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Maxwell's equations and Lorentz transformations

J M Aguirregabiria[*], **A Hernández and M Rivas**

Department of Theoretical Physics, Faculty of Science and Technology, University of the Basque Country (UPV/EHU), Bilbao, Spain

E-mail: juanmari.aguirregabiria@ehu.eus, anibal.hernandez@ehu.eus and martin.rivas@ehu.eus

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Abstract

We explore the possibility of introducing the special relativity to undergraduate students by restricting the relativity principle to Maxwell's equations in vacuum. By making some hypothesis of simplicity, we obtain the transformation equations for the electric and magnetic fields among equivalent observers. Later, the transformation equations for the charge and current density are found and, finally, the Lorentz transformations. What is left is to extend the relativity principle to all physical phenomena.

Keywords: ARTICLE KEYWORDS Maxwell's equations, special relativity, Lorentz transformations

1. Introduction

In many introductory and advanced textbooks [1] the transformation equations for the electric and magnetic fields are obtained after introducing the Lorentz transformations following previous ideas [2–7].

The object of this paper is to obtain the Lorentz coordinate transformations by a simple and heuristic procedure by assuming that Maxwell's equations in vacuum must have the same form in the equivalent reference frames. We also assume that the vacuum has the same values of the electromagnetic properties ϵ_0 and μ_0. By this method we obtain the transformation equations for the electric and magnetic fields as well as those of the charge and current densities, before the Lorentz coordinate transformations are finally found.

The method proposed here to deduce the Lorentz transformations allow undergraduate students to obtain them, after having learned Maxwell equations, by means of classical concepts

[*]Author to whom any correspondence should be addressed.

and highlighting the influence of previous ideas, in the spirit of Bell's suggestion [8] for teaching special relativity.

The method used to arrive to the Lorentz coordinate transformations requires some calculations, but they are simple from the mathematical point of view and can be performed by undergraduate students. Perhaps this is a longer method but it is a way of introducing to the students to special relativity without obscure considerations about the geometry of space-time. It has the advantage of the simplicity and clearness of the principles and concepts used. The analysis of the loss of simultaneity among equivalent observers as well as other consequences in mechanics and other fields of physics may be analysed once the Lorentz transformations are obtained. This method of obtaining the Lorentz transformations could be an alternative way of introducing special relativity and has to be compared with a previous method [9] where we obtained the transformations without reference to any electromagnetic phenomena.

In section 2 we state the basic postulates this paper is based upon. In section 3, in order that source free Maxwell's equations in vacuum have the same form in every equivalent frame, the relations between the fields $\vec{E}$ and $\vec{B}$ in the reference frame S and the fields $\vec{E}'$ and $\vec{B}'$ in the reference frame S' are established. The transformation equations for the charge and current densities are obtained in section 4. In section 5 the different values of the coefficients involved in the transformation equations, are computed. These coefficients are used later to obtain the final expressions of the transformation equations for the fields and sources, as well as the Lorentz transformations. The final section 6 contains the summary of the results.

2. Fundamental postulates

The first postulate is the restricted relativity principle for electromagnetic phenomena:

> *There exists a class of equivalent observers such that Maxwell's equations in vacuum are written in the same form in the corresponding reference frames.*

The second postulate admits the invariance of the electromagnetic properties of vacuum:

> *The permittivity ϵ_0 and permeability μ_0 of vacuum have the same values in every equivalent frame.*

Since $\epsilon_0\mu_0 = 1/c^2$, the universal constant c is also invariant and therefore the electromagnetic phenomena in vacuum propagate at the same velocity c in every equivalent reference frame.

To define the class of equivalent observers, and therefore how their relative space-time measurements are related, we assume that space-time is homogeneous, i.e. the origin of the Cartesian frames can be located anywhere, and that space is isotropic, so that the spatial reference frames can have arbitrary orientations. Therefore space-time translations and static rotations are transformations which relate their relative space-time measurements. In general the equivalent observers are related by a group of space-time transformations, in the spirit of Poincaré, so that the above assumptions imply that space-time translations and static rotations are subgroups of this general group. We also assume that some relative motions at constant velocity are also allowed among the equivalent observers. We are going to determine these transformations when there is a constant relative velocity among equivalent observers.

The equivalent reference frame S' is moving along the common axis OX with respect to the frame S, with a constant velocity v. Axis OY' and OZ' are parallel to those OY and OZ of frame S, respectively.

