Kinematical Theory of elementary particles

$$r = S \times u / mc^2$$

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- If I can't picture it, I can't understand it.
  You know, it would be sufficient to really understand the electron.

A. Einstein[1]

- If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be? What is the structure which can appear under probing with electromagnetic fields as a point charge, yet as far as spin and wave properties are concerned exhibits a size of the order of the Compton wavelength?

A.O. Barut [2]

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[1]H. Dehmelt, Proc. Natl. Acad. Sci. USA, **86**, 8618–19 (1989).

[2]A.O. Barut, *Brief History and recent developments in electron theory and Quantumelectrodynamics*, in The electron, New Theory and Experiment, D. Hestenes and A. Weingartshofer (ed.), Kluwer Academic Publishers, Dordrecht (1991).

Preface

The present notes contain some basic materials, physical and mathematical, of the general formalism for analyzing elementary particles, which under the general name of **Kinematical Theory of Elementary Particles**, I have been working during the last years. The name *kinematical* makes reference to its close relationship with the kinematical group of space-time transformations associated to the Restricted Relativity Principle which a theoretical framework must necessarily satisfy.

In a certain sense it is a revision of the basic fundamentals of the Lagrangian formalism which leads to Euler-Lagrange equations, Noether's theorem, etc., but looking for solutions which go through the postulated initial and final states of the variational formalism. This produces a classical formalism which is going to be expressed in terms of the end point variables of the dynamical evolution. This formalism is, therefore, closer to the quantum mechanical dynamical theory and it is through Feynman's path integral approach that we can find the bridge between them.

These end point variables, which I propose to call them **kinematical variables**, in the case of elementary particles will necessarilly span a homogeneous space of the kinematical group. In this way, the kinematical group not only reflects the symmetries of the system. It also supplies the necessary variables to describe elementary matter. It is crucial for the description of matter to improve in our knowledge of this kinematical group. In the present notes we shall deal mainly with the Galilei and Poincaré groups, but the formalism is so general that it can accomodate any further group we consider as the basic symmetry group of matter. But at the same time is very restrictive, because once this kinematical group is fixed the kind of classical variables which define the initial and final states of an elementary particle in a variational approach, are restricted to belong to homogeneous spaces of the group. This kinematical group is the fundamental object of the formalism and must be defined as a preliminary statement.

For the Galilei and Poincaré groups, a general spinning elementary particle is just a localized and orientable mechanical system. By localized we mean that to analyse its evolution in space we have just to describe the evolution of a single point r, where the charge is located and in terms of which the possible interactions are determined. This point r also represents the centre of mass of the system for spinless particles, while for spinning ones must necessarily be a different point than q, the centre of mass, very well defined classically and where we can locate the mass of the particle. It is the motion of the charge around the centre of mass which gives rise to a classical interpretation of the *zitterbewegung*, or trembling motion in Schroedinger's words, and also to the dipole structure of the particle. By orientable we mean that in addition to the description of the evolution of the center of charge we also need to describe the change of orientation of the system by analyzing the evolution of a local comoving and rotating frame attached to that point.

The notes pretend to be selfcontained and in this way we have included at the end of the

chapters some mathematical appendices which contain not very well spread materials. The lecture notes are organised as follows. We begin with a Preamble which suggest that the center of charge of an elementary particle moves in a helical motion at the speed of light, so that this point will satisfy, in general, fourth order differential equations. This implies that in a Lagrangian approach we shall have a Lagrangian depending on higher order derivatives.

We begin with the general Lagrangian formalism just to enhance the role of the kinematical variables in defining a concept of elementary particle. Lecture two will be devoted to the analysis of several relativistic and nonrelativistic models, to show how the standard methods of analyzing symmetries leads to the definition of the relevant observables. In particular, we shall pay attention to the definition of the spin. The spin, as any other observable, will be defined in the classical case in terms of the degrees of freedom and their derivatives, and we shall analyze its mathematical structure.

The next two lectures will cover the quantization of the formalism and the analysis of some relativistic and nonrelativistic examples. The separate fourth chapter is devoted to the model which satisfies Dirac's equation. Special attention is paid to the analysis of Dirac's algebra and its relationship with the classical observables and to show a geometrical interpretation of the difference in chirality between matter and antimatter. This chapter ends with the analysis and enlargement of the spacetime symmetry group of the Dirac particle and a plausible interaction Lagrangian which describes a short and long range interaction between two Dirac particles, which has a Coulomb-like behaviour when the spin of the particles is supressed. Equal charged spinning particles can form metastable bound states provided some boundary conditions are fulfilled.

The electromagnetic structure of the model which satisfies Dirac's equation when quantized, is analyzed in a separate fifth chapter.

Finally, some physical features which are related to the spin of the elementary particles, are described. In some places, the lectures will be complemented with numerical simulations whenever the theoretical solution is not available or very difficult to interpret because of the mathematical complexity.

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Martín Rivas

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Preamble: Helical motion of the charge

In this preliminary chapter we shall give three different kinds of arguments suggesting that the center of charge of an elementary particle moves in a helical motion, and it thus satisfies, in general, a fourth order differential equation. This means that a Lagrangian formalism for describing elementary particles has to depend, at least, up to the acceleration of the position of the charge, to properly obtain fourth order dynamical equations. By this reason, we shall start our formalism by describing in chapter 1, the way the generalized Lagrangian formalism produces the general results of Euler-lagrange equations, conserved quantities through Noether theorem, and the generalized canonical formalism.

Rigid body arguments

Let us consider that an elementary particle were described as a rigid body. A rigid body is a mechanical system of six degrees of freedom. Three represent the position of a point and the other three the orientation of a body frame attached to that point. Usually, it is described by the location of the center of mass, which is represented by the point $\boldsymbol{q}$, and the orientation by the principal axis of inertia located around $\boldsymbol{q}$. The center of mass satisfies second order dynamical equations and moves like a point of mass m, the total mass of the system, under the total external force. In this way a rigid body moves and rotates.
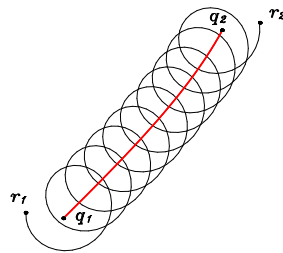
If instead of considering the description of the center of mass we take a different point $\boldsymbol{r}$, it will follow a helical trajectory around the center of mass, like the one depicted in the figure.

If an elementary particle is a charged rigid body, it is clear that we also need to know its electromagnetic structure, which can be reduced to the knowledge of the center of charge and the different multipoles. If assumed a spherical symmetry for the electric field produced by the particle we are left with the location of the center of charge to compute the actions of the external fields. In general, depending how the mass and charge are distributed, these two

points will be different points as we shall assume here. Therefore, if we try to describe the evolution of the center of mass we have to determine also at any time the location of the center of charge to compute the external forces. Newton's dynamical equations for the center of mass will be written as

$$m\frac{d^2\boldsymbol{q}}{dt^2} = e\left(\boldsymbol{E}(t,\boldsymbol{r}) + \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{B}(t,\boldsymbol{r})\right) = \boldsymbol{F}(t,\boldsymbol{r},d\boldsymbol{r}/dt). \tag{1}$$

The electromagnetic force $\boldsymbol{F}$ depends, in general, on the electric and magnetic external fields defined at the charge position $\boldsymbol{r}$ and on the velocity of the charge $d\boldsymbol{r}/dt$ which appears in the magnetic term. For the relative motion of the center of charge around the center of mass we have

$$\boldsymbol{r}(t) - \boldsymbol{q}(t) = R(t)(\boldsymbol{r}(0) - \boldsymbol{q}(0)) \tag{2}$$

We also need to compute the rotation $R(t)$ and thus to solve Euler's equations of the rotational motion in terms of the external torques. If this relative motion between $\boldsymbol{r}$ and $\boldsymbol{q}$ is a kind of circular motion, in particular in the free case, we can define a unit vector $\boldsymbol{n}$ in the direction of the normal acceleration $d^2\boldsymbol{r}/dt^2$ of point $\boldsymbol{r}$, and thus

$$\boldsymbol{n} = \frac{1}{\omega^2 R}\frac{d^2\boldsymbol{r}}{dt^2},$$

where R is the radius of the circular motion. Then the center of mass position can be written as

$$\boldsymbol{q}(t) = \boldsymbol{r}(t) + \frac{1}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \tag{3}$$

Then, it will be simpler, from a theoretical point of view, just to describe the evolution of a single point, the center of charge $\boldsymbol{r}$, instead of the center of mass $\boldsymbol{q}$, which will be in some average position of the other, and obtained from (3) once the trajectory of $\boldsymbol{r}$ is computed. The elimination of the $d^2\boldsymbol{q}/dt^2$ among equations (1) and (3) will give us, in general, a fourth order differential equation for the variable $\boldsymbol{r}$. Because the angular velocity is also orthogonal to the plane subtended by the velocity and acceleration of point $\boldsymbol{r}$,

$$\boldsymbol{\omega} = \frac{1}{u^2}\frac{d\boldsymbol{r}}{dt} \times \frac{d^2\boldsymbol{r}}{dt^2}$$

we have also solved the problem of the rotation of the charged rigid body by analyzing the evolution of just the center of charge. We see again that a Lagrangian depending on the acceleration of point $\boldsymbol{r}$, could reproduce such equations.

Invariance arguments

Let us consider the trajectory $\boldsymbol{r}(t)$, $t \in [t_1, t_2]$ followed by a point of a mechanical system for an arbitrary inertial observer O. Any other inertial observer O' is related to the previous one by a transformation of the kinematical group such that their relative space-time measurements of any space-time event are given by

$$t' = T(t,\boldsymbol{r};g_1,\ldots,g_\alpha), \quad \boldsymbol{r}' = \boldsymbol{R}(t,\boldsymbol{r};g_1,\ldots,g_\alpha),$$

where the functions T and $\boldsymbol{R}$ define the corresponding transformation of the kinematical group G, of parameters $(g_1, \ldots, g_\alpha)$, among any two observers. Then the description of the trajectory of that point for observer O' is obtained from

$$t'(t) = T(t, \boldsymbol{r}(t); g_1, \ldots, g_\alpha), \quad \boldsymbol{r}'(t) = \boldsymbol{R}(t, \boldsymbol{r}(t); g_1, \ldots, g_\alpha), \quad \forall t \in [t_1, t_2].$$

If we eliminate t as a function of t' from the first equation and substitute into the second we shall get

$$\boldsymbol{r}'(t') = \boldsymbol{r}'(t'; g_1, \ldots, g_\alpha). \tag{4}$$

Since observer O' is arbitrary, equation (4) represents the complete set of trajectories of the point for all inertial observers. Elimination of the α group parameters among the function $\boldsymbol{r}'(t')$ and their time derivatives will give us the differential equation satisfied by all the trajectories of the point. Let us assume that the trajectory is unrestricted in such a way that the above group parameters are essential in the sense that no smaller number of them gives the same family of trajectories. This differential equation is invariant by construction because it is independent of the group parameters and therefore independent of any inertial observer. If G is either the Galilei or Poincaré group, it is a ten-parameter group so that we have to work out in general up to the fourth derivative to obtain sufficient equations to eliminate the group parameters. Therefore the order of the differential equation is dictated by the number of parameters and the structure of the kinematical group. If the point $\boldsymbol{r}$ represents the position of the center of charge of an elementary particle we get again that it satisfies, in general, a fourth order differential equation. But at the same time it is telling us that to obtain the invariant differential equation satisfied by the center of charge of an elementary particle, it is sufficient to obtain its trajectory in an arbitrary reference frame, and to follow the above procedure of elimination of the group parameters.

Geometrical arguments

As is well known in differential geometry, a continuous and differentiable curve in three-dimensional space, $\boldsymbol{r}(s)$, has associated three orthogonal unit vectors, $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$, called respectively the tangent, normal and binormal. If using the arc length s as the curve parameter, they satisfy the Frenet-Serret equations

$$\dot{\boldsymbol{t}} = \kappa \boldsymbol{n}, \quad \dot{\boldsymbol{n}} = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \quad \dot{\boldsymbol{b}} = -\tau \boldsymbol{n},$$

where the overdot means $\dot{} \equiv d/ds$. The knowledge of the curvature $\kappa(s)$ and torsion $\tau(s)$, together the boundary values $\boldsymbol{r}(0)$, $\boldsymbol{t}(0)$, $\boldsymbol{n}(0)$ and $\boldsymbol{b}(0)$, completely determine the curve, because the above equations are integrable. If we define the vector $\boldsymbol{\omega} = \tau \boldsymbol{t} + \kappa \boldsymbol{b}$, known as Darboux vector, the Frenet-Serret equations can be rewritten as

$$\dot{\boldsymbol{t}} = \boldsymbol{\omega} \times \boldsymbol{t}, \quad \dot{\boldsymbol{n}} = \boldsymbol{\omega} \times \boldsymbol{n}, \quad \dot{\boldsymbol{b}} = \boldsymbol{\omega} \times \boldsymbol{b},$$

so that, in units of arc length, Darboux vector represents the instantaneous angular velocity of the local frame.

If we call $\boldsymbol{r}^{(k)}(s) \equiv d^k \boldsymbol{r}/ds^k$, and, in particular

$$\boldsymbol{r}^{(1)} = \boldsymbol{t}, \quad \boldsymbol{r}^{(2)} = \kappa \boldsymbol{n}, \quad \boldsymbol{r}^{(3)} = \dot{\kappa} \boldsymbol{n} + \kappa(-\kappa \boldsymbol{t} + \tau \boldsymbol{b})$$

and eliminate the three unit vectors $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$, in terms of the derivatives $\boldsymbol{r}^{(k)}$, $k = 1, 2, 3$,

$$\boldsymbol{t} = \boldsymbol{r}^{(1)}, \quad \boldsymbol{n} = \frac{1}{\kappa}\boldsymbol{r}^{(2)}, \quad \boldsymbol{b} = \frac{\kappa^2}{\tau}\boldsymbol{r}^{(1)} - \frac{\dot{\kappa}}{\kappa\tau}\boldsymbol{r}^{(2)} + \frac{1}{\kappa\tau}\boldsymbol{r}^{(3)}$$

and thus

$$\kappa = |\boldsymbol{r}^{(1)} \times \boldsymbol{r}^{(2)}|, \qquad \tau = \frac{1}{\kappa^2}(\boldsymbol{r}^{(1)} \times \boldsymbol{r}^{(2)}) \cdot \boldsymbol{r}^{(3)}$$

If we replace the three Frenet-Serret unit vectors in the next order derivative, one obtains that the most general differential equation satisfied by the point $\boldsymbol{r}$, is the fourth order differential system

$$\boldsymbol{r}^{(4)} - \frac{2\dot{\kappa}\tau + \dot{\tau}\kappa}{\kappa\tau}\boldsymbol{r}^{(3)} + \left(\kappa^2 + \tau^2 + \frac{\dot{\kappa}\dot{\tau} - \tau\ddot{\kappa}}{\kappa\tau} + \frac{2\dot{\kappa}^2}{\kappa^2}\right)\boldsymbol{r}^{(2)} + \frac{\kappa}{\tau}(\dot{\kappa}\tau - \dot{\tau}\kappa)\boldsymbol{r}^{(1)} = 0. \qquad (5)$$

where the coefficients are only functions of the derivatives of $\boldsymbol{r}$ up to fourth order.

Let us consider that an elementary particle, instead of being a rigid body, is just a localized mechanical system. By localized we mean that, at least, it is described by the evolution of a single point $\boldsymbol{r}$. This point could be the center of mass, but, as mentioned before, in order to determine the external forces to obtain the center of mass evolution, we also need to know the location of the center of charge to compute the actions of the external fields. Let us assume that the elementary particle is charged. By the previous arguments, if assumed spherical symmetry of its electric field, we are reduced to know the evolution just of the center of charge. The particle will have a center of mass but we make the assumption that the center of mass and the center of charge are not necessarily the same point.

Then, the center of charge of an elementary particle will satisfy, in general, a fourth order differential equation of the form (5) where $\kappa(s)$ and $\tau(s)$ will depend on the external forces and torques. Let us assume now that the motion of the particle is free. This means that we cannot distinguish one point of the evolution from another, so that the above equations (5) must be explicitly independent of the parameter s. Darboux vector is a constant vector in the comoving frame. The curvature and torsion are necessarily constants of the motion.

Thus $\dot{\kappa} = \dot{\tau} = 0$, and, in the free case, these equations are reduced to

$$\boldsymbol{r}^{(4)} + \left(\kappa^2 + \tau^2\right)\boldsymbol{r}^{(2)} = \frac{d^2}{ds^2}\left(\boldsymbol{r}^{(2)} + \left(\kappa^2 + \tau^2\right)\boldsymbol{r}\right) = 0.$$

If the curvature and torsion are constant the curve is a helix, which can be factorized in terms of a central point

$$\boldsymbol{q} = \boldsymbol{r} + \frac{1}{\kappa^2 + \tau^2}\boldsymbol{r}^{(2)}, \quad \frac{d^2\boldsymbol{q}}{ds^2} = 0,$$

which is moving in a straight trajectory, while the point $\boldsymbol{r}$ satisfies

$$\boldsymbol{r}^{(2)} + \omega^2\left(\boldsymbol{r} - \boldsymbol{q}\right) = 0,$$

an isotropic harmonic motion of frequency $\omega = \sqrt{\kappa^2 + \tau^2}$, around point $\boldsymbol{q}$. The point $\boldsymbol{q}$ clearly represents the centre of mass position of the free particle. Going further, let us assume that the free evolution is analyzed by some inertial observer. Then this observer cannot distinguish one instant from another, so that, the arc length $ds = |\boldsymbol{u}|dt$, where $\boldsymbol{u} = d\boldsymbol{r}/dt$ is the velocity of the charge, must be also independent of the time t. Otherwise, if ds is not the same we can distinguish one instant of the evolution from another, as far as the displacement of the

charge is concerned. The center of charge of a free elementary particle is describing a helix at a constant velocity for any inertial observer.

If we make a nonrelativistic analysis, the relationship of the velocity measurements among two arbitrary inertial observers O and O', is given by $\boldsymbol{u}' = R\boldsymbol{u} + \boldsymbol{v}$, where $\boldsymbol{v}$ is the constant velocity of O as measured by O' and the constant rotation matrix R is their relative orientation. Now,

$$u'^2 = u^2 + v^2 + 2\boldsymbol{v} \cdot R\boldsymbol{u}.$$

If u' has to be also constant for observer O', irrespective of $\boldsymbol{v}$ and of the rotation matrix R, this means that the vector $\boldsymbol{u}$ must be a constant vector. The center of charge necessarily moves along a straight trajectory at a constant velocity, for every inertial observer, and the above general helix degenerates into a straight line and $\boldsymbol{q} = \boldsymbol{r}$. This is the usual description of the spinless or pointlike free elementary charged particle, whose center of charge and center of mass are represented by the same point.

In the relativistic case we get simmilarly

$$\boldsymbol{u}' = \frac{R\boldsymbol{u} + \gamma(v)\boldsymbol{v} + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R\boldsymbol{u})\boldsymbol{v}}{\gamma(1 + \boldsymbol{v} \cdot R\boldsymbol{u}/c^2)}, \quad u'^2 = \frac{u^2 - c^2}{\gamma^2 \left(1 + \boldsymbol{v} \cdot R\boldsymbol{u}/c^2\right)^2} + c^2.$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, and taking the time derivative we also obtain that $\boldsymbol{v} \cdot R\dot{\boldsymbol{u}} = 0$, and thus $\boldsymbol{u}$ has to be a constant vector, for any time t, irrespective of the values of $\boldsymbol{v}$ and R.

