points on an ellipse and free to move along it. The goal is to find the values of the charges q_i for which every charge remains at rest in its initial position. We shall use for the ellipse a generic parameter p in such a way that the locations of the N charges are determined by the values p_i (see Fig. 1). Since in the equilibrium configuration the electric field on each charge must be normal to the ellipse, the unknown values q_i must satisfy the equations

$$\frac{1}{4\pi\epsilon_0} \sum_{\substack{j=1 \\ j\neq 1}}^{N} \frac{q_j(\mathbf{r}_i - \mathbf{r}_j)\cdot\boldsymbol{\tau}_i}{|\mathbf{r}_i - \mathbf{r}_j|^3} = 0 \quad (i = 1,2,...,N), \tag{3}$$

where τ_i is the unit vector tangent to the ellipse at point $\mathbf{r}_i = \mathbf{r}(p_i)$.

First, we need to know if this linear system has a solution and if this is unique, i.e., we must compute the rank of the $N \times N$ matrix A with elements

$$A_{ij} = (\mathbf{r}_i - \mathbf{r}_j)\cdot\boldsymbol{\tau}_i/|\mathbf{r}_i - \mathbf{r}_j|^3 \quad (i,j = 1,2,...,N). \tag{4}$$

The computation of this rank is an interesting application of singular value decomposition techniques.[11] The results we found can be summarized as follows:

(1) If the charges are located at random positions, rank $A = N$ and the only solution is the trivial one: $q_1 = q_2 = \cdots = q_N = 0$.

(2) If the sites are symmetrically distributed—at points with polar angles $\theta_i = 2\pi i/N$, for instance—and the number of charges is odd, rank $A = N - 1$. The indeterminacy resulting from this can easily be removed if, after giving the total charge Q, we replace any one of the equations in system (3) with Eq. (1b).

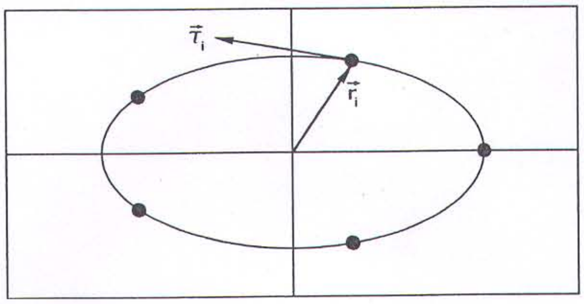

FIG. 1. In all figures N point charges are located on an ellipse at equidistant values of the parameter p (here, $p = \phi$). The semiaxes are $a = 2$ and $b = 1$.

(3) If the charges are symmetrically distributed and their number is even, rank $A = N - 2$. In these cases, there are an infinity of solutions, even after removing the invariance associated with the total charge. The reason for this is the high symmetry, as can be easily understood in the simplest cases. Indeed, any two charges, q and $Q - q$, located opposite each other at the ends of the major—or minor—axis will stay at rest. The same happens with four charges, two of the same arbitrary value q at the opposite sides of the major axis, and the other two of value $Q/2 - q$ at the ends of the minor axis. In the same way, it is easy to convince oneself that the solution will not be unique in cases with more charges but the same high symmetry.

So, in order to have a unique solution, we shall only consider an odd number of charges distributed in a symmetric fashion. Although this can still be made in many different forms, we shall only analyze three of the most natural ones. To do that, we shall consider three families of physical systems constructed by parametrizing the ellipse by three different parameters: the polar angle θ (as made by Ross)[1]; the angle ϕ for which the equations of the ellipse are $x = a\cos\phi$, $y = b\sin\phi$ (a and b being the semiaxes of the ellipse); and the curvilinear abscissa s given by the length measured along the ellipse and scaled in such a way that the total length of the ellipse is always 2π. In any case, the charges are located at points $p_i = 2\pi i/N$ ($i = 1,2,...,N$) and the linear system (3)—with an equation replaced by Eq. (1b) with $Q = 1$—can be solved, for instance, by means of the LU decomposition.[11] This gives us the values of the charges necessary to have an equilibrium configuration.

If we want to compare the results obtained for different physical systems with different values of N and for different parameters, we need to define a common quantity for all of them. The most natural candidate—especially if we expect the results to have some relation to the original continuous problem—is the linear charge density. So the next step is to assign to each charge q_i an arc of the ellipse $p_{i-1/2} < p < p_{i+1/2}$. Then we must compute the length of the arc by using Romberg integration or a routine that approximates the elliptic functions,

$$l_i = \int_{p_{i-1/2}}^{p_{i+1/2}} \frac{ds}{dp}\, dp, \tag{5}$$

and, finally, we shall define the linear charge density at site i as $\lambda_i = q_i/l_i$.