By the homogeneity and isotropy of space and also the homogeneity of time, we assume that the transformation equations are of the form

$$x' = f(x, t; v), \qquad y' = y, \qquad z' = z, \qquad t' = g(t, x; v), \tag{1}$$

because the motion takes place along OX axis. In [10] the authors claim they have obtained a solid proof of the linearity of Lorentz transformations, which is not assumed here.

Poincaré obtained the invariance of Maxwell's equations by assuming that space-time coordinates transform under the linear Lorentz transformations. What we are going to determine is the opposite: to obtain the linear Lorentz transformations by assuming the invariance of Maxwell's equations.

3. Transformation equations of the fields

Source free Maxwell's equations in vacuum in Cartesian coordinates in the reference frame S, are

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, \tag{2}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}, \tag{3}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}, \tag{4}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \tag{5}$$

We are looking for coordinate transformations such that the above equations (2)–(5) will remain in the same form in every other equivalent frame S', with the fields replaced by $\vec{E}'$ and $\vec{B}'$ and the coordinates x, y, z and t by x', y', z' and t' of the frame S', respectively.

To obtain the form of the Maxwell's equations in the frame S', we have to use (1) to replace the following differential operators:

$$\frac{\partial}{\partial x} = a\frac{\partial}{\partial x'} + m\frac{\partial}{\partial t'}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}, \qquad \frac{\partial}{\partial t} = b\frac{\partial}{\partial x'} + n\frac{\partial}{\partial t'}, \tag{6}$$

where we have used the following coefficients, which are not assumed to be constants:

$$a = \frac{\partial f}{\partial x}, \qquad b = \frac{\partial f}{\partial t}, \qquad m = \frac{\partial g}{\partial x}, \qquad n = \frac{\partial g}{\partial t}. \tag{7}$$

From (5) and (6) we obtain

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y'} - \frac{\partial B_z}{\partial z'}, \tag{8}$$

and by substituting $\partial/\partial x$ for its expression in (6)

$$a\frac{\partial B_x}{\partial x'} + m\frac{\partial B_x}{\partial t'} = -\frac{\partial B_y}{\partial y'} - \frac{\partial B_z}{\partial z'}. \tag{9}$$

Equation (2) written in terms of the primed derivatives, using the relations (6), becomes

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = -b\frac{\partial B_x}{\partial x'} - n\frac{\partial B_x}{\partial t'}. \tag{10}$$

By replacing here $\partial B_x/\partial x'$ by the expression obtained from (9) one gets

$$\frac{\partial}{\partial y'}\left(aE_z - bB_y\right) - \frac{\partial}{\partial z'}\left(aE_y + bB_z\right) = -\frac{\partial}{\partial t'}\left(an - bm\right)B_x. \tag{11}$$

The simplest way for expression (11) in terms of the primed fields to have the same form in S' as does in (2) and also the requirement that if both electric and magnetic fields vanish in a frame they also vanish in any other frame, is to assume that the fields transform as [11]

$$E'_y = aE_y + bB_z, \quad E'_z = aE_z - bB_y, \tag{12}$$

$$B'_x = (an - bm)B_x. \tag{13}$$

Similarly, the expressions (3) and (4) will have the same form in S' if the fields transform as:

$$E'_x = E_x, \qquad E'_z = aE_z - bB_y, \qquad B'_y = nB_y - mE_z, \tag{14}$$

$$E'_x = E_x, \qquad E'_y = aE_y + bB_z, \qquad B'_z = nB_z + mE_y. \tag{15}$$

We assume from now on that the fields transform as:

$$E'_x = E_x, \qquad E'_y = aE_y + bB_z, \qquad E'_z = aE_z - bB_y, \tag{16}$$

$$B'_x = (an - bm)B_x, \qquad B'_y = nB_y - mE_z, \qquad B'_z = nB_z + mE_y, \tag{17}$$

in terms of the unknown coefficients a, b, m and n. The inverse transformations are:

$$E_x = E'_x, \qquad E_y = \frac{n}{D}E'_y - \frac{b}{D}B'_z, \qquad E_z = \frac{n}{D}E'_z + \frac{b}{D}B'_y, \tag{18}$$

$$B_x = \frac{1}{D}B'_x, \qquad B_y = \frac{a}{D}B'_y + \frac{m}{D}E'_z, \qquad B_z = \frac{a}{D}B'_z - \frac{m}{D}E'_y, \tag{19}$$

where $D = an - bm$.

