Embedding of a Demianski cavity with small rotation parameter in a perturbation of a Friedmann universe with cosmological constant

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The problem of embedding a Demianski cavity with small rotation parameter in an appropriate rotational perturbation of a pressureless Friedmann universe with a Λ term is considered. The relation between the coordinate change introduced by Schücking [Z. Phys. **137**, 595 (1954)] for this kind of problems and that used for the simple model of Oppenheimer and Snyder [Phys. Rev. **56**, 455 (1939)] for gravitational collapse is also discussed.

I. INTRODUCTION

The problem of matching the two most important exact solutions of the Einstein equations, those of Friedmann–Robertson–Walker and Schwarzschild, was considered by Einstein and Straus,[1] who analyzed the influence of the universe expansion on the gravitational field surrounding an individual star. In the Einstein and Straus model, a spherical vacuum region containing at its center a Schwarzschild mass is cut out inside a pressureless cosmological fluid.The matching of metrics found by Einstein and Straus depends on the unknown solutions of some differential equations. A more explicit solution for the same problem was presented by Schücking[2] and his work has been recently extended to the case of the non-null cosmological constant by Balbinot *et al.*[3]

Related problems of embedding the Schwarzschild solution in cosmological backgrounds have been considered by McVittie,[4] Dirac,[5] and Gautreau.[6] Other spherical inhomogeneities in cosmology has been considered in the so-called "Swiss cheese" models.[7] In inflationary cosmology the dynamics of false-vacuum spherical bubbles with a domain wall have also been analyzed.[8]

On the other hand, the original approach of Einstein and Straus has been extended to the case of a small rotation by Chamorro,[9] keeping in mind that almost all large aggregations of matter in the universe have some form of rotation.In Chamorro's paper, the Kerr solution developed to first order in the rotation parameter is substituted for the Schwarzschild solution and a rotational perturbation of the Friedmann–Robertson–Walker solution is used as the exterior metric. This perturbation decays to zero as the spatial distance increases.

In this paper we simultaneously extend the works of Balbinot *et al.*[3] and Chamorro[9] by considering the matching of a Demianski solution[10] with a small rotation parameter in a spherical cavity cut out inside an external rotational perturbation of a Friedmann–Robertson–Walker universe with zero pressure and a non-null cosmological constant. Our results are valid to first order of perturbation theory. Instead of the original approach of Einstein and Straus[1] we shall start from the equivalent, but more explicit method of Schücking.[2]

In addition, the local equivalence of two problems which correspond to very different physical and topological conditions seems to have been largely overlooked (one exception would be the book by Stephani[11]). For example, the strictly local problems of matching the Schwarzschild and Robertson–Walker metrics in the Einstein and Straus vacuole and in the model for a gravitational collapse of Oppenheimer and Snyder,[12] where the exterior metric is Schwarzschild and the interior one is Friedmann, are exactly identical. In fact, the latter work has been extended to the case of the collapse of a slowly rotating dust cloud by Kegeles.[13]

We shall explicitly show the equivalence between the matching methods used in cosmological problems[2,3] and in the simplest models for gravitational collapse.[12–14]

II. THE PROBLEM

We shall consider a spherical cavity where the space-time metric is the generalization of the Kerr solution to the case of the non-null cosmological constant given by Demianski.[10] Since we shall always keep only the first term in the expansions in the small rotation parameter ϵ, this metric reads, to this first approximation, as

$$ds^2 = -b\,dt^2 + (1/b)dr^2 + r^2\,d\omega^2 - 2\epsilon \sin^2\theta(1-b)dt\,d\varphi, \quad (1)$$

with

$$b = 1 - 2M/r - (\Lambda/3)r^2, \quad d\omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2. \quad (2)$$

This metric satisfies, to first order in ϵ, the vacuum field equations with a Λ term,

$$R_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0, \quad (3)$$

and reduces to the Schwarzschild–de Sitter metric used in Ref. 3 when there is no rotation ($\epsilon = 0$) and to the expansion of the Kerr metric used in Ref. 9 when $\Lambda = 0$.

In the exterior of the cavity the metric will be a rotational perturbation of the Robertson–Walker metric in the form[13,9]

$$ds_+^2 = -d\tau^2 + R^2(C^{-2}d\rho^2 + \rho^2\,d\omega^2) - 2\epsilon\rho^2R^2\sin^2\theta(W\,d\tau + X\,d\rho)d\varphi, \quad (4)$$

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where $C=\sqrt{1-k\rho^2}$ ($k=-1,0,1$), the scale factor R depends on τ, and the functions W and X depend on τ and ρ.