However, in the relativistic analysis, there is one alternative not included in the nonrelativistic approach. The possibility that the charge of an elementary particle will be moving at the speed of light and, in that case, $u = u' = c$, for any inertial observer. This means that the center of the helix is always moving at a velocity $|d\boldsymbol{q}/dt| < c$, and, if it represents the center of mass, this particle is a massive particle. In a variational description of this system the Lagrangian should depend up to the acceleration of the point $\boldsymbol{r}$ in order to obtain fourth order differential equations. This dependence on the acceleration will give a contribution to the spin of the particle and there is also another contribution from the rotation of the system, because the body frame rotates with angular velocity $\boldsymbol{\omega}$. The motion of the charge around the center of mass produces the magnetic moment of the particle.

In summary, there are only two possibilities for a free motion of the center of charge of an elementary particle. One, the charge is moving along a straight line at any constant velocity, and the system has no magnetic moment. In the other, the particle has spin and magnetic moment, and the charge moves along a helix at the speed of light. Because all known elementary particles, quarks and leptons, are spin $1/2$ particles, we are left only with the last possibility. This is consistent with Dirac's theory of the electron, because the eigenvalues of the components of Dirac's velocity operator are $\pm c$. This means that Dirac's spinor $\psi(t, \boldsymbol{r})$ is expressed in terms of the position of the charge $\boldsymbol{r}$, because the external fields $A^\mu(t, \boldsymbol{r})$ are defined and computed at this point.

This last possiblity is the description of the center of charge of a relativistic spinning elementary particle obtained in the kinematical formalism to be developed in this course, and which satisfies Dirac's equation when quantized.

In this formalism Dirac particles are localized and also orientable mechanical systems. By orientable we mean that we have to attach to the above point $\boldsymbol{r}$, a local cartesian frame to describe its spatial orientation. This frame could be the Frenet-Serret triad. The rotation of the frame will also contribute to the total spin of the particle. When quantizing the system, the spin $1/2$ is coming from the presence of the orientation variables. Otherwise, if there are

no orientation variables, no spin 1/2 structure is described when quantizing the system. This twofold structure of the classical spin has produced a pure kinematical interpretation of the gyromagnetic ratio [3]. The dependence of the Lagrangian on the acceleration is necessary for the particle to have magnetic moment and for the separation between the center of mass and center of charge.

Formulación de Newton

El primer planteamiento de una teoría de partículas elementales se lo debemos a Newton. Por definición, el sistema más simple es el punto. Para Newton, la materia está formada por agregados de puntos de masa m, sin fijar su valor. Cada partícula elemental puntual se mueve con una ley dinámica tal que

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{F}$$

siendo $\boldsymbol{F}$ la fuerza total externa que actúa sobre ella.

Admitiendo que las fuerzas entre partículas satisfacen la tercera ley de Newton, entonces llegamos a que cualquier agregado material posee un punto característico, el centro de masa $\boldsymbol{q}$, definido mediante

$$\boldsymbol{q} = \frac{\sum m_i \boldsymbol{r}_i}{\sum m_i}, \quad m = \sum m_i$$

y que satisface

$$\sum \boldsymbol{F}_{\text{ext}} = m\frac{d^2\boldsymbol{q}}{dt^2},$$

que se conoce como **teorema del centro de masa:** *El centro de masa de cualquier agregado material se comporta como una partícula puntual de masa la suma de todas las masas del sistema, sometido únicamente a la suma de todas la fuerzas externas.*

Newton postula que toda la materia se atrae con la ley de la gravitación universal, la cual satisface la tercera ley de Newton. Según esto, si intentamos separar un folio en dos partes, la fuerza que deberemos hacer será, supuestas las dos partes separadas 10 cm y de 1 g de masa cada una,

$$F = G\frac{m^2}{d^2} = 6.672 \times 10^{-11} \times 0.001^2/0.1^2 = 6.672 \times 10^{-15} \text{ N}.$$

Es claro que las fuerzas de cohesión de la materia no son de naturaleza gravitatoria, por lo que entre las partículas elementales deben existir otro tipo de fuerzas. La teoría Newtoniana deja total libertad al tipo de fuerzas que pueden existir.

Si la partícula posee una propiedad llamada carga, ésta está localizada en el mismo punto $\boldsymbol{r}$. Tenemos por lo tanto que toda la materia está construida a partir de puntos materiales de masas arbitrarias, y por supuesto, cargas arbitrarias, que además de interaccionar de forma gravitatoria, se atraen, y a veces se repelen, con otro tipo de fuerza más intensa.

El formalismo Newtoniano es poco restrictivo y para las fuerzas $\boldsymbol{F}$ entre partículas no establece ninguna condición, por lo que todo tipo de interacciones son permitidas. Es la teoría gauge la que limitará el tipo de interacciones en la formulación cuántica y el principio atómico en el caso clásico. Tampoco establece ningún criterio mediante el cual las cargas, masas y

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[3]M. Rivas, J.M. Aguirregabiria and A. Hernández, "A pure kinematical explanation of the gyromagnetic ratio $g = 2$ of leptons and charged bosons", Phys. Lett. **A 257**, 21–25 (1999).

momentos angulares de las partículas elementales tengan los valores que tienen. Es la teoría cuántica la que predecirá estos valores. Sin embargo, hasta el momento, la teoría cuántica solo ha predicho los valores del espín de las partículas elementales.

Chapter 1

Lagrangian formalism

1.1 Fundamental principles

Because all known elementary particles, the quarks and leptons, are spinning particles and it seems that there are no spinless elementary particles in nature, we take the challenge of obtaining a classical formalism for describing spin. The interest of a classical description of spinning matter is not important in itself, because matter, at this level, behaves according to the laws of quantum mechanics. But finer a classical description of elementary matter a deeper quantum mechanical formalism, because we will have at hand, when quantizing the system, more classical variables to deal with, and therefore with a more clear physical and/or geometrical interpretation. A second feature is that a classical formalism supplies models. Both goals, in my opinion, have been succesfully achieved.

Feynman, in the first chapter of his *Lectures on Physics* [1], states that *"If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic fact or whatever you wish to call it) that all things are made of atoms-little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another."*

If the atomic hypothesis is such an important principle, physics has to take advantage of this fact, and, properly formulated, should be included as a preliminary fundamental principle of elementary particle physics, as we shall do in what follows. It is a very restrictive principle which will suggest a kind of minimal coupling interaction when analyzing compound systems of elementary particles.

The kinematical formalism for describing elementary spinning particles, previously aimed for the classical spin description of matter, has proven to be a general framework for the description of elementary particles, because it supplies a very precise definition of a classical elementary particle which has, as a quantum counterpart, Wigner's definition. All elementary systems described within this formalism have the feature that, when quantized, their Hilbert space of pure states carries a projective unitary irreducible representation of the kinematical group. It is through Feynman's path integral approach that both formalisms complement each other.

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[1]Feynman RP, Leighton RB and Sands M 1968 *The Feynman Lectures on Physics*, (NY: Addison Wesley) Vol 1, Sec 1-2.

It is based upon the four fundamental principles:

- Restricted relativity principle,

- atomic principle,

- variational principle,

- quantization principle.

The **restricted relativity principle** states that, in absence of gravitation, there exists a set of equivalent observers, historically called *inertial observers*, for whom the laws of physics must be the same. They are defined with respect to each other by a spacetime transformation group, usually called the kinematical group of the formalism. We shall deal mainly with the Galilei and Poincaré groups but in chapter 4 we shall analyze some enlargement of the Poincaré group as the spacetime symmetry group of a Dirac particle. In addition to the Poincaré group transformations it also contains spacetime dilations and local rotations.

The **atomic principle** admits that matter cannot be divided indefinitely. Matter does not satisfy the hypothesis of the continuum. After a finite number of steps in the division of matter we reach an ultimate and indivisible object, an *elementary particle*. The distinction between an elementary particle and any other finite system is that an elementary particle, in addition of being indivisible, has no excited states and, if not destroyed, it can never be modified, so that all possible states are only kinematical modifications of any one of them. This implies that the states of an elementary can be described by a finite set of variables. This will imply a restriction in the kind of classical variables we shall use to describe the initial and final states in the variational dynamical description.

The **variational principle** recognizes that the the action of the evolution of any mechanical system between some initial and final states must be stationary. It will be described in terms of a Lagrangian function which will be defined in terms of the time, the independent degrees of freedom and their subsequent time derivatives. But we are not going to restrict Lagrangians to depend only on the first order time derivatives of the independent degrees of freedom. The above atomic principle only restricts the Lagrangian to depend on a finite number of degrees of freedom and also of a finite maximum order in their derivatives. In this way we shall start in the next section with the generalized Lagrangian formalism to obtain the main results in general form. This completes the classical framework.

For the quantum description we must substitute this last variational principle by the **quantization principle**, in the form proposed by Feynman [2]: all paths of the evolution of any mechanical system between some initial and final states are equally probable. For each path a probability amplitude is defined, which is a complex number of the same magnitude but whose phase is the action of the system between the end points along the corresponding path. The probability amplitude for finding the system in any classical state, *i.e*, the quantum wave function, will be a squared integrable and normalized complex function of the variables which define the states in the variational approach. In this way, classical and quantum mechanics are described in terms of exactly the same set of classical variables.

This formalism will determine that these variables, which define the initial and final states of the evolution in the variational description, are a finite set of variables which necessarily span a homogeneous space of the kinematical group. We shall call them the kinematical

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[2]R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, MacGraw Hill, N.Y., (1965).

variables of the particle. The manifold they span is larger than the configuration space and in addition to the time and the independent degrees of freedom it also includes the derivatives of the independent degrees of freedom up to one order less the highest order they have in the Lagrangian. The Lagrangian for describing these systems will be thus dependent on these kinematical variables and their next order time derivative. If the evolution is described in terms of some group invariant evolution parameter τ, then, when writting the Lagrangian not in terms of the independent degrees of freedom but as a function of the kinematical variables and their $\tau-$derivatives, it becomes a homogeneous function of first degree of the $\tau-$derivatives of all kinematical variables.

Feynman's path integral method seems to be inspired in a Dirac's paper [3]. In this article Dirac states, when comparing the Lagrangian approach with the canonical aproach, that: *the two formulations are, of course, closely related, but there are reasons for believing that the Lagrangian one is more fundamental.* Later, he expresses that *we ought to consider the classical Lagrangian, not as a function of the coordinates and velocities, but rather as a function of the coordinates at time t and at time $t + dt$.* Here, he is clearly suggesting the use of boundary variables, i.e, the kinematical variables for the expression of the Lagrangian.

In the Preface of Feynman and Hibbs book, it is mentioned that Feynman, in a private conversation with a European colleague, became aware of the mentioned Dirac's paper, suggesting that the wave function at time $t + \epsilon$ would be related to the wave function at time t in the form

$$\psi(t + \epsilon) \sim e^{i\epsilon L/\hbar} \psi(t).$$

What Feynman did was to postulate that the above relation is an identity. There is a quotation in the book [4] that the European colleague was Herbert Jehle, while visiting Princeton in 1941.

We shall analyze several examples of spinning particles. But we shall be surprised that, for the description of free elementary particles, in particular a Dirac particle, is not necessary to postulate any Lagrangian. The analysis of Noether's theorem and conservation laws, and the group invariants will be sufficient to describe the dynamics of a free spinning elementary particle.

1.2 Generalized Lagrangian formalism

The Lagrangian formalism of generalized systems depending on higher order derivatives was already worked out by Ostrogradsky. [5] We shall outline it briefly here, mainly to analyze the generalized Lagrangians not only in terms of the independent degrees of freedom but also as functions of what we shall call the **kinematical variables** of the system, *i.e.*, of the end point variables of the variational formulation.

Let us consider a mechanical system of n degrees of freedom, characterized by a Lagrangian that depends on time t and on the n essential coordinates $q_i(t)$, that represent the n independent degrees of freedom, and their derivatives up to a finite order k. Because we can have time derivatives of arbitrary order we use a superindex enclosed in brackets to represent the

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[3]P.A.M. Dirac, *The Lagrangian in quantum mechanics*, Phys. Zeitsch. der Sowjetunion, **3**, 64–72 (1933).

[4]L.M. Brown (editor), Feynman's thesis: A new approach to quantum theory, (World Scientific 2005)

[5] M. Ostrogradsky, *Mémoire sur les équations différentielles relatives au problème des isopérimètres*, Mem. Acad. St. Petersburg, **6**(4), 385-517 (1850).

corresponding k-th derivative, *i.e.*, $q_i^{(k)}(t) = d^k q_i(t)/dt^k$. The **action functional** is defined by:

$$\mathcal{A}[q] = \int_{t_1}^{t_2} L(t, q_i(t), q_i^{(1)}(t), \ldots, q_i^{(k)}(t)) dt, \qquad (1.1)$$

where $i = 1, \ldots, n$. Using a more compact notation we define $q_i^{(0)} \equiv q_i$, and therefore we shall write

$$L(t, q_i(t), q_i^{(1)}(t), \ldots, q_i^{(k)}(t)) \equiv L(t, q_i^{(s)}(t)),$$

for $s = 0, \ldots, k$.

The trajectory followed by the mechanical system is that path which passing through the fixed end-points at initial and final times t_1 and t_2, $q_i^{(s)}(t_1)$ and $q_i^{(s)}(t_2)$, $i = 1, ..., n$, $s = 0, 1, ..., k-1$, makes extremal the action functional (1.1). Note that we need to fix as boundary values of the variational principle some particular values of time t, the n degrees of freedom q_i and their derivatives up to order $k - 1$, *i.e.*, one order less than the highest derivative of each variable q_i in the Lagrangian, at both end points of the problem. In other words we can say that the Lagrangian of any arbitrary generalized system is in general an explicit function of the variables we keep fixed as end points of the variational formulation and also of their next order time derivative.

Once the action functional (1.1) is defined for some particular path $q_i(t)$, to analyze its variation let us produce an infinitesimal modification of the functions $q_i(t)$, $q_i(t) \rightarrow q_i(t) + \delta q_i(t)$ while leaving fixed the end-points of the variational problem, *i.e.*, such that at t_1 and t_2 the modification of the generalized coordinates and their derivatives up to order $k - 1$ vanish, and thus $\delta q_i^{(s)}(t_1) = \delta q_i^{(s)}(t_2) = 0$, for $i = 1, \ldots, n$ and $s = 0, 1, \ldots, k-1$. Then, the variation of the derivatives of the $q_i(t)$ is given by $q_i^{(s)}(t) \rightarrow q_i^{(s)}(t) + \delta q_i^{(s)}(t) = q_i^{(s)}(t) + d^s \delta q_i(t)/dt^s$, since the modification of the s-th derivative function is just the s-th derivative of the modification of the corresponding function. This produces a variation in the action functional $\delta \mathcal{A} = \mathcal{A}[q + \delta q] - \mathcal{A}[q]$, given by:

$$\delta \mathcal{A} = \int_{t_1}^{t_2} L(t, q_i^{(s)}(t) + \delta q_i^{(s)}(t)) dt - \int_{t_1}^{t_2} L(t, q_i^{(s)}(t)) dt$$

$$= \int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial q_i^{(1)}} \delta q_i^{(1)} + \cdots + \frac{\partial L}{\partial q_i^{(k)}} \delta q_i^{(k)} \right], \qquad (1.2)$$

after expanding to lowest order the first integral. The term

$$\frac{\partial L}{\partial q_i^{(1)}} \delta q_i^{(1)} = \frac{\partial L}{\partial q_i^{(1)}} \frac{d}{dt} \delta q_i = \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \delta q_i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) \delta q_i,$$

and by partial integration of this expression between t_1 and t_2, it gives:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i^{(1)}} \delta q_i^{(1)} dt = \frac{\partial L}{\partial q_i^{(1)}} \delta q_i(t_2) - \frac{\partial L}{\partial q_i^{(1)}} \delta q_i(t_1) - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) \delta q_i dt$$

$$= - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(1)}} \right) \delta q_i \, dt,$$

because the variations $\delta q_i(t_1)$ and $\delta q_i(t_2)$, vanish. Similarly for the next term:

$$\frac{\partial L}{\partial q_i^{(2)}} \delta q_i^{(2)} = \frac{\partial L}{\partial q_i^{(2)}} \frac{d}{dt} \delta q_i^{(1)} = \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(2)}} \delta q_i^{(1)} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(2)}} \right) \delta q_i^{(1)},$$

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i^{(2)}} \delta q_i^{(2)} dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}}\right)\delta q_i^{(1)} dt = \int_{t_1}^{t_2} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q_i^{(2)}}\right)\delta q_i \, dt,$$

because δq_i and $\delta q_i^{(1)}$ vanish at t_1 and t_2, and finally for the last term

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q_i^{(k)}} \delta q_i^{(k)} dt = (-1)^k \int_{t_1}^{t_2} \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_i^{(k)}}\right)\delta q_i \, dt,$$

so that each term of (1.2) is written only in terms of the variations of the degrees of freedom δq_i and not of their higher order derivatives. Remark that to reach these final expressions, it has been necessary to assume the vanishing of all $\delta q_i^{(s)}$, for $s = 0,\ldots,k-1$, at times t_1 and t_2. By collecting all terms we get

$$\delta\mathcal{A} = \int_{t_1}^{t_2} dt \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}}\right) + \cdots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_i^{(k)}}\right)\right]\delta q_i.$$

If the action functional is extremal along the path $q_i(t)$, its variation must vanish, $\delta\mathcal{A} = 0$. The variations δq_i are arbitrary and therefore all terms between squared brackets cancel out. We obtain a system of n differential equations,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}}\right) + \cdots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_i^{(k)}}\right) = 0, \quad i = 1,\ldots,n, \tag{1.3}$$

the Euler-Lagrange equations, which can be written in condensed form as:

$$\sum_{s=0}^{k}(-1)^s \frac{d^s}{dt^s}\left(\frac{\partial L}{\partial q_i^{(s)}}\right) = 0, \qquad i = 1,\ldots,n. \tag{1.4}$$

1.3 Kinematical variables

In general, the system (1.4) is a system of n ordinary differential equations of order $2k$, and thus existence and uniqueness theorems guarantee only the existence of a solution of this system for the $2kn$ boundary conditions $q_i^{(s)}(t_1)$, $i = 1,\ldots,n$ and $s = 0,1,\ldots,2k-1$, at the initial instant t_1. However the variational problem has been stated by the requirement that the solution goes through the two fixed endpoints, a condition that does not guarantee neither the existence nor the uniqueness of the solution. Nevertheless, let us assume that with the fixed endpoint conditions of the variational problem, $q_i^{(s)}(t_1)$ and $q_i^{(s)}(t_2)$, $i = 1,\ldots,n$ and $s = 0,1,\ldots,k-1$, at times t_1 and t_2, respectively, there exists a solution of (1.4) perhaps non-unique. This implies that the $2kn$ boundary conditions at time t_1 required by the existence and uniqueness theorems, can be expressed perhaps in a non-uniform way, as functions of the kn conditions at each of the two endpoints. From now on, we shall consider systems in which this condition is satisfied. It turns out that a particular solution passing through these points will be expressed as a function of time with some explicit dependence of the end point values

$$\widetilde{q}_i(t) \equiv q_i(t; \, q_j^{(r)}(t_1), q_l^{(r)}(t_2)), \tag{1.5}$$

$i,j,l = 1,\ldots,n, \; r = 0,1,\ldots k-1$, in terms of these boundary end point conditions.