If the charges are located at points $p_i = 2\pi i/N$, the most natural choice for the division of the ellipse in pieces is

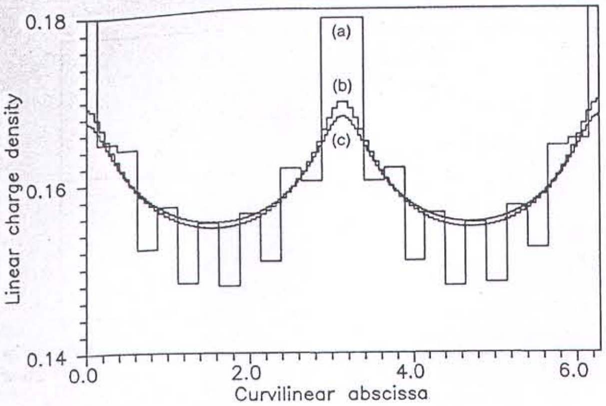

FIG. 2. Linear charge densities $\lambda(s)$ when the tangent electric field is null in discrete systems with charges located at points with curvilinear abscissa $s_i = 2\pi i/N$ and (a) $N = 25$; (b) $N = 101$; (c) $N = 175$.

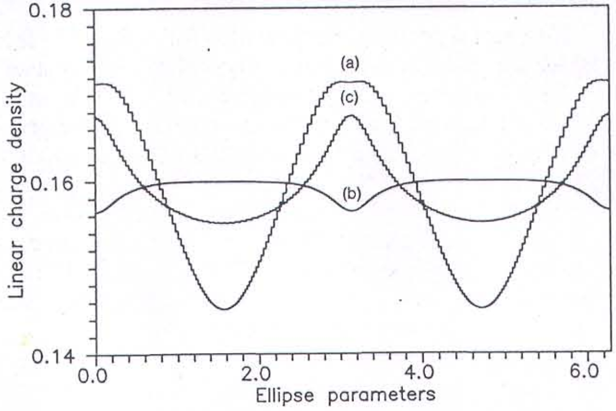

FIG. 3. Linear charge densities $\lambda(s)$ for a null tangent electric field when the $N = 175$ charges are distributed at positions $p_i = 2\pi i/N$, the parameter p being (a) the polar angle θ; (b) the angular parameter ϕ; and (c) the length s measured along the ellipse. The three different discrete systems give different results, so at most one of them can be an acceptable approximation to the continuous system.

to take $p_{i-1/2} = \frac{1}{2}(p_{i-1} + p_i)$ and $p_{i+1/2} = \frac{1}{2}(p_i + p_{i+1})$. We have made it and plotted λ vs s for different values of N. When the number of points is rather low, the result could hardly be interpreted as a meaningful density because of its up and down character [see Fig. 2(a)]. In fact, this density can even become negative for high enough eccentricities with low values of N. Though the curves are always stepwise due to the discrete nature of the problem, when $N \to \infty$ it can be seen in Fig. 2(b) and (c)—and in other curves not plotted for clarity—that they converge to a smooth limit. It is thus possible for each parameter to attribute a definite sense to the concept of linear charge density.

But, as shown by the results displayed in Fig. 3 for $N = 175$ and for the three different parameters, θ, ϕ, and s, i.e., different families of discrete systems give different limited charge distributions. So, at most, one of these families can give the linear charge density corresponding to the continuous wire.

This illustrates rather clearly how, in some instances, different discrete systems that at first glance could be thought of as equally plausible approximations for a continuous problem can give very different results. Furthermore, the difference can be not only quantitative but also qualitative, as seen in the case of parameter ϕ which gives a charge density that is smaller at points where curvature is bigger, contrary to the intuition for the continuous wire.

On the other hand, if we insist on interpreting system (3) as a mathematical approximation to an integral equation associated with the continuous system,

$$\frac{1}{4\pi\epsilon_0} \oint_C \frac{\lambda(s')[\mathbf{r}(s) - \mathbf{r}(s')]\cdot\boldsymbol{\tau}(s)\, ds'}{|\mathbf{r}(s) - \mathbf{r}(s')|^3} = 0, \qquad (6)$$

the differences between the limits obtained for the three discrete systems, which can be seen as different methods of constructing partial Darboux sums for this integral, reflect its divergent nature.