We suppose a cosmological fluid of pressureless dust moving with the four-velocity

$$u_\alpha=(-1,0,0,\epsilon L(\tau,\rho,\theta)),$$
$$u^\alpha=(1,0,0,\epsilon(W+L/\rho^2R^2\sin^2\theta)). \quad (5)$$

The stress-energy tensor is $T_{\alpha\beta}=\alpha u_\alpha u_\beta$ and its conservation gives the "total mass" $A\equiv\frac{1}{3}\alpha R^3$, which remains constant. Also, $\dot{L}\equiv\partial L/\partial\tau=0$: This condition also guarantees that the motion of the fluid is geodesic to first order in ϵ.

It can be seen that under these assumptions the field equations

$$R_{\alpha\beta}-\tfrac{1}{2}Rg_{\alpha\beta}-\Lambda g_{\alpha\beta}=-8\pi GT_{\alpha\beta} \quad (6)$$

give rise to the evolution equation for the scale factor,

$$\dot{R}=h(R)\equiv\sqrt{8\pi GAR^{-1}-k+\lambda R^2} \quad (\lambda=\Lambda/3), \quad (7)$$

and the following conditions on the functions L, W, and X:

$$L=Cl(\rho)\sin^2\theta/2\rho^2,\quad \dot{X}-W'=f(\rho)/\rho^4CR^3, \quad (8)$$

where $W'\equiv\partial W/\partial\rho$ and $l(\rho)$ and $f(\rho)$ are arbitrary except for the fact that they must satisfy

$$f'=24\pi GAl. \quad (9)$$

Finally, we shall also require the perturbation to vanish at infinite spatial distance, that is,

$$\lim_{\rho\to a}\frac{Cl(\rho)}{\rho^2}=\lim_{\rho\to a}\frac{f(\rho)}{\rho^4}$$
$$=\lim_{\rho\to a}W(\tau,\rho)$$
$$=\lim_{\rho\to a}X(\tau,\rho)=0, \quad (10)$$

where a stands for 1 when $k=1$ and for ∞ if $k=0,-1$.

The problem we face can be stated as follows: Given the values of the constants M, Λ, k, and A and the scale factor $R(\tau)$ satisfying Eq. (7), we seek a spherical surface Σ [with the equations $r=r_0(t)$ in internal coordinates and $\rho=\rho_0(\tau)$ in external coordinates] and the functions $L(\rho,\theta)$, $W(\tau,\rho)$, and $X(\tau,\rho)$ satisfying Eqs. (8)–(10) in such a way that the first and second fundamental forms are continuous across the surface.

We shall work to order ϵ throughout the paper, as indicated above, and will comment at the end on the approximate nature of our solution.

III. THE CONTINUITY OF THE METRIC

To analyze the continuity of the metric across the spherical surface, we shall closely follow the method of "curvature coordinates" of Refs. 2 and 3. Thus we shall change the coordinates for the exterior metric from (τ,ρ) to (t,r) by means of the implicit equations

$$R(\tau)=\phi(t,r),\quad \rho=r/\phi(t,r), \quad (11)$$

with $\phi(t,r)$ defined (implicitly, again) by

$$F_1(\phi(t,r))+F_2(r/\phi(t,r))=G(t), \quad (12)$$

with a function $G(t)$ to be determined later and with

$$F_1(x)=2\int\frac{dx}{8\pi GA-kx+\lambda x^3},$$
$$F_2(x)=-\frac{1}{k}\ln|1-kx^2|. \quad (13)$$

A more explicit expression for F_1 is discussed in Ref. 3.

Equations (12) and (13) are required in order to guarantee that in the new coordinates the coefficient of $dt\,dr$ vanishes. Indeed, one can easily see that in the coordinates (t,r) the exterior metric reads as

$$ds_+^2=-\frac{1}{4}G^2\frac{H^2D^2}{B}dt^2$$
$$+\frac{1}{B}dr^2+r^2d\omega^2-2\epsilon r^2\sin^2\theta$$
$$\times\left\{\left(W-\frac{rH}{\phi^2}X\right)\frac{\dot{\phi}}{H}dt\right.$$
$$\left.+\left[\left(W-\frac{rH}{\phi^2}X\right)\frac{\phi'}{H}+\frac{X}{\phi}\right]dr\right\}d\varphi, \quad (14)$$

where

$$H(t,r)=\sqrt{8\pi GA\phi^{-1}-k+\lambda\phi^2},\quad D(t,r)=\sqrt{\phi^2-kr^2},$$
$$B(t,r)=1-8\pi GAr^2\phi^{-3}-\lambda r^2=\phi^{-2}(D^2-r^2H^2). \quad (15)$$