Definition: The **Action Function** [6] of the system along a classical path is the value of the action functional (1.1) when we introduce in the integrand a particular solution (1.5) passing through those endpoints:

$$\int_{t_1}^{t_2} L\left(t, \widetilde{q}_i(t)\right) dt = A\left(t_1, q_i^{(r)}(t_1); t_2, q_i^{(r)}(t_2)\right). \tag{1.6}$$

Once the time integration is performed, we see that it will be an explicit function of the $kn + 1$ variables at the initial instant, $q_j^{(r)}(t_1)$, $r = 0, \ldots, k-1$ including the time t_1, and of the corresponding $kn + 1$ variables at final time t_2. We write it as

$$A\left(t_1, q_i^{(r)}(t_1); t_2, q_i^{(r)}(t_2)\right) \equiv A(x_1, x_2).$$

We thus arrive at the following

Definition: The **kinematical variables** of the system are the time t and the n degrees of freedom q_i and their time derivatives up to order $k - 1$. The manifold X they span is the **kinematical space** of the system.

The kinematical space for ordinary Lagrangians is just the configuration space spanned by variables q_i enlarged with the time variable t. It is usually called the **enlarged configuration space**. But for generalized Lagrangians it also includes higher order derivatives up to one order less than the highest derivative. Thus, **the action function of a system becomes a function of the values the kinematical variables take at the end points of the trajectory**, x_1 and x_2. From now on we shall consider systems for which the action function is defined and is a continuous and differentiable function of the kinematical variables at the end points of its possible evolution. This function clearly has the property $A(x, x) = 0$.

1.3.1   Replacement of time as evolution parameter

The constancy of speed of light in special relativity brings space and time variables on the same footing. So, the next step is to remove the time observable as the evolution parameter of the variational formalism and express the evolution as a function of some arbitrary parameter to be chosen properly. Then, let us assume that the trajectory of the system can be expressed in parametric form, in terms of some arbitrary evolution parameter τ, $\{t(\tau), q_i(\tau)\}$. The functional (1.1) can be rewritten in terms of the kinematical variables and their derivatives and becomes:

$$\mathcal{A}[t, q] = \int_{\tau_1}^{\tau_2} L\left(t(\tau), q_i(\tau), \frac{\dot{q}_i(\tau)}{\dot{t}(\tau)}, \ldots, \frac{\dot{q}_i^{(k-1)}(\tau)}{\dot{t}(\tau)}\right) \dot{t}(\tau) d\tau$$

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[6]Please remark that we use the same letter $A(\ )$ for the action function, followed by normal brackets containing the variables of which it depends, and for the action functional $A[\ ]$ which is followed by squared brackets to enhance that it is not a function but rather a functional.

$$= \int_{\tau_1}^{\tau_2} \widehat{L}\left(x(\tau), \dot{x}(\tau)\right) d\tau, \tag{1.7}$$

where the dot means derivative with respect to the evolution variable τ that without loss of generality can be taken dimensionless. Therefore $\widehat{L} \equiv L(t(\tau), \dot{q}_i^{(s)}/\dot{t}(\tau))\,\dot{t}(\tau)$ has dimensions of action.

It seems that (1.7) represents the variational problem of a Lagrangian system depending only on first order derivatives and of $kn + 1$ degrees of freedom. However the kinematical variables, considered as generalized coordinates, are not all independent. There exist among them the following $(k-1)n$ differential constraints

$$q_i^{(s)}(\tau) = \dot{q}_i^{(s-1)}(\tau)/\dot{t}(\tau), \quad i = 1, \ldots, n, \quad s = 1, \ldots, k-1. \tag{1.8}$$

We can also see that the integrand $\widehat{L}$ is a homogeneous function of first degree as a function of the derivatives of the kinematical variables. In fact, each time derivative function $q_i^{(s)}(t)$ has been replaced by the quotient $\dot{q}_i^{(s-1)}(\tau)/\dot{t}(\tau)$ of two derivatives with respect to τ. Even the highest order k-th derivative function $q_i^{(k)} = \dot{q}_i^{(k-1)}/\dot{t}$ is expressed in terms of the derivatives of the kinematical variables $q_i^{(k-1)}$ and t. Thus the original function L is a homogeneous function of zero degree in the derivatives of the kinematical variables. Finally, the last term $\dot{t}(\tau)$, gives to the new defined $\widehat{L}$ the character of a homogeneous function of first degree. Then, Euler's theorem on homogeneous functions gives rise to the additional relation:

$$\widehat{L}(x(\tau), \dot{x}(\tau)) = \sum_j \frac{\partial \widehat{L}}{\partial \dot{x}^j} \dot{x}^j = \sum_j F_j(x, \dot{x})\dot{x}^j. \tag{1.9}$$

With the above $(k-1)n$ differentiable constraints among the kinematical variables (1.8) and condition (1.9), it reduces to n the number of essential degrees of freedom of the system (1.7).

This possibility of expressing the Lagrangian as a homogeneous function of first degree of the derivatives was already considered in 1933 by Dirac [7] on aesthetical grounds. It is this homogeneity of first degree in terms of the derivatives which will allow us later to transform the variational formalism into a geodesic problem on the kinematical space X, but where the metric $g_{ij}(x, \dot{x})$ will be direction dependent, and thus the particle trajectory is a geodesic, not in a Riemannian manifold but rather in a Finsler space.[8]

Function $\widehat{L}$ is not an explicit function of the evolution parameter τ and thus we can see that the variational problem (1.7), is invariant with respect to any arbitrary change of evolution parameter τ. [9]

In fact, if we change the evolution parameter $\tau = \tau(\theta)$, then the derivative $\dot{t}(\tau) = (dt/d\theta)(d\theta/d\tau)$ and $\dot{q}_i^{(s)}(\tau) = (dq_i^{(s)}(\theta)/d\theta)(d\theta/d\tau)$ such that the quotients

$$\frac{\dot{q}_i^{(s)}(\tau)}{\dot{t}(\tau)} = \frac{(dq_i^{(s)}(\theta)/d\theta)\,\dot{\theta}(\tau)}{(dt(\theta)/d\theta)\,\dot{\theta}(\tau)} \equiv \frac{\dot{q}_i^{(s)}(\theta)}{\dot{t}(\theta)},$$

where once again this last dot means derivation with respect to θ. It turns out that (1.7) can be written as:

$$A[t, q] = \int_{\tau_1}^{\tau_2} L(t(\theta), q_i(\theta), \ldots, \dot{q}_i^{(k-1)}(\theta)/\dot{t}(\theta)) \frac{dt(\theta)}{d\theta} d\theta$$

[7] P.A.M. Dirac, *Proc. Cam. Phil. Soc.* **29**, 389 (1933): "a greater elegance is obtained", "a symmetrical treatment suitable for relativity."

[8] G.S. Asanov, *Finsler geometry, Relativity and Gauge theories*, Reidel Pub. Co, Dordrecht (1985).

[9] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, Interscience, N.Y. (1970); I.M. Gelfand, S.V. Fomin, *Calculus of Variations* Prentice Hall, Englewood Cliffs, N.J. (1963).

$$= \int_{\theta_1}^{\theta_2} \widehat{L}(x(\theta), \dot{x}(\theta)) d\theta. \tag{1.10}$$

1.3.2   Recovering the Lagrangian from the Action function

The formalism thus stated has the advantage that it is independent of the evolution parameter, and if we want to come back to a time evolution description, we just use our time as the evolution parameter and make the replacement $\tau = t$, and therefore $\dot{t} = 1$. From now on we shall consider those systems for which the evolution can be described in a parametric form, and we shall delete the symbol $\,\widehat{}\,$ over the Lagrangian, which is understood as written in terms of the kinematical variables and their first order derivatives.

If what we know is the action function of any system $A(x_1, x_2)$, as a function of the kinematical variables at the end points we can proceed conversely and recover the Lagrangian $L(x, \dot{x})$ by the limiting process:

$$L(x, \dot{x}) = \lim_{y \to x} \frac{\partial A(x, y)}{\partial y^j} \dot{x}^j, \tag{1.11}$$

where the usual addition convention on repeated or dummy index j, extended to the whole set of kinematical variables, has been assumed.

> If in (1.7) we consider two very close points $x_1 \equiv x$ and $x_2 \equiv x + dx$, we have that the action function $A(x, x + dx) = A(x, x + \dot{x}d\tau) = L(x, \dot{x})d\tau$ and making a Taylor expansion of the function A with the condition $A(x, x) = 0$ we get (1.11).

1.4   Generalized Noether's theorem

Noether's analysis for generalized Lagrangian systems also states the following

Theorem: To every one-parameter group of continuous transformations that transform the action function of the system, leaving dynamical equations invariant, in the form

$$A(\delta g x_1, \delta g x_2) = A(x_1, x_2) + B(x_2)\delta g - B(x_1)\delta g,$$

and where $B(x)$ is a function defined on the kinematical space, there is associated a classical observable N, which is a constant of the motion.

Let us assume the existence of a r-parameter continuous group of transformations G, of the enlarged configuration space (t, q_i), that can be extended as a transformation group to the whole kinematical space X. Let δg be an infinitesimal element of G with coordinates δg^α, $\alpha = 1, \ldots, r$ and its action on these variables be given by:

$$t \to t' \quad = \quad t + \delta t = t + M_\alpha(t, q)\delta g^\alpha, \tag{1.12}$$

$$q_i(t) \to q_i'(t') \quad = \quad q_i(t) + \delta q_i(t) = q_i(t) + M_{i\alpha}^{(0)}(t, q)\delta g^\alpha, \tag{1.13}$$

and its extension on the remaining kinematical variables by

$$q_i'^{(1)}(t') \quad = \quad q_i^{(1)}(t) + \delta q_i^{(1)}(t) = q_i^{(1)}(t) + M_{i\alpha}^{(1)}(t, q, q^{(1)})\delta g^\alpha, \tag{1.14}$$

and in general

$$q'^{(s)}_i(t') = q^{(s)}_i(t) + \delta q^{(s)}_i(t) = q^{(s)}_i(t) + M^{(s)}_{i\alpha}(t, q, \ldots, q^{(s)})\delta g^{\alpha}, \qquad s = 0, 1, \ldots, k-1, \quad (1.15)$$

where M_{α} and $M^{(0)}_{i\alpha}$ are functions only of q_i and t while the functions $M^{(s)}_{i\alpha}$ with $s \geq 1$, obtained in terms of the derivatives of the previous ones, will be functions of the time t and of the variables q_i and their time derivatives up to order s.

For instance,

$$q'^{(1)}_i(t') \equiv \frac{dq'_i(t')}{dt'} = \frac{d(q_i(t) + M^{(0)}_{i\alpha}\delta g^{\alpha})}{dt}\frac{dt}{dt'},$$

but up to first order in δg

$$\frac{dt'}{dt} = 1 + \frac{dM_{\alpha}(t, q)}{dt}\delta g^{\alpha}, \quad \frac{dt}{dt'} \approx 1 - \frac{dM_{\alpha}(t, q)}{dt}\delta g^{\alpha},$$

and thus

$$q'^{(1)}_i(t') = q^{(1)}_i(t) + \left(\frac{dM^{(0)}_{i\alpha}(t, q)}{dt} - q^{(1)}_i\frac{dM_{\alpha}(t, q)}{dt}\right)\delta g^{\alpha},$$

and comparing with (1.14) we get

$$M^{(1)}_{i\alpha}(t, q, q^{(1)}) = \frac{dM^{(0)}_{i\alpha}(t, q)}{dt} - q^{(1)}_i\frac{dM_{\alpha}(t, q)}{dt},$$

where the total time derivatives

$$\frac{dM_{\alpha}(t, q)}{dt} = \frac{\partial M_{\alpha}}{\partial t} + \sum_j \frac{\partial M^{(0)}_{\alpha}}{\partial q_j}q^{(1)}_j, \quad \frac{dM^{(0)}_{i\alpha}(t, q)}{dt} = \frac{\partial M^{(0)}_{i\alpha}}{\partial t} + \sum_j \frac{\partial M^{(0)}_{i\alpha}}{\partial q_j}q^{(1)}_j.$$

The remaining $M^{(s)}_{i\alpha}$ for $s > 1$, are obtained in the same way from the previous $M^{(s-1)}_{i\alpha}$.

Under δg the change of the action functional of the system is:

$$\begin{aligned}
\delta \mathcal{A}[q] &= \int_{t'_1}^{t'_2} L(t', q'^{(s)}_i(t'))dt' - \int_{t_1}^{t_2} L(t, q^{(s)}_i(t))dt \\
&= \int_{t'_1}^{t'_2} L(t + \delta t, q^{(s)}_i(t) + \delta q^{(s)}_i(t))dt' - \int_{t_1}^{t_2} L(t, q^{(s)}_i(t))dt.
\end{aligned}$$

By replacing in the first integral the integration range (t'_1, t'_2) by (t_1, t_2) having in mind the Jacobian of t' in terms of t, this implies that the differential $dt' = (1 + d(\delta t)/dt)dt$, and thus:

$$\begin{aligned}
\delta \mathcal{A}[q] &= \int_{t_1}^{t_2} L(t + \delta t, q^{(s)}_i + \delta q^{(s)}_i)\left(1 + \frac{d(\delta t)}{dt}\right)dt - \int_{t_1}^{t_2} L(t, q^{(s)}_i)dt \\
&= \int_{t_1}^{t_2} \left(L\frac{d(\delta t)}{dt} + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^{(s)}_i}\delta q^{(s)}_i(t)\right)dt,
\end{aligned}$$

keeping only for the Lagrangian $L(t + \delta t, q^{(s)} + \delta q^{(s)})$, first order terms in its Taylor expansion.

Now, in the total variation of $\delta q_i^{(s)}(t) = q_i'^{(s)}(t') - q_i^{(s)}(t)$ is contained a variation in the form of the function $q_i^{(s)}(t)$ and a variation in its argument t, that is also affected by the transformation of the group, *i.e.*,

$$\delta q_i^{(s)} = q_i'^{(s)}(t + \delta t) - q_i^{(s)}(t) = q_i'^{(s)}(t) - q_i^{(s)}(t) + (dq_i^{(s)}(t)/dt)\delta t$$
$$= \bar{\delta} q_i^{(s)}(t) + q_i^{(s+1)}(t)\delta t,$$

where $\bar{\delta} q_i^{(s)}(t)$ is the variation in form of the function $q_i^{(s)}(t)$ at the instant of time t. Taking into account that for the variation in form

$$\bar{\delta} q_i^{(s)}(t) = d^s(\bar{\delta} q_i(t))/dt^s = d(\bar{\delta} q_i^{(s-1)}(t))/dt,$$

it follows that

$$\delta \mathcal{A}[q] = \int_{t_1}^{t_2} \left(L \frac{d(\delta t)}{dt} + \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q_i^{(s)}}\bar{\delta} q_i^{(s)}(t) + \frac{\partial L}{\partial q_i^{(s)}}\frac{dq_i^{(s)}}{dt}\delta t \right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{d(L\delta t)}{dt} + \frac{\partial L}{\partial q_i^{(s)}}\bar{\delta} q_i^{(s)}(t) \right) dt. \tag{1.16}$$

Making the replacements

$$\frac{\partial L}{\partial q_i}\bar{\delta} q_i = \frac{\partial L}{\partial q_i}\bar{\delta} q_i,$$

$$\frac{\partial L}{\partial q_i^{(1)}}\bar{\delta} q_i^{(1)} = \frac{\partial L}{\partial q_i^{(1)}}\frac{d(\bar{\delta} q_i)}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}}\bar{\delta} q_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}} \right)\bar{\delta} q_i,$$

$$\frac{\partial L}{\partial q_i^{(2)}}\bar{\delta} q_i^{(2)} = \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}}\bar{\delta} q_i^{(1)} \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \right)\bar{\delta} q_i^{(1)}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}}\bar{\delta} q_i^{(1)} \right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \right)\bar{\delta} q_i \right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial q_i^{(2)}} \right)\bar{\delta} q_i,$$

$$\frac{\partial L}{\partial q_i^{(k)}}\bar{\delta} q_i^{(k)} = \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(k)}}\bar{\delta} q_i^{(k-1)} \right) - \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(k)}} \right)\bar{\delta} q_i^{(k-2)} \right) + \cdots,$$

and collecting terms we get

$$\delta \mathcal{A}[q] = \int_{t_1}^{t_2} dt \left\{ \frac{d(L\delta t)}{dt} \right.$$

$$+ \bar{\delta} q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(1)}} \right) + \cdots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right]$$

$$+ \frac{d}{dt}\left(\bar{\delta} q_i \left[\frac{\partial L}{\partial q_i^{(1)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(2)}} \right) + \cdots + (-1)^{k-1} \frac{d^{k-1}}{dt^{k-1}}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right] \right)$$

$$+ \frac{d}{dt}\left(\bar{\delta} q_i^{(1)} \left[\frac{\partial L}{\partial q_i^{(2)}} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{(3)}} \right) + \cdots + (-1)^{k-2} \frac{d^{k-2}}{dt^{k-2}}\left(\frac{\partial L}{\partial q_i^{(k)}} \right) \right] \right)$$

$$\left. + \cdots + \frac{d}{dt}\left(\bar{\delta} q_i^{(k-1)} \left[\frac{\partial L}{\partial q_i^{(k)}} \right] \right) \right\}.$$

The terms between squared brackets are precisely the conjugate momenta of order s, $p^i_{(s)}$, except the first one, which is the left-hand side of (1.3) and vanishes identically if the functions q_i satisfy the dynamical equations.

> In ordinary Lagrangian systems that depend only on first order derivatives of the independent degrees of freedom, the canonical approach associates to every generalized coordinate q_i a dynamical variable p_i, called its canonical conjugate momentum and defined by
>
> $$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$
>
> As a generalization of this, in Lagrangian systems with higher order derivatives, a generalized canonical formalism can be obtained by defining various canonical conjugate momenta (up to a total of k of them) associated to each of the independent degrees of freedom q_i: [10]
>
> $$p^i_{(s)} = \sum_{r=0}^{k-s} (-1)^r \frac{d^r}{dt^r}\left(\frac{\partial L}{\partial q_i^{(r+s)}}\right), \quad s = 1,\ldots,k, \quad i = 1,\ldots,n, \tag{1.17}$$
>
> which are precisely the above terms between the squared barckets. It is said that $p^i_{(s)}$ is the conjugate momentum of order s of the variable q_i.

Now if we introduce in the integrand the variables q_i that satisfy Euler-Lagrange equations, the variation of the action functional (1.16) is transformed into the variation of the action function along the classical trajectory, and therefore, the variation of the action function can be written as,

$$\delta A(x_1, x_2) = \int_{t_1}^{t_2} \frac{d}{dt}\left\{L\delta t + \left(\bar{\delta}q_i p^i_{(1)} + \bar{\delta}q_i^{(1)} p^i_{(2)} + \cdots + \bar{\delta}q_i^{(k-1)} p^i_{(k)}\right)\right\} dt, \tag{1.18}$$

with $p^i_{(s)}$ given in (1.17). If we replace in (1.18) the form variation $\bar{\delta}q_i^{(s)} = \delta q_i^{(s)} - q_i^{(s+1)}$, then

$$\delta A(x_1, x_2) = \int_{t_1}^{t_2} \frac{d}{dt}\left\{L\delta t + \delta q_i^{(s)} p^i_{(s+1)} - q_i^{(s+1)} p^i_{(s+1)}\delta t\right\} dt \tag{1.19}$$

with the usual addition convention. By substitution of the variations δt and $\delta q_i^{(s)}$ in terms of the infinitesimal element of the group δg^α, (1.13-1.15), we get:

$$\delta A(x_1, x_2) = \int_{t_1}^{t_2} \frac{d}{dt}\left\{\left(L - p^i_{(s)} q_i^{(s)}\right) M_\alpha + p^i_{(u+1)} M_{i\alpha}^{(u)}\right\} \delta g^\alpha dt, \tag{1.20}$$

with the following range for repeated indexes for the addition convention, $i = 1,\ldots,n$, $s = 1,\ldots,k$, $u = 0,1,\ldots,k-1$ and $\alpha = 1,\ldots,r$.