Another reasonable method to share an arc of the ellipse between the two neighboring charges in order to define the charge density is always to halve its length, regardless of the parameter used to locate the charges. We have also done that but, as expected, the results do not significantly change when N is big enough.

As seen in Fig. 4, the value of the potential V is not in general the same at the sites of the different charges (marked in the figure with vertical strokes), though, as required by the equilibrium condition, the tangent electric field is null at all these points. If, on the contrary, we require with Ross[1] the potential to be the same at every site, the tangent component of the electric field would not in general be zero there, as seen in Fig. 5. This fact shows that the two conditions are no longer equivalent in the discrete model and invalidates, in our opinion, the results previously mentioned. Since each charge is affected only by the fields of the others, Figs. 4 and 5 show in the neighborhood of each charge the contributions to the potential and the tangent electric field created by the remaining charges. This is the origin of the apparent discontinuities of Figs. 4 and 5.

III. A MORE NATURAL DISCRETE SYSTEM

In this section we shall consider a different discrete physical system that could be thought of as a more natural—though still naive—model for the continuous wire. Like the electrons in a conducting wire, we think of N charges that all have the same value, $q_i = 1/N$. The unknowns will now be the parameter values p_i for charge positions in equilibrium configurations. Related problems of finding the equilibrium configurations of N pointlike charges free to move on the surface of a sphere[2,3] or in the interior of a circle,[4] a sphere,[5] or other volumes,[6] or in the case of magnets,[7] have recently been considered.

The system to be solved now is a nonlinear one and

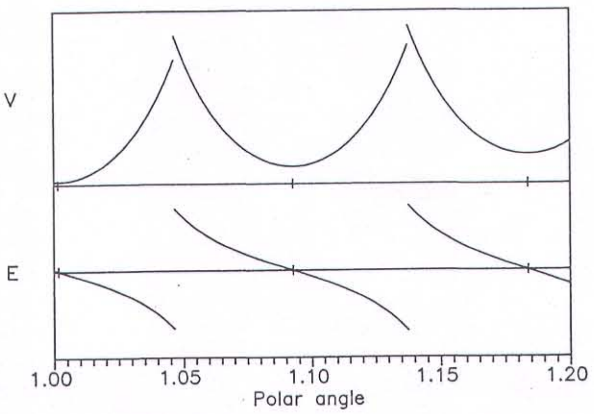

FIG. 4. Though the tangent electric field E is null at all charge positions (indicated by vertical strokes), the potential V does not have the same values at all these points. E and V are measured in arbitrary units and the charges are located at equidistant values of the polar angle.

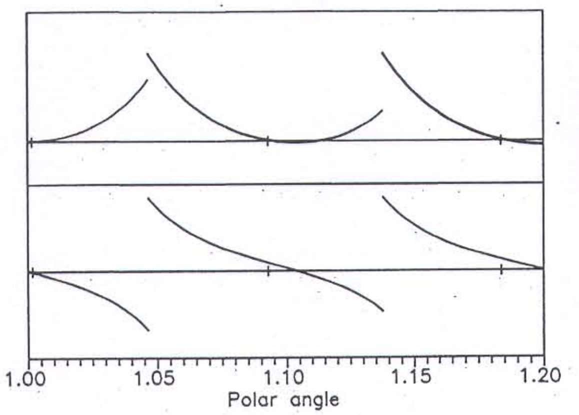

FIG. 5. The same problem as in Fig. 4, but now charge values have been computed, as in Ref. 1, in order to have the same V at all sites. Since the tangent electric field acting on charges is not null, this is not an equilibrium configuration.

thus more difficult. Though this task can be accomplished, especially if N is rather small, by means of the Newton–Raphson method for nonlinear systems of equations,[11] a better approach can be undertaken.

The condition of a null tangent field at each site is equivalent to requiring the electrostatic energy,

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{N^2}\sum_{i<j}\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{7}$$

to be extremal. Indeed we are only interested in solutions of minimal energy and these can be found by means of different methods; we have tried with success the Polak–Ribiere variant of the conjugate gradient method and simulated annealing.[11] This approach is more efficient and has the added advantage of not giving the unstable equilibrium configurations (corresponding to maximum energy) that are occasionally obtained by imposing the condition of a null tangent field.

The results obtained by means of this method and after dividing each piece of ellipse in two parts of the same length assigned to the neighboring charges is presented in Fig. 6. A comparison with Fig. 3(c) shows that the results for this system are nearly the same as those obtained with charges located at regular length intervals, which was the most natural element of the former family of discrete systems. That suggests very strongly that Figs. 3(c) and 6 are good approximations to the linear charge density of the continuous original problem.