Next, we require continuity of the line element on the spherical surface Σ at $r=r_0(t)=R(\tau)\rho_0(\tau)=\phi_0(t,r_0(t))\rho_0(\tau)$. By comparing the coefficients of dr^2 in Eqs. (1) and (14), we see that[2,3]

$$\rho_0=(2M/8\pi GA)^{1/3}=\text{const.} \quad (16)$$

Using Eqs. (12) and (13) to analyze the continuity of the coefficients of dt^2, we find

$$\dot{r}_0=r_0^{-3/2}[r_0-(2M+\lambda r_0^3)](2M+\lambda r_0^3-k\rho_0^2r_0)^{1/2}(1-k\rho_0^2)^{-1/2}. \quad (17)$$

Equations (16) and (17) give the radius of the matching surface in external and internal coordinates, respectively. By using Eq. (16) and a solution $r_0(t)$ to Eq. (17), the function $G(t)$ can be computed by means of the restriction of Eq. (12) to the surface of matching, which gives the relation

$$G(t)=F_1[r_0(t)/\rho_0]+F_2(\rho_0) \quad (18)$$

and then $\phi(t,r)$ can in principle be found from Eq. (12). This completely determines the change of coordinates in Eq. (11) and guarantees the continuity of the metric to zeroth order in ϵ.

In addition, continuity of the coefficients of $dt\,d\varphi$ and $dr\,d\varphi$, i.e., continuity of the metric to first order in ϵ, gives

$$W_0=C_0(1-B_0)/\rho_0^2R^2B_0,$$
$$X_0=\dot{R}(1-B_0)/\rho_0RC_0B_0, \quad (19)$$

where the subindex zero means that the expression is valid only on the matching surface. Thus, for example, we have

$$C_0=\sqrt{1-k\rho_0^2},$$
$$B_0=1-8\pi GAR^{-1}\rho_0^2-\lambda R^2\rho_0^2$$
$$=b_0=1-2M/r_0-\lambda r_0^2. \quad (20)$$

In the particular case in which $\Lambda = 0$, the results of Chamorro[9] are recovered.

The matching surface can be seen as made of points that slowly rotate along the geodesics, with equations given in external coordinates by (5) with $\rho = \rho_0$ and in internal coordinates by (17) and

$$\frac{d\varphi}{dt} = \epsilon \frac{1 + Nb_0}{r_0^2}, \quad N = \frac{l(\rho_0)}{2\rho_0^2} - 1 = \text{const.} \tag{21}$$

IV. THE CONTINUITY OF THE EXTRINSIC CURVATURE

Since we assume that there is no singular domain wall at the points on the matching surface—which are in free fall in both metrics—we must require not only the continuity of the metric, but also that of the extrinsic curvature.[15]

By using the results of Sec. III it is easy to see that the outward normal unit vector in internal and external coordinates is

$$n_\alpha^{(-)} = (-\rho_0 \dot{R}, C_0 B_0^{-1}, 0, 0),$$
$$n_\alpha^{(+)} = (0, RC_0^{-1}, 0, 0). \tag{22}$$

We choose the intrinsic coordinates for Σ as $(\xi_1, \xi_2, \xi_3) = (\tau, \theta, \varphi)$; the associated basis of tangent vectors in internal coordinates is

$$e_{(\tau)}^\alpha = (C_0 B_0^{-1}, \rho_0 \dot{R}, 0, 0), \quad e_{(\theta)}^\alpha = (0,0,1,0),$$
$$e_{(\varphi)}^\alpha = (0,0,0,1) \tag{23}$$

and in external coordinates the basis is

$$e_{(\tau)}^\alpha = (1,0,0,0), \quad e_{(\theta)}^\alpha = (0,0,1,0),$$
$$e_{(\varphi)}^\alpha = (0,0,0,1). \tag{24}$$

It is possible to see that the components of the extrinsic curvature

$$K_{ij} = -n_\alpha \left(\frac{\partial e_{(i)}^\alpha}{\partial \xi^j} + \Gamma_{\beta\gamma}^\alpha e_{(i)}^\beta e_{(j)}^\gamma \right), \tag{25}$$

as computed in both types of coordinates, are exactly the same at zeroth order in ϵ,[3] but now we have the following first-order terms:

$$K_{\tau\varphi}^{(-)} = \frac{1}{2} \epsilon \sin^2 \theta \times \frac{(3B_0 - 2C_0^2)(1 - B_0) - \Lambda \rho_0^2 R^2 B_0}{\rho_0 R B_0},$$
$$K_{\tau\varphi}^{(+)} = \tfrac{1}{2} \epsilon \sin^2 \theta \rho_0 R C_0 [\rho_0 (\dot{X}_0 - W_0') - 2W_0]. \tag{26}$$

In consequence, using (19) we see that one must also require that W and X satisfy, at Σ,

$$\dot{X}_0 - W_0' = 6M / \rho_0^4 R^3 C_0, \tag{27}$$

which reduces to the condition found by Chamorro[9] when $\Lambda = 0$.