In the above integral we are using the solution of the dynamical equations, and therefore the variation of the action function is

$$\delta A(x_1, x_2) = A(\delta g x_1, \delta g x_2) - A(x_1, x_2).$$

If it happens to be of first order in the group parameters in the form

$$\delta A(x_1, x_2) = B_\alpha(x_2)\delta g^\alpha - B_\alpha(x_1)\delta g^\alpha, \tag{1.21}$$

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[10] E.T.Whittaker, *Analytical Dynamics*, Cambridge University Press, Cambridge (1927), p. 265.

then equating to (1.20) we can perform the trivial time integral on the right hand side. By considering that the group parameters δg^α are arbitrary, rearranging terms depending on t_1 and t_2 on the left- and right-hand side, respectively, we get several observables that take the same values at the two arbitrary times t_1 and t_2. They are thus constants of the motion and represent the time conserved physical quantities,

$$N_\alpha = B_\alpha(x) - \left(L - p^i_{(s)} q^{(s)}_i\right) M_\alpha - p^i_{(s+1)} M^{(s)}_{i\alpha}, \qquad \alpha = 1, \ldots, r, \tag{1.22}$$

where the term within brackets $H = p^i_{(s)} q^{(s)}_i - L$ is the generalized Hamiltonian.

These are the r Noether constants of the motion related to the infinitesimal transformations (1.21) of the action function under the corresponding r-parameter Lie group.

To express the different magnitudes in terms of the kinematical variables, let us define the variables x^j according to the rule: $x^0 = t$, $x^i = q_i$, $x^{n+i} = q^{(1)}_i, \ldots, x^{(k-1)n+i} = q^{(k-1)}_i$. Since $L = \widehat{L}/\dot{x}^0$, and $q^{(s)}_i = \dot{q}^{(s-1)}_i/\dot{x}^0$, the derivatives in the definition of the canonical momenta can be written as:

$$\frac{\partial L}{\partial q^{(s)}_i} = \frac{\partial(\widehat{L}/\dot{x}^0)}{\partial\left(\dot{x}^{(s-1)n+i}/\dot{x}^0\right)} = \frac{\partial\widehat{L}}{\partial\dot{x}^{(s-1)n+i}} = F_{(s-1)n+i}, \tag{1.23}$$

in terms of the functions F_i of the expansion (1.9) of the Lagrangian. The different conjugate momenta appear in the form:

$$p^i_{(s)} = \sum_{r=0}^{k-s} (-1)^r \frac{d^r}{dt^r} F_{(r+s-1)n+i}, \tag{1.24}$$

in terms of the functions F_i and their time derivatives. Therefore the Noether constants of the motion are written as

$$N_\alpha = B_\alpha(x) - \left(F_j \frac{\dot{x}^j}{\dot{x}^0} - p^i_{(s)} \frac{\dot{x}^{(s-1)n+i}}{\dot{x}^0}\right) M_\alpha - p^i_{(s+1)} M^{(s)}_{i\alpha}. \tag{1.25}$$

We see that the Noether constants of the motion N_α are finally expressed in terms of the functions F_i and their time derivatives, of the functions $M^{(s)}_{i\alpha}$ which represent the way the different kinematical variables transform under infinitesimal transformations, and of the functions B_α which, as we shall see below, are related to the exponents of the group G. Functions F_i and their time derivatives are homogeneous functions of zero degree in terms of the derivatives of the kinematical variables $\dot{x}^i$. Functions $B_\alpha(x)$ and $M^{(s)}_{i\alpha}(x)$ depend only on the kinematical variables. Consequently, Noether constants of the motion are also homogeneous functions of zero degree in terms of the derivatives of kinematical variables and thus invariant under arbitrary changes of evolution parameter.

1.5   Lagrangian gauge functions

In the variational formulation of classical mechanics

$$\mathcal{A}[q] = \int_{t_1}^{t_2} L(t, q^{(s)}_i(t)) dt \equiv \int_{\tau_1}^{\tau_2} L(x, \dot{x}) d\tau, \tag{1.26}$$

$\mathcal{A}[q]$ is a path functional, *i.e.*, it takes in general different values for the different paths joining the fixed end points x_1 and x_2. Then it is necessary that $Ld\tau$ be a non-exact differential.

Otherwise, if $Ldt = d\lambda$, then $\mathcal{A}[\,q\,] = \lambda_2 - \lambda_1$ and the functional does not distinguish between the different paths and the action function of the system from x_1 to x_2, $A(x_1, x_2) = \lambda(x_2) - \lambda(x_1)$, is expressed in terms of the potential function $\lambda(x)$, and is thus, path independent.

If $\lambda(x)$ is a real function defined on the kinematical space X of a Lagrangian system with action function $A(x_1, x_2)$, then the function $A'(x_1, x_2) = A(x_1, x_2) + \lambda(x_2) - \lambda(x_1)$ is another action function equivalent to $A(x_1, x_2)$. In fact it gives rise by (1.11) to the Lagrangian L' that differs from L in a total τ-derivative. [11]

Using (1.11), we have

$$L'(x, \dot{x}) = L(x, \dot{x}) + \frac{d\lambda}{d\tau}, \tag{1.27}$$

and therefore L and L' produce the same dynamical equations and $A(x_1, x_2)$ and $A'(x_1, x_2)$ are termed as equivalent action functions.

Let G be a transformation group of the enlarged configuration space (t, q_i), that can be extended to a transformation group of the kinematical space X. Let $g \in G$ be an arbitrary element of G and $x' = gx$, the transform of x. Consider a mechanical system characterized by the action function $A(x_1, x_2)$ that under the transformation g is changed into $A(x'_1, x'_2)$. If G is a symmetry group of the system, *i.e.*, the dynamical equations in terms of the variables x' are the same as those in terms of the variables x, this implies that $A(x'_1, x'_2)$ and $A(x_1, x_2)$ are necessarily equivalent action functions, and thus they will be related by:

$$A(gx_1, gx_2) = A(x_1, x_2) + \alpha(g; x_2) - \alpha(g; x_1). \tag{1.28}$$

The function α will be in general a continuous function of g and x. This real function $\alpha(g; x)$ defined on $G \times X$ is called a **gauge function** of the group G for the kinematical space X. Because of the continuity of the group it satisfies $\alpha(e; x) = 0$, e being the neutral element of G. If the transformation g is infinitesimal, let us represent it by the coordinates δg^σ, then $\alpha(\delta g; x) = \delta g^\sigma B_\sigma(x)$ to first order in the group parameters. The transformation of the action function takes the form

$$A(\delta gx_1, \delta gx_2) = A(x_1, x_2) + \delta g^\sigma B_\sigma(x_2) - \delta g^\sigma B_\sigma(x_1),$$

i.e., in the form required by Noether's theorem to obtain the corresponding conserved quantities. In general, B_σ functions for gauge-variant Lagrangians are obtained by

$$B_\sigma(x) = \left. \frac{\partial \alpha(g; x)}{\partial g^\sigma} \right|_{g=0}. \tag{1.29}$$

Because of the associative property of the group law, any gauge function satisfies the identity

$$\alpha(g'; gx) + \alpha(g; x) - \alpha(g'g; x) = \xi(g', g), \tag{1.30}$$

where the function ξ, defined on $G \times G$, is independent of x and is an exponent of the group G.

This can be seen by the mentioned associative property of the group law. From (1.28) we get:

$$A(g'gx_1, g'gx_2) = A(x_1, x_2) + \alpha(g'g; x_2) - \alpha(g'g; x_1), \tag{1.31}$$

and also

$$A(g'gx_1, g'gx_2) = A(gx_1, gx_2) + \alpha(g'; gx_2) - \alpha(g'; gx_1)$$

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[11] J.M. Levy-Leblond, *Comm. Math. Phys.* **12**, 64 (1969).

$$= A(x_1, x_2) + \alpha(g; x_2) - \alpha(g; x_1) + \alpha(g'; gx_2) - \alpha(g'; gx_1),$$

and therefore by identification of this with the above (1.31), when collecting terms with the same x argument we get

$$\alpha(g'; gx_2) + \alpha(g; x_2) - \alpha(g'g; x_2) = \alpha(g'; gx_1) + \alpha(g; x_1) - \alpha(g'g; x_1),$$

and since x_1 and x_2 are two arbitrary points of X, this expression is (1.30) and defines a function $\xi(g', g)$, independent of x.

If we substitute this function $\xi(g', g)$ into (1.45) we see that it is satisfied identically. For $g' = g = e$, it reduces to $\xi(e, e) = \alpha(e; x) = 0$, and thus ξ is an exponent of G.

It is shown by Levy-Leblond in the previous reference that if X is a homogeneous space of G, i.e., if there exists a subgroup H of G such that $X = G/H$, then, the exponent ξ is equivalent to zero on the subgroup H, and gauge functions for homogeneous spaces become:

$$\alpha(g; x) = \xi(g, h_x), \tag{1.32}$$

where h_x is any group element of the coset space represented by $x \in G/H$.

For the Poincaré group $\mathcal{P}$ all its exponents are equivalent to zero and thus the gauge functions when X is a homogeneous space of $\mathcal{P}$ are identically zero. Lagrangians of relativistic systems whose kinematical spaces are homogeneous spaces of $\mathcal{P}$ can be taken strictly invariant.

However, the Galilei group $\mathcal{G}$ has nontrivial exponents, that are characterized by a parameter m that is interpreted as the total mass of the system, and thus Galilei Lagrangians for massive systems are not in general invariant under $\mathcal{G}$. In the quantum formalism, the Hilbert space of states of a massive nonrelativistic system carries a projective unitary representation of the Galilei group instead of a true unitary representation. [12]

1.6 Elementary systems

In Newtonian mechanics the simplest geometrical object is a point of mass m. Starting with massive points we can construct arbitrary systems of any mass and shape, and thus any distribution of matter. The massive point can be considered as the elementary particle of Newtonian mechanics. In the modern view of particle physics it corresponds to a spinless particle. We know that there exist spinning objects like electrons, muons, photons, neutrinos, quarks and perhaps many others, that can be considered as elementary particles in the sense that they cannot be considered as compound systems of other objects. Even more, we do not find in Nature any spinless elementary particles. It is clear that the Newtonian point does not give account of the spin structure of particles and the existence of spin is a fundamental intrinsic attribute of an elementary particle, which is lacking in Newtonian mechanics, but it has to be accounted for.

In quantum mechanics, Wigner's work [13] on the representations of the inhomogeneous Lorentz group provides a very precise mathematical definition of the concept of elementary particle. An **elementary particle** is a quantum mechanical system whose Hilbert space of pure states is the representation space of a projective unitary irreducible representation of the Poincaré group. Irreducible representations of the Poincaré group are characterized by two

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[12] see ref.7 and also J.M. Levy-Leblond, *Galilei Group and Galilean Invariance*, in E.M. Loebl, *Group Theory and its applications*, Acad. Press, NY (1971), vol. 2, p. 221.

[13] see ref.1.

invariant parameters m and S, the mass and the spin of the system, respectively. By finding the different irreducible representations, we can obtain the quantum description of massless and massive particles of any spin.

The very important expression of the above mathematical definition, with physical consequences, lies in the term **irreducible**. Mathematically it means that the Hilbert space is an invariant vector space under the group action and that it has no other invariant subspaces. But it also means that there are no other states for a single particle than those that can be obtained by just taking any arbitrary vector state, form all its possible images in the different inertial frames and finally produce the closure of all finite linear combinations of these vectors.

We see that starting from a single state and by a simple change of inertial observer, we obtain the state of the particle described in this new frame. Take the orthogonal part of this vector to the previous one and normalize it. Repeat this operation with another kinematical transformation acting on the same first state, followed by the corresponding orthonormalization procedure, as many times as necessary to finally obtain a complete orthonormal basis of the whole Hilbert space of states. All states in this basis are characterized by the physical parameters that define the first state and a countable collection of group transformations of the kinematical group G. And this can be done starting from any arbitrary state.

This idea allows us to define a concept of physical equivalence among states of any arbitrary quantum mechanical system in the following way: Two states are said to be physically equivalent if they can produce by the above method an orthonormal basis of the same Hilbert subspace, or in an equivalent way, if they belong to the same invariant subspace under the group action. It is easy to see that this is an equivalence relation. But if the representation is irreducible, all states are equivalent as basic pieces of physical information for describing the elementary system. There is one and only one single piece of basic physical information to describe an elementary object. That is what the term elementary might mean.

But this definition of elementary particle is a pure group theoretical one. The only quantum mechanical ingredient is that the group operates on a Hilbert space. Then one question arises. Can we translate this quantum mechanical definition into the classical domain and obtain an equivalent group theoretical definition for a classical elementary particle?

Following with the above idea, in classical mechanics we have no vector space structure to describe the states of a system. What we have are manifolds of points where each point represents either the configuration state, the kinematical state or the phase state of the system depending on which manifold we work. But the idea that any point that represents the state of an elementary particle is physically equivalent to any other, is in fact the very mathematical concept of homogeneity of the manifold under the corresponding group action. In this way, the irreducibility assumption of the quantum mechanical definition is translated into the realm of classical mechanics in the concept of homogeneity of the corresponding manifold under the Poincaré group or any other kinematical group we consider as the symmetry group of the theory. But, what manifold? Configuration space? Phase space? The answer as has been shown in previous works, [14] is that the appropriate manifold is the **kinematical space**.

In the Lagrangian approach of classical mechanics, the kinematical space X is the manifold where the dynamics is developed as an input-output formalism. When quantizing the system we will obtain the natural link between the classical and quantum formalisms through Feynman's path integral approach, as will be shown later. This manifold is the natural space on which to define the Hilbert space structure of the quantized system. In a formal way we can say

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[14] M. Rivas, *J. Phys.* A **18**, 1971 (1985); *J. Math. Phys.* **30**, 318 (1989); *J. Math. Phys.* **35**, 3380 (1994).

that each point $x \in X$ that represents the kinematical state of a system is spread out and is transformed through Feynman's quantization into the particle wave function $\psi(x)$ defined around x. This wave function is a squared integrable complex function defined on X.

We can also analyze the elementarity condition from a different point of view. Let us consider an inertial observer O that is measuring a certain observable $A(\tau)$ of an arbitrary system at an instant τ. This observable takes the value $A'(\tau)$ for a different inertial observer O'. It can be expressed in terms of $A(\tau)$ in the form $A'(\tau) = f(A(\tau), g)$, where g is the kinematical transformation between both observers. At instant $\tau + d\tau$, the corresponding measured values of that observable will have changed but $A'(\tau + d\tau) = f(A(\tau + d\tau), g)$ with the same g as before, and assuming that the evolution parameter τ is group invariant.

But if the system is elementary, we take as an assumption that the modifications of the observables produced by the dynamics can always be compensated by a change of inertial reference frame. Then, given an observer O, it is always possible to find at instant $\tau + d\tau$ another inertial observer O' who measures the value of an essential observable $A'(\tau + d\tau)$ with the same value as O does at instant τ, i.e., $A'(\tau + d\tau) \equiv A(\tau)$. If the system is not elementary, this will not be possible in general because the external interaction might change its internal structure, and thus it will not be possible to compensate the modification of the observable by a simple change of inertial observer. Think about a non-relativistic description of an atom that goes into some excited state. The new internal energy, which is Galilei invariant, cannot be transformed into the old one by a simple change of reference frame.

But the essential observables are the kinematical variables. From the dynamical point of view we can take as initial and final points any x_1 and $x_2 \in X$, compatible with the causality requirements. This means that any x can be considered as the initial point of the variational formalism. In this way, at any instant τ if the system is elementary, we can find an infinitesimal kinematical transformation $\delta g(\tau)$ such that

$$x'(\tau + d\tau) = f(x(\tau + d\tau), \delta g(\tau)) \equiv x(\tau),$$

or by taking the inverse of this transformation,

$$x(\tau + d\tau) = f^{-1}(x(\tau), \delta g(\tau)).$$

This equation represents the dynamical evolution equation in X space. Knowledge of the initial state x_1 and the function $\delta g(\tau)$ completely determines the evolution of the system. In general, $\delta g(\tau)$ will depend on the instant τ, because the change of the observables depends on the external interaction. But if the system is elementary and the motion is free, all $\delta g(\tau)$ have necessarily to be the same, and thus τ independent. We cannot distinguish in a free motion one instant from any other. Then, starting from x_1 we shall arrive at x_2 by the continuous action of the same infinitesimal group element δg, and the free particle motion is the action of the one-parameter group generated by δg on the initial state. Therefore, there should exist a finite group element $g \in G$ such that $x_2 = gx_1$. If the evolution is not free, the composition of all infinitesimal group elements $\delta g(\tau)$ for all intermediate values of $\tau \in [\tau_1, \tau_2]$, will also produce a finite group element g, and thus, $x_2 = gx_1$. We thus arrive at the:

> **Definition:** A **classical elementary particle** is a Lagrangian system whose kinematical space X is a **homogeneous space** of the kinematical group G.

Usually the Lagrangian of any classical Newtonian system is restricted to depend only on the first order derivative of each of the coordinates q_i that represent the independent degrees of freedom, or equivalently, that the q_i satisfy second order differential equations. But at this stage, if we do not know what are the basic variables we need to describe our elementary system, how can we state that they necessarily satisfy second order differential equations? If some of the degrees of freedom, say q_1, q_2 and q_3, represent the center of mass position of the system, Newtonian mechanics implies that in this particular case L will depend on the first order derivatives of these three variables. But what about other degrees of freedom? It is this condition on the kinematical space to be considered as a homogeneous space of G, as the mathematical statement of elementarity, that will restrict the dependence of the Lagrangian on these higher order derivatives. It is this definition of elementary particle with the proper election of the kinematical group, which will supply information about the structure of the Lagrangian.

The Galilei and Poincaré groups are ten-parameter Lie groups and therefore the largest homogeneous space we can find for these groups is a ten-dimensional manifold. The variables that define the different homogeneous spaces will share the same domains and dimensions as the corresponding variables we use to parameterize the group. Both groups, as we shall see later, are parameterized in terms of the following variables $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ with domains and dimensions respectively like $b \in \mathbb{R}$ that represents the time parameter of the time translation and $\boldsymbol{a} \in \mathbb{R}^3$, the three spatial coordinates for the space translation. Parameter $\boldsymbol{v} \in \mathbb{R}^3$ are the three components of the relative velocity between the inertial observers, restricted to $v < c$ in the Poincaré case. Finally $\boldsymbol{\alpha} \in SO(3)$ are three dimensionless variables which characterize the relative orientation of the corresponding Cartesian frames and whose compact domain is expressed in terms of a suitable parametrization of the rotation group.

In this way the maximum number of kinematical variables, for a classical elementary particle, is also ten. We represent them by $x \equiv (t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$ with the same domains and dimensions as above and interpret them respectively as the **time, position, velocity and orientation** of the particle.

Because the Lagrangian must also depend on the next order derivatives of the kinematical variables, we arrive at the conclusion that L must also depend on the acceleration and angular velocity of the particle. The particle is a system of six degrees of freedom, three $\boldsymbol{r}$, represent the position of a point and other three $\boldsymbol{\alpha}$, its orientation in space. We can visualize this by assuming a system of three orthogonal unit vectors linked to point $\boldsymbol{r}$ as a body frame. But the Lagrangian will depend up to the second time derivative of $\boldsymbol{r}$, or acceleration of that point, and on the first derivative of $\boldsymbol{\alpha}$, *i.e.*, on the angular velocity. The Galilei and Poincaré groups lead to generalized Lagrangians depending up to second order derivatives of the position.