IV. FINAL COMMENTS

Back to the conducting wire problem, is it natural to ask ourselves if it is possible to attack the continuous problem directly by means of some variant of the method of the moments.[8–10] As we have already said, the most direct approach by the point-matching approximation is not workable, because when computing the potential at a point the contribution of the division of the wire in which it is located is infinite. The solution by Ross,[1] who skips this important contribution completely, does not seem acceptable either.

Of course, another method could consist of taking into account that the wire's transversal dimensions are not strictly null. But another approach is possible, and simpler, which would consist of considering as equilibrium condition for the continuous system, instead of the constancy of the potential, the physically and mathematically equivalent condition of a null tangent electric field, as given by Eq. (6). For this new condition, it is straightforward to design

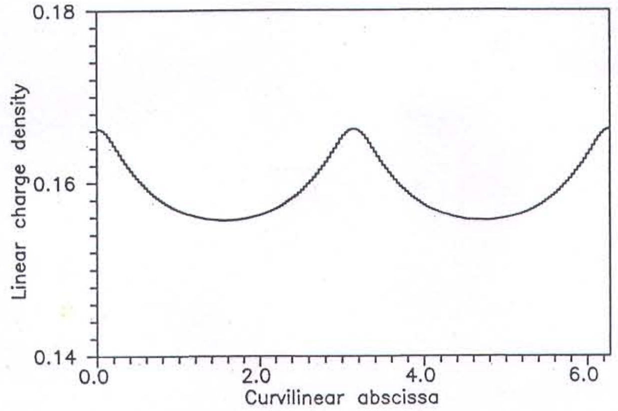

FIG. 6. Linear charge density $\lambda(s)$ corresponding to minimal electrostatic energy when $N = 175$ equal charges are free to move on the ellipse.

a variant of the point-matching approximation in the method of moments.[8] The problem is, as before, that at each point the contribution of its own wire division is *in general* infinite. But—contrary to the case of the potential, where this contribution is *necessarily* infinite—we can now carefully select each division in order to have a null self-contribution to the tangent electric field. It is sufficient to take, in the limit $N \to \infty$, the division as a very short straight segment of wire with the point where the field is computed located precisely in its geometric center. In this case the self-contribution is obviously null by symmetry.

But the method just described is exactly equivalent to the discrete system obtained when in Sec. II the selected parameter was the arc length s, and only to it. And this system was precisely the one that gave the same charge density that the more natural system considered in Sec. III. We can thus consider that discrete system as the simplest variant of the point-matching approximation in the method of moments that can be applied to the continuous wire problem.

From a mathematical point of view, that discrete system can be seen as a careful way of defining partial sums that give in the limit a sense to the divergent integral (6) in a way very similar to Cauchy's principal value. The same cannot be done for integral (2a) because its integrand is definite.

We think these are convincing reasons to accept Fig. 3(c) or 6, as a meaningful approximation to the linear charge density of the continuous conducting wire problem.

ACKNOWLEDGMENT

One of the authors (J.M.A.) acknowledges financial support from an "Acción integrada" from the Spanish and French Governments.

REFERENCES

1. J. B. Ross, Am. J. Phys. **55**, 948 (1987).
2. A. A. Berezin, Nature **315**, 104 (1985); Nature **317**, 208 (1985); Am. J. Phys. **53**, 1036 (1985); Am. J. Phys. **54**, 403 (1986).
3. B. W. Van de Waal, Am. J. Phys. **56**, 583 (1988); Am. J. Phys. **56**, 750 (1988).
4. L. T. Wille and J. Vennik, J. Phys. A **18**, L1113 (1985).
5. M. G. Calkin, D. Kiang, and D. A. Tindall, Nature **319**, 454 (1986).
6. A. A. Berezin, Am. J. Phys. **55**, 199 (1987).
7. D. Pescetti and E. Piano, Am. J. Phys. **56**, 1106 (1988).
8. R. F. Harrington, *Field Computation by Moment Methods* (Krieger, Malabar, FL, 1968).
9. *Computer Techniques for Electromagnetics*, edited by R. Mittra (Hemisphere, Washington, DC, 1987).
10. C. W. Steele, *Numerical Computations of Electric and Magnetic Fields* (Van Nostrand, New York, 1987).
11. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing* (Cambridge U.P., New York, 1986).