By a straightforward extension of the analysis in Ref. 9 it is possible to show that there exist the functions $L(\rho,\theta)$, $W(\tau,\rho)$, and $X(\tau,\rho)$ satisfying Eqs. (8)–(10), (19), and (27). This solves the proposed embedding problem.

V. FINAL COMMENTS

To match both metrics we have passed from the coordinates (τ,ρ) to (t,r) by means of (11). It is equally possible, of course, to express the coordinates (t,r) in terms of (τ,ρ). In fact, the necessary inverse change of coordinates is a slight extension of the change used by several authors[14,13] to deal with the model of gravitational collapse of Oppenheimer and Snyder[2] and its rotational perturbation.[13] Although the topology and physical meaning of both problems are completely different, the local problem of matching both metrics at a spherical surface is mathematically the same as the one discussed above. The only different minor details are the relative positions of the vacuum and dust solutions and the fact that in the problem of collapse one selects length units to have $k = 8\pi GA > 0$. Of course, there is no Λ term in the latter problem and instead of Eq. (10) other conditions[13] must be imposed.

In order to establish the relation between these two changes of coordinates used in different kinds of problems, we shall sketch the procedure to match both metrics in the coordinates (τ,ρ). The change from the coordinates (t,r) to (τ,ρ) is given by

$$t = -C_0 \int \frac{dR}{B_0(R) h(R)} \bigg|_{R = S(\tau,\rho)}, \quad r = \rho R(\tau), \tag{28}$$

where the function $S(\tau,\rho)$ must be determined in the matching process. The latter can be accomplished in a way similar to that used in Secs. II–IV to obtain, obviously, the same final results. The relation between the changes (11) and (28) is given in terms of the functions defined in (13) by

$$S(\tau,\rho) = U(F_1(R(\tau)) + F_2(\rho)), \tag{29}$$

where the function U is implicitly defined by means of function $G(t)$:

$$G\left(-C_0 \int \frac{dR}{B_0(R) h(R)} \bigg|_{R = U(x)} \right) = x. \tag{30}$$

Finally, we want to comment on the *approximate* nature of our solution. The problem has been solved to first order in the rotation parameter ϵ. If higher orders were considered, new features would appear in the situation. In fact, it is to be expected that the solution to second order in ϵ should require in general a nonspherical shape for the boundary of the Demianski cavity. This is so because the centrifugal force only becomes effective to second order in the angular velocity of the dust (second order in ϵ), therefore distorting to this order the originally spherical shape of the boundary. The rate of expansion of the boundary should also be in general latitude dependent in the second order, with that dependence determined by the initial conditions of the dust. This bears some resemblance to the results obtained by Brill and Cohen[16] and Pfister and Braun[17] in their studies of the Machian induction of the inertial forces by a rotating shell. Pfister and Braun were able to extend Brill and Cohen's induction of the Coriolis force (first order in the angular velocity of the shell) to the centrifugal force (second order in the angular velocity of the shell) by allowing for a prolate shell and a latitude-dependent mass density instead of the spherical and homogeneous shell considered by the latter authors.

It is perhaps worth stressing that our results do not guarantee the existence of an exact exterior cosmological solution matched to the Demianski cavity. However, the possibility of obtaining first-order results is a necessary condition for such a solution to exist at all.

We do not see *a priori* reasons suggesting that an infinitesimally thin wall at the boundary of the cavity must become necessary at higher orders. Postulating singular domain walls in embedding problems relaxes the requirement of the matching of the extrinsic curvatures and therefore makes the embedding much easier. However, under the usual conditions of our present universe, most embeddings with an infinitesimal wall should be regarded as limiting cases of the more realistic smooth embeddings, where continuity of the extrinsic curvature holds in addition to that of the metric. An example in this direction is that of expanding voids in the universe: Their relativistic treatment has usually been undertaken within the context of the thin wall approximation.[18] However, the existence of expanding voids without thin walls smoothly embedded in asymptotically Friedmann–Tolman universes can be shown.[19]

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