By this definition it is the kinematical group G that implements the special Relativity Principle that completely determines the structure of the kinematical space where the Lagrangians that represent classical elementary particles have to be defined.Point particles are particular cases of the above definition and their kinematical spaces are just the quotient structures between the group G and subgroup of rotations and boosts, and thus their kinematical variables reduce only to time and position $(t, \boldsymbol{r})$. Therefore, the larger the kinematical group of space-time transformations, the greater the number of allowed classical variables to describe elementary objects with a more detailed and complex structure. In this way, the proposed formalism can be accommodated to any symmetry group. It is the proper definition of this group which contains the physical information of the elementary particles.

1.6.1   Elementary Lagrangian systems

An elementary Lagrangian system will be characterized by the Lagrangian function $L(x, \dot{x})$ where the variables $x \in X$ lie in a homogeneous space X of G. L is a homogeneous function of first degree of the derivatives of the kinematical variables, and this allows us to write

$$L(x, \dot{x}) = F_i(x, \dot{x})\, \dot{x}^i. \tag{1.33}$$

Functions $F_i(x, \dot{x})$ are therefore homogeneous functions of zero degree in the variables $\dot{x}^i$ and summation convention on repeated indexes as usual is assumed.

Under G, x transforms as $x' = gx$ or more explicitly its coordinates by $x'^i = f^i(g, x)$, and their derivative variables

$$\dot{x}'^i = \frac{\partial x'^i}{\partial x^j}\dot{x}^j, \tag{1.34}$$

transform like the components of a contravariant vector.

The Lagrangian transforms under G,

$$L(x'(x), \dot{x}'(x, \dot{x})) = L(x, \dot{x}) + \frac{d\alpha(g; x)}{d\tau}, \tag{1.35}$$

i.e.,

$$F_i(x', \dot{x}')\dot{x}'^i = F_j(x, \dot{x})\dot{x}^j + \frac{\partial\alpha(g; x)}{\partial x^j}\dot{x}^j. \tag{1.36}$$

Taking into account the way the different variables transform, we thus arrive at:

$$F_i(x', \dot{x}') = \frac{\partial x^j}{\partial x'^i}\left[F_j(x, \dot{x}) + \frac{\partial\alpha(g; x)}{\partial x^j}\right]. \tag{1.37}$$

In the case when $\alpha(g; x) = 0$, they transform like the components of a covariant vector over the kinematical space X. But in general this will not be the case and $\alpha(g; x)$ contains basic physical information about the system.

We thus find that for a fixed kinematical space X, the knowledge of the group action of G on X, and the gauge function $\alpha(g; x)$, will give us information about the possible structure of the functions $F_i(x, \dot{x})$, and therefore about the structure of the Lagrangian.

In practice, if we restrict ourselves to the Galilei $\mathcal{G}$ and Poincaré $\mathcal{P}$ groups, we see that $\mathcal{P}$ has gauge functions equivalent to zero and thus Poincaré Lagrangians that describe elementary particles can be taken strictly invariant. In the case of the Galilei group, it has only one class of gauge functions that define the mass of the system, and thus nonrelativistic Lagrangians will be in general not invariant. In the particular case of Galilei invariant Lagrangians, they will describe massless systems.

1.7   Appendix: Lie groups of transformations

Let us introduce the notation and general features of the action of Lie groups on continuous manifolds to analyze the transformation properties of the different magnitudes we can work with in either classical or quantum mechanics. We shall use these features all throughout this book.

Let us consider the transformation of an n-dimensional manifold X, $x' = gx$ given by n continuous and differentiable functions depending on a set $g \in G$ of r continuous parameters of the form

$$x'^i = f^i(x^j; g^\sigma), \quad \forall x \in X, \quad \forall g \in G, \quad i, j = 1, \ldots, n, \quad \sigma = 1, \ldots, r.$$

This transformation is said to be the action of a Lie group of transformations if it fulfils the two conditions:

(i) G is a Lie group, *i.e.*, there exists a group composition law $c = \phi(a, b) \in G$, $\forall a, b \in G$, in terms of r continuous and differentiable functions ϕ^σ.

(ii) The transformation equations satisfy

$$x'' = f(x'; b) = f(f(x; a); b) = f(x; c) = f(x; \phi(a, b)).$$

The group parametrization can be chosen such that the coordinates that characterize the neutral element e of the group are $e \equiv (0, \ldots, 0)$, so that an infinitesimal element of the group is the one with infinitesimal coordinates $\delta g^\sigma, \sigma = 1, \ldots, r$.

Under the action of an infinitesimal element δg of the group G, the change in the coordinates x^i of a point $x \in X$ is given by

$$x^i + dx^i = f^i(x; \delta g) = x^i + \left.\frac{\partial f^i(x; g)}{\partial g^\sigma}\right|_{g=e} \delta g^\sigma,$$

after a Taylor expansion up to first order in the group parameters and with $x^i = f^i(x; 0)$. There are nr auxiliary functions of the group that are defined as

$$u^i_\sigma(x) = \left.\frac{\partial f^i(x; g)}{\partial g^\sigma}\right|_{g=e}, \tag{1.38}$$

and therefore to first order in the group parameters, $dx^i = u^i_\sigma(x)\delta g^\sigma$.

The group action on the manifold X can be extended to the action on the set $\mathcal{F}(X)$ of continuous and differentiable functions defined on X by means of:

$$g : h(x) \to h'(x) \equiv h(gx). \tag{1.39}$$

If the group element is infinitesimal, then

$$h'(x) = h(x^i + dx^i) = h(x^i + u^i_\sigma(x)\delta g^\sigma) = h(x) + \frac{\partial h(x)}{\partial x^i}\, u^i_\sigma(x)\delta g^\sigma,$$

after a Taylor expansion to first order in the infinitesimal group parameters. The infinitesimal transformation on $\mathcal{F}(X)$ can be represented by the action of a differential operator in the form

$$h'(x) = \left(\mathbb{I} + \delta g^\sigma\, u^i_\sigma(x)\frac{\partial}{\partial x^i}\right) h(x) = (\mathbb{I} + \delta g^\sigma X_\sigma)\, h(x) = U(\delta g)h(x),$$

where $\mathbb{I}$ is the identity operator and the linear differential operators

$$X_\sigma = u^i_\sigma(x)\frac{\partial}{\partial x^i}. \tag{1.40}$$

In particular, when acting with the operator $U(\delta g) \equiv (\mathbb{I} + \delta g^\sigma X_\sigma)$ on the coordinate x^j we get $x^j + dx^j = x^j + u_\sigma^j(x)\delta g^\sigma$.

The operators X_σ are called the **generators** of the infinitesimal transformations. They are r linearly independent operators that span an r-dimensional real vector space such that its commutator $[X_\sigma, X_\lambda]$ also belongs to the same vector space, *i.e.*,

$$[X_\sigma, X_\lambda] = c_{\sigma\lambda}^\alpha \, X_\alpha, \quad \alpha, \sigma, \lambda = 1, \ldots, r. \tag{1.41}$$

The coefficients $c_{\sigma\lambda}^\alpha$ are a set of real constant numbers, called the **structure constants** of the group, and the vector space spanned by the generators is named the **Lie algebra** $\mathcal{L}(G)$, associated to the Lie group G. The structure constants are antisymmetric in their lower indexes $c_{\sigma\lambda}^\alpha = -c_{\lambda\sigma}^\alpha$, and satisfy Jacobi's indentitites:

$$c_{\sigma\lambda}^\alpha c_{\mu\alpha}^\beta + c_{\lambda\mu}^\alpha c_{\sigma\alpha}^\beta + c_{\mu\sigma}^\alpha c_{\lambda\alpha}^\beta = 0, \quad \forall \sigma, \lambda, \mu, \beta = 1, \ldots, r.$$

Equations (1.41) are the commutation relations that characterize the structure of the Lie algebra of the group.

If a finite group transformation of parameters g^σ can be done in n smaller steps of parameters g^σ/n, with n sufficiently large, then a finite transformation $U(g)h(x)$ can be obtained as

$$U(g)h(x) \equiv \lim_{n \to \infty} \left(\mathbb{I} + \frac{g^\sigma}{n} X_\sigma \right)^n h(x) = \exp(g^\sigma X_\sigma) \, h(x).$$

This defines the exponential mapping and in this case the group parameters g^σ are called **normal** or **canonical** parameters. In the normal parameterization the composition law of one-parameter subgroups reduces to the addition of the corresponding parameters of the involved group elements.

Consider $\mathcal{F}(X)$ a Hilbert space of states of a quantum system; (1.39) can be interpreted as the transformed wave function under the group element g. Then if the operator $U(g)$ is unitary it is usually written in the explicit form

$$U(g) = \exp\left(\frac{i}{\hbar} \, g^\sigma \widetilde{X}_\sigma \right),$$

in terms of the imaginary unit i and Planck's constant $\hbar$, such that in this case the new $\widetilde{X}_\sigma$ above are self-adjoint operators and therefore represent certain observables of the system. The physical dimensions of these observables depend on the dimensions of the group parameters g^σ, since the argument of the exponential function is dimensionless and because of the introduction of Planck's constant this implies that $g^\sigma \widetilde{X}_\sigma$ has dimensions of action. These observables, taking into account (1.40), are represented in a unitary representation by the differential operators

$$\widetilde{X}_\sigma = \frac{\hbar}{i} u_\sigma^i(x) \frac{\partial}{\partial x^i}. \tag{1.42}$$

However, (1.39) is not the most general form of transformation of the wave function of a quantum system, as we shall see in Chapter 3, but once we know the way it transforms we shall be able to obtain the explicit expression of the group generators by a similar procedure as the one developed so far. In general the wave function transforms under continuous groups with what is called a projective unitary representation of the group, which involves in general some additional phase factors.

1.7.1 Casimir operators

When we have a representation of a Lie group either by linear operators or by matrices acting on a linear space, we can define there what are called the Casimir operators. They are operators C that can be expressed as functions of the generators X_σ of the Lie algebra with the property that they commute with all of them, i.e., they satisfy $[C, X_\sigma] = 0, \quad \forall \sigma = 1, \ldots, r$. In general they are not expressed as real linear combinations of the X_σ and therefore they do not belong to the Lie algebra of the group. They belong to what is called the **group algebra**, i.e., the associative, but in general non-commutative algebra, spanned by the real or complex linear combinations of products of the X_σ, in the corresponding group representation.

In those representations where the X_σ are represented by self-adjoint operators as in a quantum formalism, the Casimir operators may be also self-adjoint and will represent those observables that remain invariant under the group transformations. In particular, when we consider later the kinematical groups that relate the space-time measurements between inertial observers, the Casimir operators of these groups will represent the intrinsic properties of the system. They are those properties of the physical system whose measured values are independent of the inertial observers.

For semisimple groups, i.e., for groups that do not have Abelian invariant subgroups like the rotation group $SO(3)$, the unitary groups $SU(n)$ and many others, it is shown that the Casimir operators are real homogeneous polynomials of the generators X_σ, but this is no longer the case for general Lie groups. Nevertheless, for most of the interesting Lie groups in physics, like Galilei, Poincaré, De Sitter, $SL(4, \mathbb{R})$, the inhomogeneous $ISL(4, \mathbb{R})$ and Conformal $SU(2, 2)$ groups, the Casimir operators can be taken as real polynomial functions of the generators.

1.7.2 Exponents of a group

The concept of exponent of a continuous group G was developed by Bargmann in his work on the projective unitary representations of continuous groups. [15]

Wigner's theorem about the symmetries of a physical system is well known in Quantum Mechanics. [16]

It states that if $\mathcal{H}$ is a Hilbert space that characterizes the pure quantum states of a system, and the system has a symmetry S, then there exists a unitary or antiunitary operator $U(S)$, defined up to a phase, that implements that symmetry on $\mathcal{H}$, i.e., if ϕ and $\psi \in \mathcal{H}$ are two possible vector states of the system and $| < \phi|\psi > |^2$ is the transition probability between them and $U(S)\phi$ and $U(S)\psi$ represent the transformed states under the operation S, then

$$| < U(S)\phi|U(S)\psi > |^2 = | < \phi|\psi > |^2.$$

If the system has a whole group of symmetry operations G, then to each element $g \in G$ there is associated an operator $U(g)$ unitary or antiunitary, but if G is a continuous group, in that case $U(g)$ is necessarily unitary. This can be seen by the fact that the product of two antiunitary operators is a unitary one.

Because there is an ambiguity in the election of the phase of the unitary operator $U(g)$, it implies that in general $U(g_1)U(g_2) \neq U(g_1 g_2)$ and therefore the transformation of the wave function is not given by an expression of the form (1.39), but it also involves in general a phase

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[15] V.Bargmann, *Ann. Math.* **59**, 1 (1954).

[16] E.P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, Acad. Press, NY (1959); V. Bargmann, *J. Math. Phys.* **5**, 862 (1964).

factor. However in the case of continuous groups we can properly choose the corresponding phases of all elements in such a way that

$$U(g_1)U(g_2) = \omega(g_1, g_2)U(g_1 g_2), \tag{1.43}$$

where $\omega(g_1, g_2) = \exp\{i\xi(g_1, g_2)\}$ is a phase that is a continuous function of its arguments. The real continuous function on $G \times G$, $\xi(g_1, g_2)$ is called an **exponent** of G. The operators $U(g)$ do not reproduce the composition law of the group G and (1.43) represents what Bargmann calls a **projective representation** of the group.

If we use the associative property of the group law, we get

$$
\begin{aligned}
(U(g_1)U(g_2))\, U(g_3) &= \omega(g_1, g_2)U(g_1 g_2)U(g_3) \\
&= \omega(g_1, g_2)\omega(g_1 g_2, g_3)U(g_1 g_2 g_3),
\end{aligned}
$$

and also

$$
\begin{aligned}
U(g_1)\,(U(g_2)U(g_3)) &= U(g_1)\omega(g_2, g_3)U(g_2 g_3) \\
&= \omega(g_1, g_2 g_3)\omega(g_2, g_3)U(g_1 g_2 g_3).
\end{aligned}
$$

Therefore

$$\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega(g_2, g_3), \tag{1.44}$$

which in terms of the exponents becomes:

$$\xi(g_1, g_2) + \xi(g_1 g_2, g_3) = \xi(g_1, g_2 g_3) + \xi(g_2, g_3). \tag{1.45}$$

Because of the continuity of the exponents,

$$\xi(g, e) = \xi(e, g) = 0, \quad \forall g \in G, \tag{1.46}$$

where e is the neutral element of the group.

Any continuous function on G, $\phi(g)$, with the condition $\phi(e) = 0$, can generate a trivial exponent by

$$\xi(g, g') = \phi(gg') - \phi(g) - \phi(g'),$$

that satisfies (1.45) and (1.46). All trivial exponents are equivalent to zero exponents, and in a unitary representation (1.43) can be compensated into the phases of the factors, thus transforming the projective representation (1.43) into a true unitary one.

Given a continuous group, the existence or not of non-trivial exponents is an intrinsic group property related to the existence or not of central extensions of the group. [17]

1.7.3   Homogeneous space of a group

A manifold X is called a homogeneous space of a group G, if $\forall\, x_1,\, x_2 \in X$ there exists at least one element $g \in G$ such that $x_2 = gx_1$. In that case it is said that G acts on X in a transitive way. The term homogeneous reminds us that the local properties of the manifold at a point x are translated to any other point of the manifold by means of the group action, and therefore all points of X share the same local properties.

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[17] see ref.7 and J.M. Levy-Leblond, *Comm. Math. Phys.*, **12**, 64 (1969); A.A. Kirillov, *Éléments de la theorie des représentations*, Mir, Moscow (1974).

The **orbit** of a point x is the set of points of the form gx, $\forall g \in G$, such that if X is a homogeneous space of G, then the whole X is the orbit of any of its points.

Given a point $x_0 \in X$, the **stabilizer group** (little group) of x_0 is the subgroup H_{x_0} of G, that leaves invariant the point x_0, i.e., $\forall h \in H_{x_0}$, $hx_0 = x_0$.

If H is a subgroup of G, then every element $g \in G$ can be written as $g = g'h$, where $h \in H$, and g' is an element of G/H, the set of left cosets generated by the subgroup H. If X is a homogeneous space of G, it can be generated by the action of G on an arbitrary point $x_0 \in X$. Then $\forall x \in X$, $x = gx_0 = g'hx_0 = g'x_0$, and thus the homogeneous space X is isomorphic to the manifold G/H_{x_0}.

The homogeneous spaces of a group can be constructed as quotient manifolds of the group by all its possible continuous subgroups. Conversely, it can also be shown that if X a homogeneous space of a group G, then there exists a subgroup H of G such that X is isomorphic to G/H. Therefore, the largest homogeneous space of a group is the group itself.

Chapter 2

Soluble examples of spinning particles

Nonrelativistic particles

2.1 Nonrelativistic point particle

See the Appendix about the Galilei group at the end of this chapter for the notation used through this section.

Let us consider a mechanical system whose kinematical space is the manifold $X = \mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is the six-dimensional subgroup of the homogeneous Galilei transformations of elements of the form $(0, \mathbf{0}, \boldsymbol{v}, \boldsymbol{\mu})$. See the Appendix at the end of this chapter for the notation related to the Galilei group. Then X is a four-dimensional manifold spanned by the variables $(t, \boldsymbol{r}) \equiv x$, with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$, similar to the group parameters b and $\boldsymbol{a}$ respectively. We assume that they are functions of some evolution parameter τ and at any instant τ of the evolution two different inertial observers relate their measurements by:

$$
\begin{aligned}
t'(\tau) &= t(\tau) + b, & (2.1)\\
\boldsymbol{r}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}. & (2.2)
\end{aligned}
$$

Because of the way they transform, we can interpret them respectively as the time and position of the system. If we assume that the evolution parameter τ is group invariant, by taking the $\tau-$derivative of both sides of the above expressions, it turns out that the derivatives of the kinematical variables at any instant τ transform as:

$$
\begin{aligned}
\dot{t}'(\tau) &= \dot{t}(\tau), & (2.3)\\
\dot{\boldsymbol{r}}'(\tau) &= R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \boldsymbol{v}\dot{t}(\tau). & (2.4)
\end{aligned}
$$

There are no constraints among these variables. It is only the homogeneity of the Lagrangian in terms of their derivatives (1.9) which reduces to three the number of independent degrees of freedom. This homogeneity leads to the general form:

$$
L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}}, \tag{2.5}
$$

where $T = \partial L/\partial \dot{t}$ and $R_i = \partial L/\partial \dot{r}_i$ are still some unknown functions of the kinematical variables and their derivatives, which are homogeneous of zero degree in terms of the derivatives.