4. Transformation equations of the charge and current densities

The source dependent Maxwell's equations in vacuum and in Cartesian coordinates in the frame S, are

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 j_x + \epsilon_0\mu_0\frac{\partial E_x}{\partial t}, \tag{20}$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 j_y + \epsilon_0\mu_0\frac{\partial E_y}{\partial t}, \tag{21}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 j_z + \epsilon_0\mu_0\frac{\partial E_z}{\partial t}, \tag{22}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}, \tag{23}$$

where ρ is the macroscopic charge density and $\vec{j}$ the current density.

Equation (20), since from (18) $E_x = E'_x$, can be written as

$$\frac{\partial B_z}{\partial y'} - \frac{\partial B_y}{\partial z'} = \mu_0 j_x + \frac{1}{c^2}\left(b\frac{\partial E'_x}{\partial x'} + n\frac{\partial E'_x}{\partial t'}\right). \tag{24}$$

From (23) we write on the lhs

$$\frac{\partial E_x}{\partial x} \equiv \frac{\partial E'_x}{\partial x} = a\frac{\partial E'_x}{\partial x'} + m\frac{\partial E'_x}{\partial t'} = \frac{\rho}{\epsilon_0} - \left(\frac{\partial E_y}{\partial y'} + \frac{\partial E_z}{\partial z'}\right), \tag{25}$$

so that if we consider the expressions of E_y and E_z from (18), eliminate $\partial E'_x/\partial x'$ from here and substitute the result in (24), while for B_y and B_z we use their expressions (19) in terms of the primed fields, we finally get

$$\frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} = \mu_0 H\left(j_x + \frac{b}{a}\rho\right) + K\frac{\partial E'_y}{\partial y'} + K\frac{\partial E'_z}{\partial z'} + L\frac{\partial E'_x}{\partial t'}, \tag{26}$$

where

$$H = \frac{Dac^2}{a^2c^2 - b^2}, \qquad K = \frac{mac^2 - nb}{a^2c^2 - b^2}, \qquad L = \frac{D^2}{a^2c^2 - b^2}. \tag{27}$$

If we proceed similarly with the equations (21) and (22), we obtain

$$\frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'}\left(a^2 - \frac{b^2}{c^2}\right) = \mu_0 D j_y + M\frac{\partial E'_y}{\partial t'} + N\frac{\partial E'_y}{\partial x'} - N\frac{\partial B'_z}{\partial t'}, \tag{28}$$

$$\left(a^2 - \frac{b^2}{c^2}\right)\frac{\partial B'_y}{\partial x'} - \frac{\partial B'_x}{\partial y'} = \mu_0 D j_z + M\frac{\partial E'_z}{\partial t'} + N\frac{\partial E'_z}{\partial x'} - N\frac{\partial B'_y}{\partial t'}, \tag{29}$$

where

$$M = \frac{n^2}{c^2} - m^2, \qquad N = \frac{nb}{c^2} - am. \tag{30}$$

Finally, equation (23) can be rewritten in terms of the primed variables and fields as:

$$\frac{\partial E'_x}{\partial x'} + \frac{n}{aD}\frac{\partial E'_y}{\partial y'} + \frac{n}{aD}\frac{\partial E'_z}{\partial z'} = \frac{\rho}{a\epsilon_0} + \frac{b}{aD}\left(\frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'}\right), \tag{31}$$

and by substitution of the term in the brackets by the expression (26) we get

$$
\begin{aligned}
&\frac{\partial E'_x}{\partial x'} + \frac{n}{aD}\left(\frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'}\right) \\
&\qquad = \frac{\mu_0 bH}{aD}\left(j_x + \rho\frac{ac^2}{b}\right) + \frac{bK}{aD}\left(\frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'}\right) \\
&\quad + \left(\frac{bL}{aD} - \frac{m}{a}\right)\frac{\partial E'_x}{\partial t'}.
\end{aligned}
\tag{32}
$$