Associated to this manifold X, the gauge function for this system is

$$
\alpha(g; x) = \xi(g, x) = m\left(v^2 t/2 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}\right), \tag{2.6}
$$

where the parameter m is interpreted as the mass of the system and $\xi(g, g')$ is the exponent of $\mathcal{G}$, so that the transformation of the Lagrangian under the Galilei group is

$$L(x', \dot{x}') = L(x, \dot{x}) + m \left(v^2 \dot{t}/2 + \boldsymbol{v} \cdot R(\boldsymbol{\mu}) \dot{\boldsymbol{r}} \right). \tag{2.7}$$

Then

$$T' = \frac{\partial L'}{\partial \dot{t}'} = \left(\frac{\partial L}{\partial \dot{t}} + \frac{1}{2} m v^2 \right) \frac{\partial \dot{t}}{\partial \dot{t}'} + \left(\frac{\partial L}{\partial \dot{r}_i} + m v_j R(\boldsymbol{\mu})_{ji} \right) \frac{\partial \dot{r}_i}{\partial \dot{t}'}, \tag{2.8}$$

but from (2.3) and (2.4) we get $\partial \dot{t}/\partial \dot{t}' = 1$ and $\partial \dot{r}_i/\partial \dot{t}' = -R^{-1}(\boldsymbol{\mu})_{ik} v_k$, respectively, and thus

$$T' = T - \frac{1}{2} m v^2 - \boldsymbol{v} \cdot R(\boldsymbol{\mu}) \boldsymbol{R}. \tag{2.9}$$

Similarly

$$\boldsymbol{R}' = R(\boldsymbol{\mu}) \boldsymbol{R} + m \boldsymbol{v}. \tag{2.10}$$

The conjugate momenta of the independent degrees of freedom $q_i = r_i$, are $p_i = \partial L/\partial \dot{r}_i$, and consequently Noether's theorem leads to the following constants of the motion:

a) Under time translations the gauge function (2.6) vanishes, $\delta t = \delta b$, $M = 1$, while $\delta r_i = 0$ and the constant reduces to the following expression $\boldsymbol{R} \cdot d\boldsymbol{r}/dt - L/\dot{t} = -T$.

b) Under space translations also $\alpha(g; x) \equiv 0$, $\delta t = 0$, $M = 0$, while $\delta r_i = \delta a_i$, $M_{ij} = \delta_{ij}$ and the conserved observable is $\boldsymbol{R}$.

c) Under pure Galilei transformations $\delta t = \delta b$ and $M = 0$, while $\delta r_i = t \delta v_i$ and $M_{ij} = t \delta_{ij}$, but now the gauge function to first order in the velocity parameters is $\alpha(\delta \boldsymbol{v}; x) = m \boldsymbol{r} \cdot \delta \boldsymbol{v}$, and we get $m \boldsymbol{r} - \boldsymbol{P} t$.

d) Under rotations $\alpha(g; x) \equiv 0$, $\delta t = 0$ and $M = 0$, while $\delta r_i = -\varepsilon_{ijk} r_j n_k \delta \alpha$ and $M_{ik} = -\varepsilon_{ijk} r_j$ the conserved quantity is $\boldsymbol{r} \times \boldsymbol{R}$.

Collecting all terms we can give them the following names:

$$
\begin{aligned}
\text{Energy} \quad H &= -T, & (2.11) \\
\text{linear momentum} \quad \boldsymbol{P} &= \boldsymbol{R} = \boldsymbol{p}, & (2.12) \\
\text{kinematical momentum} \quad \boldsymbol{K} &= m \boldsymbol{r} - \boldsymbol{P} t, & (2.13) \\
\text{angular momentum} \quad \boldsymbol{J} &= \boldsymbol{r} \times \boldsymbol{P}. & (2.14)
\end{aligned}
$$

We reserve for these observables the same symbols as the corresponding group generators which produce the space-time transformations that leave dynamical equations invariant. Even their names make reference to the corresponding group transformation parameter, except the energy which in this context should be called the 'temporal momentum'. For the kinematical momentum we can find in the literature alternative names like 'Galilei momentum' or 'static momentum'. Being consistent with this notation, we should call it 'Poincaré or Lorentz momentum' in a relativistic approach. Nevertheless we shall use the name of kinematical momentum for this observable $\boldsymbol{K}$ in either relativistic or non-relativistic formalism. None of these conserved quantities associated to the symmetry under one-parameter subgroups are definite positive so that the observable H can take any sign.

The linear momentum takes the general expression $\boldsymbol{P} = m \dot{\boldsymbol{r}}/\dot{t} = m \boldsymbol{u}$ because taking the τ-derivative in (2.13) of the kinematical momentum, $\dot{\boldsymbol{K}} = 0$, implies $\boldsymbol{P} = m \boldsymbol{u}$, where $\boldsymbol{u}$ is the time derivative of the position of the system, *i.e.*, the velocity of the particle.

The six conditions $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$, imply $\boldsymbol{u} = 0$ and $\boldsymbol{r} = 0$, so that the system is at rest and placed at the origin of the observer's reference frame. There is still an arbitrary rotation and a time translation to fix a unique inertial observer. Nevertheless we call this class of observers, for which $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$, the center of mass observer. These six conditions will be also used as the definition of the center of mass observer for any other system even in a relativistic approach.

From (2.9) and (2.10) we see that the energy and linear momentum transform as:

$$
\begin{aligned}
H' &= H + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{P} + \frac{1}{2}mv^2, & (2.15)\\
\boldsymbol{P}' &= R(\boldsymbol{\mu})\boldsymbol{P} + m\boldsymbol{v}. & (2.16)
\end{aligned}
$$

Then, if H_0 and $\boldsymbol{P} = 0$ are the energy and linear momentum measured by the center of mass observer, for any arbitrary observer who sees the particle moving with velocity $\boldsymbol{u}$, it follows from (2.15) and (2.16) that

$$
H = H_0 + \frac{1}{2}mu^2 = H_0 + \boldsymbol{P}^2/2m, \quad \boldsymbol{P} = m\boldsymbol{u}.
$$

The Lagrangian for the point particle is thus

$$
L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} = -H\dot{t} + \boldsymbol{P} \cdot \dot{\boldsymbol{r}} = -H_0\dot{t} + \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}}, \tag{2.17}
$$

with H_0 an arbitrary constant which plays no role in the dynamics and can be taken $H_0 = 0$. It will be related to the mc^2 term of the relativistic point particle.

If we define the spin of the system, as in (2.173), by

$$
\boldsymbol{S} \equiv \boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P} = \boldsymbol{J} - \boldsymbol{r} \times \boldsymbol{P} = 0, \tag{2.18}
$$

it represents the angular momentum of the system with respect to the center of mass $\boldsymbol{r}$. It vanishes, so that the point particle is a spinless system.

2.2 Galilei free spinning particle

The most general nonrelativistic particle [1] is the system whose kinematical space X is the whole Galilei group $\mathcal{G}$. Then the kinematical variables are the ten real variables $x(\tau) \equiv (t(\tau), \boldsymbol{r}(\tau), \boldsymbol{u}(\tau), \boldsymbol{\rho}(\tau))$ with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$, $\boldsymbol{u} \in \mathbb{R}^3$ and $\boldsymbol{\rho} \in \mathbb{R}^3_c$ similarly as the corresponding group parameters. The relationship between the values $x'(\tau)$ and $x(\tau)$ they take at any instant τ for two arbitrary inertial observers, is given by:

$$
\begin{aligned}
t'(\tau) &= t(\tau) + b, & (2.19)\\
\boldsymbol{r}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}, & (2.20)\\
\boldsymbol{u}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \boldsymbol{v}, & (2.21)\\
\boldsymbol{\rho}'(\tau) &= \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu} \times \boldsymbol{\rho}(\tau)}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau)}. & (2.22)
\end{aligned}
$$

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[1] M. Rivas, *J. Phys.* **A 18**, 1971 (1985).

Among these kinematical variables there exist the differential constraints $\boldsymbol{u}(\tau) = \dot{\boldsymbol{r}}(\tau)/\dot{t}(\tau)$, that together with the homogeneity condition of the Lagrangian L in terms of the derivatives of the kinematical variables:

$$L(x, \dot{x}) = (\partial L/\partial \dot{x}_i)\dot{x}_i, \tag{2.23}$$

reduce from ten to six the essential degrees of freedom of the system.

These degrees of freedom are the position $\boldsymbol{r}(t)$ and the orientation $\boldsymbol{\rho}(t)$. The Lagrangian depends on the second derivative of $\boldsymbol{r}(t)$ and the first derivative of $\boldsymbol{\rho}(t)$. Expression (2.23) is explicitly given by:

$$L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}} + \boldsymbol{V} \cdot \dot{\boldsymbol{\rho}}, \tag{2.24}$$

where the functions $T = \partial L/\partial \dot{t}$, $R_i = \partial L/\partial \dot{r}^i$, $U_i = \partial L/\partial \dot{u}^i$, $V_i = \partial L/\partial \dot{\rho}^i$ will be in general functions of the ten kinematical variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$ and homogeneous functions of zero degree in terms of the derivatives $(\dot{t}, \dot{\boldsymbol{r}}, \dot{\boldsymbol{u}}, \dot{\boldsymbol{\rho}})$. By assuming that the evolution parameter τ is group invariant, these derivatives transform under $\mathcal{G}$:

$$\dot{t}'(\tau) = \dot{t}(\tau), \tag{2.25}$$

$$\dot{\boldsymbol{r}}'(\tau) = R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \boldsymbol{v}\dot{t}(\tau), \tag{2.26}$$

$$\dot{\boldsymbol{u}}'(\tau) = R(\boldsymbol{\mu})\dot{\boldsymbol{u}}(\tau), \tag{2.27}$$

$$\dot{\boldsymbol{\rho}}'(\tau) = \frac{(\dot{\boldsymbol{\rho}}(\tau) + \boldsymbol{\mu} \times \dot{\boldsymbol{\rho}}(\tau))(1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau))}{(1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau))^2} +$$
$$\frac{\boldsymbol{\mu} \cdot \dot{\boldsymbol{\rho}}(\tau)(\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu} \times \boldsymbol{\rho}(\tau))}{(1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau))^2}. \tag{2.28}$$

Instead of the derivative $\dot{\boldsymbol{\rho}}(\tau)$ that transforms in a complicated way, we can define the angular velocity of the particle $\boldsymbol{\omega}$ as a function of it in the form

$$\boldsymbol{\omega} = \frac{2}{1 + \boldsymbol{\rho}^2}(\dot{\boldsymbol{\rho}} + \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}). \tag{2.29}$$

It is a linear function of $\dot{\boldsymbol{\rho}}$, and transforms as:

$$\boldsymbol{\omega}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{\omega}(\tau). \tag{2.30}$$

We interpret the rotation matrix $R(\boldsymbol{\rho})$ as the rotation that carries the initial frame linked to the body at instant $\tau = 0$ to the frame at instant τ, as in a rigid body. Then, the three columns of matrix $R(\boldsymbol{\rho})$ represent the Cartesian components of the three unit vectors linked to the body when chosen parallel to the laboratory frame at instant $\tau = 0$.

If $\boldsymbol{k}(\tau)$ is any internal vector of a rigid body with origin at point $\boldsymbol{r}$, then its dynamics is contained in the expression $\boldsymbol{k}(\tau) = R(\boldsymbol{\rho}(\tau))\boldsymbol{k}(0)$. The velocity of point $\boldsymbol{k}$ is

$$\dot{\boldsymbol{k}}(\tau) = \dot{R}(\boldsymbol{\rho}(\tau))\boldsymbol{k}(0) = \dot{R}(\boldsymbol{\rho}(\tau))R^{-1}(\boldsymbol{\rho}(\tau))\boldsymbol{k}(\tau) = \Omega(\tau)\boldsymbol{k}(\tau)$$

where matrix $\Omega = \dot{R}R^{-1} = \dot{R}R^T$ is an antisymmetric matrix. At any instant τ, $R(\boldsymbol{\rho}(\tau))R^T(\boldsymbol{\rho}(\tau)) = \mathbb{I}$, where superscript T means the transposed matrix and $\mathbb{I}$ is the 3×3 unit matrix. Taking the τ-derivative of this expression, $\dot{R}R^T + R\dot{R}^T = \Omega + \Omega^T = 0$, and thus the three essential components of the antisymmetric matrix Ω define a three-vector $\boldsymbol{\omega}$

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix},$$

such that we can also write $\dot{\boldsymbol{k}}(\tau) = \Omega(\tau)\boldsymbol{k}(\tau) \equiv \boldsymbol{\omega}(\tau) \times \boldsymbol{k}(\tau)$ and $\boldsymbol{\omega}$ is interpreted as the instantaneous angular velocity. The different components of $\boldsymbol{\omega}$, expressed as functions of the variables $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ are given in (2.29).

Expression (2.22) corresponds to $R(\boldsymbol{\rho}'(\tau)) = R(\boldsymbol{\mu})R(\boldsymbol{\rho}(\tau))$. Therefore

$$\begin{aligned} \Omega' &=& \dot{R}(\boldsymbol{\rho}'(\tau))R^T(\boldsymbol{\rho}'(\tau)) = R(\boldsymbol{\mu})\dot{R}(\boldsymbol{\rho}(\tau))R^T(\boldsymbol{\rho}(\tau))R^T(\boldsymbol{\mu}) \\ &=& R(\boldsymbol{\mu})\Omega R^{-1}(\boldsymbol{\mu}), \end{aligned}$$

and this leads to the equation (2.30) in terms of the essential components $\boldsymbol{\omega}$ of the antisymmetric matrix Ω.

In this way the last part of the Lagrangian $(\partial L/\partial \dot{\rho}^i)\dot{\rho}^i$ can be writen as

$$\boldsymbol{V} \cdot \dot{\boldsymbol{\rho}} \equiv \frac{\partial L}{\partial \dot{\rho}^i}\dot{\rho}^i = \frac{\partial L}{\partial \omega^j}\frac{\partial \omega^j}{\partial \dot{\rho}^i}\dot{\rho}^i = \boldsymbol{W} \cdot \boldsymbol{\omega}, \tag{2.31}$$

due to the linearity of $\boldsymbol{\omega}$ in terms of $\dot{\boldsymbol{\rho}}$ and where $W_i = \partial L/\partial \omega^i$. Thus the most general form of the Lagrangian of a nonrelativistic particle can also be written instead of (2.24) as:

$$L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}} + \boldsymbol{W} \cdot \boldsymbol{\omega}. \tag{2.32}$$

Since X is the whole Galilei group $\mathcal{G}$ the most general gauge function is just the group exponent:

$$\alpha(g; x) = \xi(g, h_x) = m(\boldsymbol{v}^2 t(\tau)/2 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau)), \tag{2.33}$$

similar to (2.6), and this allows us to interpret the parameter m as the mass of the system. Under the action of an arbitrary element of the Galilei group, the Lagrangian L transforms according to:

$$L(gx(\tau), d(gx(\tau))/d\tau) = L(x(\tau), \dot{x}(\tau)) + d\alpha(g; x(\tau))/d\tau. \tag{2.34}$$

This leads through some straightforward calculations, similar to the ones performed in (2.8)-(2.10), to the following form of transformation of the functions:

$$\begin{aligned} T'(\tau) &=& T(\tau) - \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{R}(\tau) - m\boldsymbol{v}^2/2, \tag{2.35} \\ \boldsymbol{R}'(\tau) &=& R(\boldsymbol{\mu})\boldsymbol{R}(\tau) + m\boldsymbol{v}, \tag{2.36} \\ \boldsymbol{U}'(\tau) &=& R(\boldsymbol{\mu})\boldsymbol{U}(\tau), \tag{2.37} \\ \boldsymbol{W}'(\tau) &=& R(\boldsymbol{\mu})\boldsymbol{W}(\tau). \tag{2.38} \end{aligned}$$

2.2.1 Noether constants of the motion

Using the action of the Galilei group on the kinematical space given by (2.19)-(2.22), Noether's theorem defines the following constants of the motion:

a) Under time translation the action function is invariant, $B(x) = 0$, and as usual we call the corresponding conserved quantity, the **total energy** of the system H. Since $\delta t = \delta b$ and $\delta q_i^{(s)} = 0$, $M = 1$ and $M_i^{(s)} = 0$, by applying (1.25) we have:

$$H = -(L - p_{(s)}^i q_i^{(s)})M = -(\hat{L}/\dot{t} - p_{(s)}^i q_i^{(s)}) = -T - \boldsymbol{R} \cdot \boldsymbol{u} - \boldsymbol{U} \cdot \dot{\boldsymbol{u}}/\dot{t} - \boldsymbol{W} \cdot \boldsymbol{\omega}/\dot{t}$$

$$+(\boldsymbol{R} - d\boldsymbol{U}/dt) \cdot \boldsymbol{u} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}}/\dot{t} + \boldsymbol{V} \cdot \dot{\boldsymbol{\rho}}/\dot{t},$$

and since $\boldsymbol{W} \cdot \boldsymbol{\omega} = \boldsymbol{V} \cdot \dot{\boldsymbol{\rho}}$, it turns out that

$$H = -T - \frac{d\boldsymbol{U}}{dt} \cdot \boldsymbol{u}. \tag{2.39}$$

b) Under spatial translation, $A(x_1, x_2)$ is invariant, $B_i(x) = 0$, and this defines the **total linear momentum** of the system. We have now:

$$\delta t = 0, \ M = 0, \ \delta r_i = \delta a_i, \ M_{ij}^{(0)} = \delta_{ij}, \ \delta u_i = 0, \ M_{ij}^{(1)} = 0,$$

$$\delta \rho_i = 0, \ M_{ij}^{(\rho)} = 0,$$

and then

$$\boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt}. \tag{2.40}$$

c) Under a pure Galilei transformation of velocity $\delta\boldsymbol{v}$, $A(x_1, x_2)$ is no longer invariant but taking into account (1.28) and the gauge function (2.33), it transforms as $\delta A = m\boldsymbol{r}_2 \cdot \delta\boldsymbol{v} - m\boldsymbol{r}_1 \cdot \delta\boldsymbol{v}$ and thus, $B_i(x) = mr_i$, and this defines the **total kinematical momentum $\boldsymbol{K}$**, in the following way:

$$\delta t = 0, \ M = 0, \ \delta r_i = \delta v_i t, \ M_{ij}^{(0)} = \delta_{ij} t, \ \delta u_i = \delta v_i, \ M_{ij}^{(1)} = \delta_{ij},$$

$$\delta \rho_i = 0, \ M_{ij}^{(\rho)} = 0,$$

and thus

$$\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}\,t - \boldsymbol{U}. \tag{2.41}$$

From $\dot{\boldsymbol{K}} = 0$, this leads to $\boldsymbol{P} = m\boldsymbol{u} - d\boldsymbol{U}/dt$, and thus by identification with (2.40), the function $\boldsymbol{R} = m\boldsymbol{u}$ irrespective of the particular Lagrangian. The total linear momentum does not lie along the velocity of point $\boldsymbol{r}$.

d) Finally, under rotations $A(x_1, x_2)$ remains invariant, $B_i(x) = 0$, and the corresponding constant of the motion, the **total angular momentum** of the system, with respect to the origin of observer's frame, comes from the infinitesimal transformation of value $\delta\mu_i = \delta\alpha_i/2$, i.e., half of the rotated infinitesimal angle, and then

$$\delta t = 0, \ M_i = 0, \delta r_i = \epsilon_{ikj}\delta\alpha_j r_k, \ M_{ij}^{(0)} = \epsilon_{ikj} r_k,$$

$$\delta u_i = \epsilon_{ikj}\delta\alpha_j u_k, \quad M_{ij}^{(1)} = \epsilon_{ikj} u_k,$$

$$\delta \rho_i = \delta\alpha^j (\delta_{ij} + \epsilon_{ikj}\rho^k + \rho_i\rho_j)/2, \quad M_{ij}^{(\rho)} = (\delta_{ij} + \epsilon_{ikj}\rho^k + \rho_i\rho_j)/2,$$

which leads to

$$V_i M_{ij}^{(\rho)} = \frac{\partial L}{\partial\omega^k}\frac{\partial\omega^k}{\partial\dot\rho_i} M_{ij}^{(\rho)} = W_j,$$

and therefore

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}. \tag{2.42}$$

We are tempted to consider $\boldsymbol{S}$ as the spin of the system. Since $\dot{\boldsymbol{J}} = 0$, this function $\boldsymbol{S}$ satisfies $d\boldsymbol{S}/dt = \boldsymbol{P} \times \boldsymbol{u}$ and is not a constant of the motion for a free particle. It is the classical angular momentum equivalent to Dirac's spin operator in the quantum case. Because $\boldsymbol{J}$ is the angular momentum of the particle with respect to the origin of the observer frame, $\boldsymbol{S}$ represents the angular momentum of the particle with respect to the point $\boldsymbol{r}$. We can also consider the spin for a free particle as the angular momentum with respect to its center of mass, once we accurately identify the center of mass of the particle. It is this angular momentum with respect to the center of mass, which is a constant of the motion and its absolute value an invariant property. In any case we shall keep the notation $\boldsymbol{S}$ for the classical angular momentum equivalent to Dirac's spin operator.

The center of mass observer is defined as that inertial observer for whom $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$. These six conditions do not define uniquely an inertial observer but rather a class of them up to a rotation and an arbitrary time translation. In fact, the condition $\boldsymbol{P} = 0$ establishes the class of observers for which the center of mass is at rest, and $\boldsymbol{K} = 0$ is the additional condition to locate it at the origin of coordinates. This comes from the analysis of (2.41), where $\boldsymbol{k} = \boldsymbol{U}/m$ is an observable with dimensions of length, and taking the derivative with respect to τ of both sides, taking into account that $\dot{\boldsymbol{P}} = 0$, we have:

$$\dot{\boldsymbol{K}} = 0 = m\dot{\boldsymbol{r}} - \boldsymbol{P}\dot{t} - m\dot{\boldsymbol{k}}, \qquad i.e., \qquad \boldsymbol{P} = m\frac{d(\boldsymbol{r} - \boldsymbol{k})}{dt}. \tag{2.43}$$

Then the point $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$ is moving at constant speed and we say that it represents the position of the center of mass of the system. Thus, the observable $\boldsymbol{k} = \boldsymbol{r} - \boldsymbol{q}$ is just the relative position of point $\boldsymbol{r}$ with respect to the center of mass. Therefore $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$ give rise to $d\boldsymbol{q}/dt = 0$, and $\boldsymbol{r} = \boldsymbol{k}$, i.e., $\boldsymbol{q} = 0$, as we pointed out. With this definition, the kinematical momentum can be written as $\boldsymbol{K} = m\boldsymbol{q} - \boldsymbol{P}t$, in terms of the center of mass position $\boldsymbol{q}$ and the total linear momentum $\boldsymbol{P}$.

The spin of the system, with respect to the center of mass, is defined as the difference between the total angular momentum $\boldsymbol{J}$ and the orbital angular momentum of the center of mass motion $\boldsymbol{q} \times \boldsymbol{P}$, and thus

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q} \times \boldsymbol{P} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P} = \boldsymbol{S} + \boldsymbol{k} \times \boldsymbol{P} = -m\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} + \boldsymbol{W}. \tag{2.44}$$

The spin $\boldsymbol{S}_{CM}$, expressed in terms of the constants of the motion $\boldsymbol{J}$, $\boldsymbol{K}$ and $\boldsymbol{P}$, is also a constant of the motion.

It is the sum of two terms, one $\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W}$, coming from the new degrees of freedom and another $\boldsymbol{k} \times \boldsymbol{P}$, which is the angular momentum of the linear momentum located at point $\boldsymbol{r}$ with respect to the center of mass. Alternatively we can describe the spin with respect to the center of mass, according to the last expression in which the term $-\boldsymbol{k} \times md\boldsymbol{k}/dt$ suggests a contribution of (anti)orbital type coming from the motion around the center of mass. It is related to the zitterbewegung or more precisely to the function $\boldsymbol{U} = m\boldsymbol{k}$ which reflects the dependence of the Lagrangian on the acceleration. The other term $\boldsymbol{W}$ comes from the dependence on the other three degrees of freedom ρ_i, and thus on the angular velocity. This zitterbewegung is the motion of the center of charge around the center of mass. Point $\boldsymbol{r}$, as representing the position of the center of charge, has been also suggested in previous works for the relativistic electron. [2]

Because $\dot{\boldsymbol{J}} = 0$, and that $d\boldsymbol{W}/d\tau = \boldsymbol{\omega} \times \boldsymbol{W}$ and the expression of $\boldsymbol{P}$, (2.40), this implies the general relation for a free particle

$$\dot{\boldsymbol{r}} \times \boldsymbol{R} + \dot{\boldsymbol{u}} \times \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{W} = 0, \tag{2.45}$$

which reflects the fact that velocity, acceleration and angular velocity are not independent magnitudes, and taking into account that $\boldsymbol{R}$ and $\dot{\boldsymbol{r}}$ have the same direction, it reduces to

$$\dot{\boldsymbol{u}} \times \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{W} = 0. \tag{2.46}$$

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[2] A.O. Barut and A.J. Bracken, *Phys. Rev. D* **23**, 2454 (1981).

2.2.2   Galilei spinning particle of (anti)orbital spin

To analyze the spin structure of the particle, and therefore the different contributions to spin coming from these functions $\boldsymbol{U}$ and $\boldsymbol{W}$, let us consider the following simpler example.

Consider a Galilei particle whose kinematical space is $X = \mathcal{G}/SO(3)$, so that any point $x \in X$ can be characterized by the seven variables $x \equiv (t, \boldsymbol{r}, \boldsymbol{u})$, $\boldsymbol{u} = d\boldsymbol{r}/dt$, which are interpreted as time, position and velocity of the particle respectively. In this example we have no orientation variables. The Lagrangian will also depend on the next order derivatives, *i.e.*, on the velocity which is already considered as a kinematical variable and on the acceleration of the particle. Rotation and translation invariance implies that L will be a function of only $\boldsymbol{u}^2$, $(d\boldsymbol{u}/dt)^2$ and $\boldsymbol{u} \cdot d\boldsymbol{u}/dt = d(u^2/2)/dt$, but this last term is a total time derivative and it will not be considered here.

Since from condition (2.46) $\boldsymbol{U} \sim \dot{\boldsymbol{u}}$, let us assume that our elementary system is represented by the following Lagrangian, which when written in terms of the three degrees of freedom and their derivatives is expressed as

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2. \tag{2.47}$$

Parameter m is the mass of the particle because the first term is gauge variant in terms of the gauge function (2.33) defined by this constant m, while parameter ω of dimensions of time^{-1} represents an internal frequency. It is the frequency of the internal zitterbewegung.

In terms of the kinematical variables and their derivatives, and in terms of some group invariant evolution parameter τ, the Lagrangian can also be written as

$$L = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} - \frac{m}{2\omega^2}\frac{\dot{\boldsymbol{u}}^2}{\dot{t}}, \tag{2.48}$$

where the dot means τ-derivative. If we consider that the evolution parameter is dimensionless, all terms in the Lagrangian have dimensions of action. Because the Lagrangian is a homogeneous function of first degree in terms of the derivatives of the kinematical variables, L can also be written as

$$L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}}, \tag{2.49}$$

where the functions accompanying the derivatives of the kinematical variables are defined and explicitly given by

$$
\begin{aligned}
T &= \frac{\partial L}{\partial \dot{t}} = -\frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2, \\
\boldsymbol{R} &= \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\frac{d\boldsymbol{r}}{dt}, \tag{2.50}\\
\boldsymbol{U} &= \frac{\partial L}{\partial \dot{\boldsymbol{u}}} = -\frac{m}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \tag{2.51}
\end{aligned}
$$

Dynamical equations obtained from Lagrangian (2.47) are:

$$\frac{1}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4} + \frac{d^2\boldsymbol{r}}{dt^2} = 0, \tag{2.52}$$

whose general solution is:

$$\boldsymbol{r}(t) = \boldsymbol{A} + \boldsymbol{B}t + \boldsymbol{C}\cos\omega t + \boldsymbol{D}\sin\omega t, \tag{2.53}$$

in terms of the 12 integration constants $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$.

When applying Noether's theorem to the invariance of dynamical equations under the Galilei group, the corresponding constants of the motion can be written in terms of the above functions in the form:

$$\text{Energy} \quad H \;=\; -T - \boldsymbol{u} \cdot \frac{d\boldsymbol{U}}{dt}, \tag{2.54}$$

$$\text{linear momentum} \quad \boldsymbol{P} \;=\; \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt} = m\boldsymbol{u} - \frac{d\boldsymbol{U}}{dt}, \tag{2.55}$$

$$\text{kinematical momentum} \quad \boldsymbol{K} \;=\; m\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{U}, \tag{2.56}$$

$$\text{angular momentum} \quad \boldsymbol{J} \;=\; \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{u} \times \boldsymbol{U}. \tag{2.57}$$

It is the presence of the $\boldsymbol{U}$ function that distinguishes the features of this system with respect to the point particle case. We find that the total linear momentum is not lying along the direction of the velocity $\boldsymbol{u}$, and the spin structure is directly related to the dependence of the Lagrangian on the acceleration.

If we substitute the general solution (2.53) in (2.54-2.57) we see in fact that the integration constants are related to the above conserved quantities

$$H \;=\; \frac{m}{2}\boldsymbol{B}^2 - \frac{m\omega^2}{2}(\boldsymbol{C}^2 + \boldsymbol{D}^2), \tag{2.58}$$

$$\boldsymbol{P} \;=\; m\boldsymbol{B}, \tag{2.59}$$

$$\boldsymbol{K} \;=\; m\boldsymbol{A}, \tag{2.60}$$

$$\boldsymbol{J} \;=\; \boldsymbol{A} \times m\boldsymbol{B} - m\omega\boldsymbol{C} \times \boldsymbol{D}. \tag{2.61}$$

We see that the kinematical momentum $\boldsymbol{K}$ in (2.56) differs from the point particle case (2.13) in the term $-\boldsymbol{U}$, such that if we define the vector $\boldsymbol{k} = \boldsymbol{U}/m$, with dimensions of length, then $\dot{\boldsymbol{K}} = 0$ leads from (2.56) to the equation:

$$\boldsymbol{P} = m\frac{d(\boldsymbol{r} - \boldsymbol{k})}{dt},$$

and $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$, defines the position of the center of mass of the particle that is a different point than $\boldsymbol{r}$ and using (2.51) is given by

$$\boldsymbol{q} = \boldsymbol{r} - \frac{1}{m}\boldsymbol{U} = \boldsymbol{r} + \frac{1}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \tag{2.62}$$

In terms of it, dynamical equations (2.52) can be separated into the form:

$$\frac{d^2\boldsymbol{q}}{dt^2} \;=\; 0, \tag{2.63}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} \;+\; \omega^2(\boldsymbol{r} - \boldsymbol{q}) = 0, \tag{2.64}$$

where (2.63) is just eq. (2.52) after twice differentiating (2.62), and Equation (2.64) is (2.62) after collecting all terms on the left hand side.

From (2.63) we see that point $\boldsymbol{q}$ moves in a straight trajectory at constant velocity while the motion of point $\boldsymbol{r}$, given in (2.64), is an isotropic harmonic motion of angular frequency ω around point $\boldsymbol{q}$.

The spin of the system with respect to the center of mass, $\boldsymbol{S}_{CM}$ is defined as

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q} \times \boldsymbol{P} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P}, \tag{2.65}$$

and since it is written in terms of constants of the motion it is clearly a constant of the motion, and its magnitude S^2 is also a Galilei invariant quantity that characterizes the system. In terms of the integration constants it is expressed as

$$\boldsymbol{S}_{CM} = -m\omega\,\boldsymbol{C} \times \boldsymbol{D}. \tag{2.66}$$

From its definition we get

$$\boldsymbol{S}_{CM} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{k} \times \boldsymbol{P} = -m(\boldsymbol{r} - \boldsymbol{q}) \times \frac{d}{dt}(\boldsymbol{r} - \boldsymbol{q}) = -\boldsymbol{k} \times m\frac{d\boldsymbol{k}}{dt}, \tag{2.67}$$

which appears as the (anti)orbital angular momentum of the relative motion of point $\boldsymbol{r}$ around the center of mass position $\boldsymbol{q}$ at rest, so that the total angular momentum can be written as

$$\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{P} + \boldsymbol{S}_{CM} = \boldsymbol{L} + \boldsymbol{S}_{CM}. \tag{2.68}$$

It is the sum of the orbital angular momentum $\boldsymbol{L}$ associated to the motion of the center of mass and the spin part $\boldsymbol{S}_{CM}$. For a free particle both $\boldsymbol{L}$ and $\boldsymbol{S}_{CM}$ are separately constants of the motion. We use the term (anti)orbital to suggest that if vector $\boldsymbol{k}$ represents the position of a point mass m, the angular momentum of this motion is in the opposite direction as the obtained spin observable. But as we shall see in a moment, vector $\boldsymbol{k}$ does not represent the position of the mass m but rather the position of the charge e of the particle.

2.2.3 Interacting with an external electromagnetic field

But if $\boldsymbol{q}$ represents the center of mass position, then what position does point $\boldsymbol{r}$ represent? Point $\boldsymbol{r}$ represents the position of the charge of the particle. This can be seen by considering some interaction with an external field. The homogeneity condition of the Lagrangian in terms of the derivatives of the kinematical variables leads us to consider an interaction term of the form

$$L_I = -e\phi(t,\boldsymbol{r})\dot{t} + e\boldsymbol{A}(t,\boldsymbol{r}) \cdot \dot{\boldsymbol{r}}, \tag{2.69}$$

which is linear in the derivatives of the kinematical variables t and $\boldsymbol{r}$ and where the external potentials are only functions of t and $\boldsymbol{r}$. We can also consider more general interaction terms of the form $\boldsymbol{N}(t,\boldsymbol{r},\boldsymbol{u}) \cdot \dot{\boldsymbol{u}}$, and also more general terms in which functions ϕ and $\boldsymbol{A}$ also depend on $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$. If the interaction Lagrangian depends on $\dot{\boldsymbol{u}}$ this implies that the interaction modifies the definition of the observable $\boldsymbol{U} = m\boldsymbol{k}$ which defines the spin of the free system. But if the system is elementary the spin definition cannot be changed, so that (2.69) is the most general interaction term.

Dynamical equations obtained from $L + L_I$ are

$$\frac{1}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4} + \frac{d^2\boldsymbol{r}}{dt^2} = \frac{e}{m}\left(\boldsymbol{E}(t,\boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}(t,\boldsymbol{r})\right), \tag{2.70}$$

where the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are expressed in terms of the potentials in the usual form, $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t$, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Dynamical equations (2.70) can again be

separated into the form

$$\frac{d^2\boldsymbol{q}}{dt^2} \;\; = \;\; \frac{e}{m}\left(\boldsymbol{E}(t,\boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}(t,\boldsymbol{r})\right), \tag{2.71}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} \;\; + \;\; \omega^2(\boldsymbol{r} - \boldsymbol{q}) = 0. \tag{2.72}$$

The center of mass $\boldsymbol{q}$ satisfies Newton's equations under the action of the total external Lorentz force, while point $\boldsymbol{r}$ still satisfies the isotropic harmonic motion of angular frequency ω around point $\boldsymbol{q}$. But the external force and the fields are defined at point $\boldsymbol{r}$ and not at point $\boldsymbol{q}$. It is the velocity $\boldsymbol{u}$ of point $\boldsymbol{r}$ that appears in the magnetic term of the Lorentz force. Point $\boldsymbol{r}$ clearly represents the position of the charge. In fact, this minimal coupling we have considered is the coupling of the electromagnetic potentials with the particle current, that in the relativistic case can be written as $j_\mu A^\mu$, but the current j_μ is associated to the motion of a charge e at point $\boldsymbol{r}$.
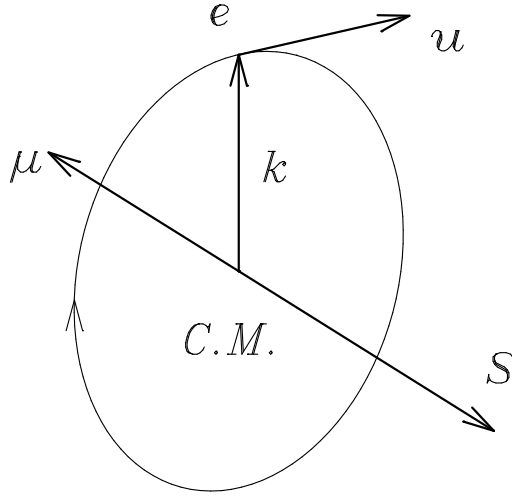
Figure 2.1: **Charge motion in the C.M. frame.**

This charge has an oscillatory motion of very high frequency ω that, in the case of the relativistic electron is $\omega = 2mc^2/\hbar \simeq 1.55 \times 10^{21}\mathrm{s}^{-1}$. The average position of the charge is the center of mass, but it is this internal orbital motion, usually known as the zitterbewegung, that gives rise to the spin structure for this model and also to the magnetic properties of the particle, as we shall see later.

When analyzed in the center of mass frame (see Fig. 2.1), $\boldsymbol{q} = 0$, $\boldsymbol{r} = \boldsymbol{k}$, the system reduces to a point charge whose motion is in general an ellipse, but if we choose $C = D$, and $\boldsymbol{C} \cdot \boldsymbol{D} = 0$, it reduces to a circle of radius $a = C = D$, orthogonal to the spin. Then if the particle has charge e, it has a magnetic moment that according to the usual classical definition is: [3]

$$\boldsymbol{\mu} = \frac{1}{2}\int \boldsymbol{r} \times \boldsymbol{j}\, d^3\boldsymbol{r} = \frac{e}{2}\,\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{S}_{CM}, \tag{2.73}$$

---

[3] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, NY (1998), 3rd. ed. p.186.

where $\boldsymbol{j} = e\delta^3(\boldsymbol{r} - \boldsymbol{k})d\boldsymbol{k}/dt$ is the current associated to the motion of a charge e located at point $\boldsymbol{k}$. The magnetic moment is orthogonal to the zitterbewegung plane and opposite to the spin if $e > 0$. It also has a non-vanishing oscillating electric dipole $\boldsymbol{d} = e\boldsymbol{k}$, orthogonal to $\boldsymbol{\mu}$ and therefore to $\boldsymbol{S}_{CM}$ in the center of mass frame, such that its time average value vanishes for times larger than the natural period of this internal motion. Although this is a nonrelativistic example it is interesting to point out and compare with Dirac's relativistic analysis of the electron, [4] in which both momenta $\boldsymbol{\mu}$ and $\boldsymbol{d}$ appear, giving rise to two possible interacting terms in Dirac's Hamiltonian. We shall come back to this analysis later when we study the elementary relativistic particles.

2.2.4   Spinning Galilei particle with orientation

Another simple example of spinning particles is the one in which the spin is related only to the angular variables that describe orientation.

Let us assume now a dynamical system whose kinematical space is $X = \mathcal{G}/\mathbb{R}_v^3$, where $\mathbb{R}_v^3 \equiv \{\mathbb{R}^3, +\}$ is the 3-parameter Abelian subgroup of pure Galilei transformations. Then, the kinematical variables are $x \equiv (t, \boldsymbol{r}, \boldsymbol{\rho})$, which are interpreted as the time, position and orientation respectively.