5. Calculation of the coefficients

If equations (26), (28), (29) and (32) have the same form in S' as the corresponding ones of (20)–(23) one should have

$$K = 0, \qquad mac^2 - nb = 0 \quad \Rightarrow \quad N = 0, \tag{33}$$

$$L = \frac{1}{c^2}, \qquad D^2c^2 = a^2c^2 - b^2, \tag{34}$$

$$M = \frac{1}{c^2}, \qquad n^2 - m^2c^2 = 1, \tag{35}$$

$$1 = a^2 - \frac{b^2}{c^2}, \tag{36}$$

$$1 = \frac{n}{aD}. \tag{37}$$

From (34) and (36) we get $D = an - bm = 1$ and from (37) $n = a$. From (33) we obtain that $b = mc^2$, and from (35) that $a^2 - m^2c^2 = 1$. In terms of the coefficient a (which is a possible function of v) one writes

$$m = \pm\frac{1}{c}\sqrt{a^2 - 1}, \tag{38}$$

$$b = \pm c\sqrt{a^2 - 1}, \tag{39}$$

$$H = a. \tag{40}$$

To obtain the transformation equations of j'_x and ρ' we use (26) and compare it with (20) to get

$$j'_x = a\,j_x + b\rho. \tag{41}$$

A similar calculation leads to $j'_y = j_y$ and $j'_z = j_z$.

If we compare (32) with (23) we also get

$$\frac{\rho'}{\epsilon_0} = \mu_0 b\left(j_x + \frac{\rho ac^2}{b}\right) = \frac{b}{\epsilon_0 c^2}\left(j_x + \frac{\rho ac^2}{b}\right), \tag{42}$$

i.e.

$$\rho' = \frac{b}{c^2} j_x + a\rho. \tag{43}$$

The transformation of (41) and (43) is

$$\rho = a\rho' - \frac{b}{c^2} j'_x, \quad j_x = a j'_x - b\rho'. \tag{44}$$

Since

$$a = \frac{\partial f}{\partial x} \quad \Longleftrightarrow \quad f(x, t; v) = ax + f_1(t; v) \tag{45}$$

and

$$b = -av = \frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t} \quad \Longleftrightarrow \quad f_1(t; v) = -avt + f_2(v), \tag{46}$$

one gets $f(x, t;\ v) = a(x - vt) + f_2(v)$ and function $f_2(v)$ vanishes if the event $x = t = 0$ is transformed into $x' = t' = 0$. Thus

$$x' = a(x - vt). \tag{47}$$

Similarly,

$$a = n = \frac{\partial g}{\partial t} \quad \Longleftrightarrow \quad g = at + g_1(x; v), \tag{48}$$

and

$$m = \frac{b}{c^2} = \frac{-av}{c^2} = \frac{\partial g}{\partial x} = \frac{\partial g_1}{\partial x} \quad \Longleftrightarrow \quad g_1(x; v) = -\frac{av}{c^2} + g_2(v), \tag{49}$$

and like the previous case $g_2(v)$ vanishes with the choice that $x' = t' = 0$ when $x = t = 0$ and the solution is

$$t' = a\left(t - \frac{vx}{c^2}\right). \tag{50}$$

To determine the value of the coefficient a, let us consider a unique charge carrier of density ρ' at rest in the frame S', where $j'_x = \rho' u'_x = 0$, since the velocity of the charge carrier is $u'_x = 0$. In frame S, the charge density is ρ and the current density $j_x = \rho v$. From (41) we get

$$0 = a\rho v + b\rho, \quad av = -b = c\sqrt{a^2 - 1}, \quad a = \frac{\pm 1}{\sqrt{1 - v^2/c^2}}. \tag{51}$$

We have to take in (51) the positive value for a because if $x = 0$, $t' = at$ and the two times have to rise evenly.

6. Summary

To summarize, the linear transformation equations of the space-time coordinates between two equivalent reference frames S and S', with S' moving with respect to S along OX axis with velocity v, and such that Maxwell's equations in vacuum take the same form in both frames are

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma(t - vx/c^2), \tag{52}$$

where $\gamma \equiv a = (1 - v^2/c^2)^{-1/2}$.

The transformation equations for the fields are

$$E'_x = E_x, \qquad E'_y = \gamma(E_y - vB_z), \qquad E'_z = \gamma(E_z + vB_y), \tag{53}$$

$$B'_x = B_x, \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right). \tag{54}$$

Finally, the transformation equations for the charge and current densities are

$$\rho' = \gamma\left(\rho - \frac{v}{c^2}j_x\right), \qquad j'_x = \gamma\left(j_x - \frac{v}{c^2}\rho\right), \qquad j'_y = j_y, \qquad j'_z = j_z. \tag{55}$$

ORCID iDs

J M Aguirregabiria https://orcid.org/0000-0002-3009-914X
M Rivas https://orcid.org/0000-0001-9169-9678

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