The Lagrangian for this model takes the general form

$$L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{W} \cdot \boldsymbol{\omega}.$$

Because of the structure of the exponent (2.169), the gauge function for this system can be taken the same as before. The general relationship (2.46) leads to $\boldsymbol{W} \times \boldsymbol{\omega} = 0$, because the Lagrangian is independent of $\dot{\boldsymbol{u}}$, and therefore $\boldsymbol{W}$ and $\boldsymbol{\omega}$ must be collinear. According to the transformation properties of the Lagrangian, the third term $\boldsymbol{W} \cdot \boldsymbol{\omega}$ is Galilei invariant and since $\boldsymbol{W}$ and $\boldsymbol{\omega}$ are collinear, we can take $\boldsymbol{W} \sim \boldsymbol{\omega}$ and one possible Lagrangian that describes this model is of the form:

$$L = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} + \frac{I}{2}\frac{\boldsymbol{\omega}^2}{\dot{t}}. \tag{2.74}$$

The different Noether's constants are

$$H = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{I}{2}\boldsymbol{\Omega}^2, \quad \boldsymbol{P} = m\boldsymbol{u},$$

$$\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t, \quad \boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{W},$$

where $\boldsymbol{u} = d\boldsymbol{r}/dt$ is the velocity of point $\boldsymbol{r}$, and $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is the time evolution angular velocity. Point $\boldsymbol{r}$ is moving at a constant speed and it also represents the position of the center of mass. The spin is just the observable $\boldsymbol{S} \equiv \boldsymbol{W}$ that satisfies the dynamical equation $d\boldsymbol{S}/dt = \boldsymbol{\omega} \times \boldsymbol{S} = 0$, and thus the frame linked to the body rotates with a constant angular velocity $\boldsymbol{\Omega}$.

The spin takes the constant value $\boldsymbol{S} = I\boldsymbol{\Omega}$, whose absolute value is independent of the inertial observer and also the angular velocity $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is constant. The parameter I plays the role of a principal moment of inertia, suggesting a linear relationship between the spin and the angular velocity, which corresponds to a particle with spherical symmetry. The particle can also be considered as an extended object of gyration radius R_0, related to the other particle parameters by $I = mR_0^2$.

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[4] P.A.M. Dirac, *The Principles of Quantum mechanics*, Oxford Univ. Press, 4th ed. (1967).

This system corresponds classically to a rigid body with spherical symmetry where the orientation variables $\boldsymbol{\rho}$ can describe for instance, the orientation of its principal axes of inertia in a suitable parametrization of the rotation group. This is a system of six degrees of freedom. Three represent the position of the center of charge $\boldsymbol{r}$ and the other three $\boldsymbol{\rho}$, represent the orientation of a Cartesian frame linked to that point $\boldsymbol{r}$. Since for this system there is no dependence on the acceleration, the centers of mass and charge will be represented by the same point.

In the center of mass frame there is no current associated to this particle and therefore it has neither magnetic nor electric dipole structure. As seen in previous examples, all magnetic properties seem therefore to be related to the zitterbewegung part of spin and are absent in this rigid body-like model.

Relativistic particles

2.3   Relativistic point particle

See the Appendix about the Poincaré group at the end of this chapter for the group notation used throughout this section.

The kinematical space is the quotient structure $X = \mathcal{P}/\mathcal{L}$, where $\mathcal{P}$ is the Poincaré group and the subgroup $\mathcal{L}$ is the Lorentz group. Then every point $x \in X$ is characterized by the variables $x \equiv (t(\tau), \boldsymbol{r}(\tau))$, with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$ as the corresponding group parameters, in such a way that under the action of a group element $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ of $\mathcal{P}$ they transform as:

$$
\begin{aligned}
t'(\tau) &= \gamma t(\tau) + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))/c^2 + b, & (2.75)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{r}'(\tau) &= R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \gamma \boldsymbol{v} t(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))\boldsymbol{v} + \boldsymbol{a}, & (2.76)
\end{aligned}
$$

and are interpreted as the time and position of the system. If, as usual, we assume that the evolution parameter τ is invariant under the group, taking the τ-derivatives of (2.75) and (2.76) we get

$$
\begin{aligned}
\dot{t}'(\tau) &= \gamma \dot{t}(\tau) + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau))/c^2, & (2.77)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{r}}'(\tau) &= R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \gamma \boldsymbol{v} \dot{t}(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau))\boldsymbol{v}. & (2.78)
\end{aligned}
$$

The homogeneity condition of the Lagrangian, in terms of the derivatives of the kinematical variables, reduces to three the number of degrees of freedom of the system. This leads to the general expression

$$
L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}}, \tag{2.79}
$$

where $T = \partial L/\partial \dot{t}$ and $R_i = \partial L/\partial \dot{r}_i$, will be functions of t and $\boldsymbol{r}$ and homogeneous functions of zero degree of $\dot{t}(\tau)$ and $\dot{\boldsymbol{r}}(\tau)$. Because the Lagrangian is invariant under $\mathcal{P}$, the functions T and $\boldsymbol{R}$ transform under the group $\mathcal{P}$ in the form:

$$
\begin{aligned}
T' &= \gamma T - \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{R}), & (2.80)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{R}' &= R(\boldsymbol{\mu})\boldsymbol{R} - \gamma \boldsymbol{v} T/c^2 + \frac{\gamma^2}{1+\gamma}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{R})\boldsymbol{v}/c^2. & (2.81)
\end{aligned}
$$

We thus see that T and $\boldsymbol{R}$ are invariant under translations and therefore they must be functions independent of t and $\boldsymbol{r}$.

The conjugate momenta of the independent degrees of freedom $q_i = r_i$ are $p_i = \partial L/\partial \dot{r}_i$, and consequently Noether's theorem (1.25) leads to the following constants of the motion, that are calculated similarly as in the Galilei case except for the invariance under pure Lorentz transformations. We have now no gauge function and the variations are $\delta t = \boldsymbol{r} \cdot \delta \boldsymbol{v}/c^2$, $M_i = r_i/c^2$ and $\delta \boldsymbol{r} = t\delta \boldsymbol{v}$, $M_{ij} = t\delta_{ij}$ and thus we get:

$$\text{Energy} \quad H \;=\; -T, \tag{2.82}$$

$$\text{linear momentum} \quad \boldsymbol{P} \;=\; \boldsymbol{R} = \boldsymbol{p}, \tag{2.83}$$

$$\text{kinematical momentum} \quad \boldsymbol{K} \;=\; H\boldsymbol{r}/c^2 - \boldsymbol{P}t, \tag{2.84}$$

$$\text{angular momentum} \quad \boldsymbol{J} \;=\; \boldsymbol{r} \times \boldsymbol{P}. \tag{2.85}$$

The energy and the linear momentum transform as:

$$H'(\tau) \;=\; \gamma H(\tau) + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{P}(\tau)), \tag{2.86}$$

$$\boldsymbol{P}'(\tau) \;=\; R(\boldsymbol{\mu})\boldsymbol{P}(\tau) + \frac{\gamma\boldsymbol{v}}{c^2}H(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{P}(\tau))\boldsymbol{v}. \tag{2.87}$$

They transform like the contravariant components of a four-vector $P^\mu \equiv (H/c, \boldsymbol{P})$. The observables $c\boldsymbol{K}$ and $\boldsymbol{J}$ are the essential components of the antisymmetric tensor $J^{\mu\nu} = -J^{\nu\mu} = x^\mu P^\nu - x^\nu P^\mu$, $cK_i = J^{i0}$ and $J_k = \epsilon_{kil}J^{il}/2$.

Taking the τ derivative of the kinematical momentum, $\dot{\boldsymbol{K}} = 0$, we get $\boldsymbol{P} = H\dot{\boldsymbol{r}}/c^2\dot{t} = H\boldsymbol{u}/c^2$, where $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$ is the velocity of the particle and the point $\boldsymbol{r}$ represents both the center of mass and center of charge position of the particle.

The six conditions $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$, imply $\boldsymbol{u} = 0$ and $\boldsymbol{r} = 0$, so that the system is at rest and placed at the origin of the reference frame, similarly as in the nonrelativistic case. We again call this class of observers the center of mass observer.

From (2.86) and (2.87) we see that the magnitude $(H/c)^2 - \boldsymbol{P}^2 = m^2c^2$ is a Poincaré invariant and a constant of the motion. This defines the mass of the particle which we take as a positive number m. By using the expression of $\boldsymbol{P} = H\boldsymbol{u}/c^2$, we get

$$H = \pm mc^2(1 - u^2/c^2)^{-1/2},$$

and the sign of H, which is another Poincaré invariant property, can be either positive or negative. The velocity $u < c$, otherwise H will be imaginary. If $u > c$ the invariant $(H/c)^2 - P^2 < 0$ and it is not possible to define the rest mass of the system. By substitution of the found expressions for T and $\boldsymbol{R}$ in (2.79), there are two possible Lagrangians for a point particle of mass m, characterized by the sign of H

$$L = \mp mc\sqrt{c^2\dot{t}^2 - \dot{\boldsymbol{r}}^2}. \tag{2.88}$$

Expansion of this Lagrangian to lowest order in u/c, in the case of positive H, we get

$$L = -mc^2\dot{t} + \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}},$$

where the first term $-mc^2\dot{t}$ that can be withdrawn is just the equivalent to the Galilei internal energy term $-H_0\dot{t}$ of (2.17). The Lagrangian with $H < 0$ has as nonrelativistic limit $-(m/2)\dot{\boldsymbol{r}}^2/\dot{t}$ which is not obtained in the Galilei case.

The spin of this system, defined similarly as in the nonrelativistic case,

$$\boldsymbol{S} \equiv \boldsymbol{J} - \boldsymbol{q} \times \boldsymbol{P} = \boldsymbol{J} - \frac{c^2}{H} \boldsymbol{K} \times \boldsymbol{P} = 0, \tag{2.89}$$

vanishes, so that the relativistic point particle is also a spinless system.

2.4 Relativistic spinning particles

There are three maximal homogeneous spaces of $\mathcal{P}$, all of them at first parameterized by the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$, where the velocity variable $\boldsymbol{u}$ can be either $u < c$, $u = c$ or $u > c$. We shall call these kinds of particles by the following names: The first one, since the motion of the position of the charge $\boldsymbol{r}$ satisfies $u < c$, we call a **Bradyon**, from the Greek term $\beta\rho\alpha\delta\upsilon\varsigma \equiv$ slow. Bradyons are thus particles for which point $\boldsymbol{r}$ never reaches the speed of light. The second class of particles ($u = c$) will be called **Luxons** because point $\boldsymbol{r}$ is always moving at the speed of light for every observer, and finally those of the third group, because $u > c$, are called **Tachyons**, from the Greek $\tau\alpha\chi\upsilon\varsigma \equiv$ fast.

For the second class we use the Latin denomination Luxons in spite of the Greek one of photons, because this class of particles will supply the description not only of classical photons but also a classical model of the electron. This class of models is very important and it has no nonrelativistic limit. Therefore the models this manifold produce have no nonrelativistic equivalent.

The first class corresponds to a kinematical space that is the Poincaré group itself and produces models equivalent to the ones analyzed in the non-relativistic case. To describe the classical electron and the photon we shall consider next the case of luxons.

2.5 Luxons

Let us consider those mechanical systems whose kinematical space is the manifold X generated by the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$ with domains $t \in \mathbb{R}$, $\boldsymbol{r} \in \mathbb{R}^3$, $\boldsymbol{\rho} \in \mathbb{R}_c^3$ as in the previous case, and $\boldsymbol{u} \in \mathbb{R}^3$ but now with $u = c$. Since $u = c$ we shall call this kind of particles **Luxons**. This manifold is in fact a homogeneous space of the Poincaré group $\mathcal{P}$, and therefore, according to our definition of elementary particle has to be considered as a possible candidate for describing the kinematical space of an elementary system. In fact, if we consider the point in this manifold $x \equiv (0, 0, \boldsymbol{u}, 0)$, the little group that leaves x invariant is the one-parameter subgroup $\mathcal{V}_u$ of pure Lorentz transformations in the direction of the vector $\boldsymbol{u}$. Then $X \sim \mathcal{P}/\mathcal{V}_u$, is a nine-dimensional homogeneous space.

For this kind of systems the variables t, $\boldsymbol{r}$ transform according to (2.75) and (2.76), respectively and the derivatives as in (2.77) and (2.78). For the velocity $\boldsymbol{u}$ the transformation is obtained from (2.180) and is

$$\boldsymbol{u}'(\tau) = \frac{R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \gamma\boldsymbol{v} + \dfrac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau))\boldsymbol{v}}{\gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\,\boldsymbol{u}(\tau)/c^2)}. \tag{2.90}$$

The general transformation of the orientation variables $\boldsymbol{\rho}$ are obtained from (2.181) but now the functions $\boldsymbol{F}$ and G, which involve some $\gamma(u)$ factors, become infinite and in the limit $u \to c$

they take the form

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu} \times \boldsymbol{\rho}(\tau) + \boldsymbol{F}_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}(\tau), \boldsymbol{\rho}(\tau))}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau) + G_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}(\tau), \boldsymbol{\rho}(\tau))}, \tag{2.91}$$

where the functions $\boldsymbol{F}_c$ and G_c are given now by:

$$\begin{aligned}
\boldsymbol{F}_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}, \boldsymbol{\rho}) = \ & \frac{\gamma(v)}{(1 + \gamma(v))c^2} \left[\boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{\mu}) + \boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{\rho}) \right. \\
& + \ \boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{\mu}) + (\boldsymbol{u} \times \boldsymbol{\rho}) \times \boldsymbol{v} + (\boldsymbol{u} \cdot \boldsymbol{\rho})(\boldsymbol{v} \times \boldsymbol{\mu}) \\
& + \ \left. (\boldsymbol{u} \times \boldsymbol{\rho})(\boldsymbol{v} \cdot \boldsymbol{\mu}) + (\boldsymbol{u} \times \boldsymbol{\rho}) \times (\boldsymbol{v} \times \boldsymbol{\mu}) \right],
\end{aligned} \tag{2.92}$$

$$\begin{aligned}
G_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}, \boldsymbol{\rho}) = \ & \frac{\gamma(v)}{(1 + \gamma(v))c^2} \left[\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{\mu}) + \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{\rho}) \right. \\
& - \ \left. (\boldsymbol{u} \cdot \boldsymbol{\rho})(\boldsymbol{v} \cdot \boldsymbol{\mu}) + (\boldsymbol{u} \times \boldsymbol{\rho}) \cdot (\boldsymbol{v} \times \boldsymbol{\mu}) \right].
\end{aligned} \tag{2.93}$$

Since $u' = u = c$, the absolute value of the velocity vector is conserved and it means that $\boldsymbol{u}'$ can be obtained from $\boldsymbol{u}$ by an orthogonal transformation, so that the transformation equations of the velocity under $\mathcal{P}$ can be expressed as:

$$\boldsymbol{u}' = R(\boldsymbol{\phi})\boldsymbol{u}, \tag{2.94}$$

where the kinematical rotation of parameter $\boldsymbol{\phi}$ is

$$\boldsymbol{\phi} = \frac{\boldsymbol{\mu} + \boldsymbol{F}_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}(\tau), 0)}{1 + G_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}(\tau), 0)}. \tag{2.95}$$

In this case there also exist among the kinematical variables the constraints $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$.

Equation (2.91) also corresponds to

$$R(\boldsymbol{\rho}') = R(\boldsymbol{\phi})R(\boldsymbol{\rho}), \tag{2.96}$$

with the same $\boldsymbol{\phi}$ in both cases, as in (2.95).

Since the variable $u(\tau) = c$, during the whole evolution, we can distinguish two different kinds of systems, because, by taking the derivative with respect to τ of this expression we get $\dot{\boldsymbol{u}}(\tau) \cdot \boldsymbol{u}(\tau) = 0$, i.e., systems for which $\dot{\boldsymbol{u}} = 0$ or massless systems as we shall see, and systems where $\dot{\boldsymbol{u}} \neq 0$ but always orthogonal to $\boldsymbol{u}$. These systems will correspond to massive particles whose charge internal motion occurs at the constant velocity c, although their center of mass moves with velocity below c.

2.5.1 Massless particles. (The photon)

If $\dot{\boldsymbol{u}} = 0$, $\boldsymbol{u}$ is constant and the system follows a straight trajectory with constant velocity, and therefore the kinematical variables reduce simply to $(t, \boldsymbol{r}, \boldsymbol{\rho})$ with domains and physical meaning as usual as, time, position and orientation, respectively. The derivatives $\dot{t}$ and $\dot{\boldsymbol{r}}$ transform like (2.77) and (2.78) and instead of the variable $\dot{\boldsymbol{\rho}}$ we shall consider the linear function $\boldsymbol{\omega}$ defined in (2.29) that transforms under $\mathcal{P}$:

$$\boldsymbol{\omega}'(\tau) = R(\boldsymbol{\phi})\boldsymbol{\omega}(\tau), \tag{2.97}$$

where, again, $\boldsymbol{\phi}$ is given by (2.95).

In fact, from (2.96), since $\dot{\boldsymbol{u}} = 0$, taking the τ-derivative,

$$\dot{R}(\boldsymbol{\rho}') = R(\boldsymbol{\phi})\dot{R}(\boldsymbol{\rho}),$$

the antisymmetric matrix $\Omega = \dot{R}(\boldsymbol{\rho})R^T(\boldsymbol{\rho})$ has as essential components the angular velocity $\boldsymbol{\omega}$,

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{2.98}$$

It transforms as

$$\Omega' = \dot{R}(\boldsymbol{\rho}')R^T(\boldsymbol{\rho}') = R(\boldsymbol{\phi})\dot{R}(\boldsymbol{\rho})R^T(\boldsymbol{\rho})R^T(\boldsymbol{\phi}) = R(\boldsymbol{\phi})\Omega R^T(\boldsymbol{\phi}),$$

and this matrix transformation leads for its essential components to (2.97).

For this system there are no constraints among the kinematical variables, and, since $\dot{\boldsymbol{u}} = 0$, the general form of its Lagrangian is

$$L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{W} \cdot \boldsymbol{\omega}. \tag{2.99}$$

Funtions $T = \partial L/\partial \dot{t}$, $R_i = \partial L/\partial \dot{r}^i$, $W_i = \partial L/\partial \omega^i$, will depend on the variables $(t, \boldsymbol{r}, \boldsymbol{\rho})$ and are homogeneous functions of zero degree in terms of the derivatives of the kinematical variables $(\dot{t}, \dot{\boldsymbol{r}}, \boldsymbol{\omega})$. Since $\dot{t} \neq 0$ they will be expressed in terms of $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t}$ and $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$, which are the true velocity and angular velocity of the particle respectively.

Invariance of the Lagrangian under $\mathcal{P}$ leads to the following transformation form of these functions under the group $\mathcal{P}$:

$$T' = \gamma T - \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{R}), \tag{2.100}$$

$$\boldsymbol{R}' = R(\boldsymbol{\mu})\boldsymbol{R} - \gamma \boldsymbol{v}T/c^2 + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{R})\boldsymbol{v}, \tag{2.101}$$

$$\boldsymbol{W}' = R(\boldsymbol{\phi})\boldsymbol{W}. \tag{2.102}$$

They are translation invariant and therefore independent of t and $\boldsymbol{r}$. They will be functions of only $(\boldsymbol{\rho}, \boldsymbol{u}, \boldsymbol{\Omega})$, with the constraint $u = c$. Invariance under rotations forbids the explicit dependence on $\boldsymbol{\rho}$, so that the dependence of these functions on $\boldsymbol{\rho}$ and $\dot{\boldsymbol{\rho}}$ variables is only through the angular velocity $\boldsymbol{\omega}$.

Noether's theorem gives rise, as before, to the following constants of the motion:

$$\begin{aligned} \text{Energy} \quad H &= -T, & (2.103) \\ \text{linear momentum} \quad \boldsymbol{P} &= \boldsymbol{R}, & (2.104) \\ \text{kinematical momentum} \quad \boldsymbol{K} &= H\boldsymbol{r}/c^2 - \boldsymbol{P}t - \boldsymbol{W} \times \boldsymbol{u}/c^2, & (2.105) \\ \text{angular momentum} \quad \boldsymbol{J} &= \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{W}. & (2.106) \end{aligned}$$

In this case the system has no zitterbewegung because the Lagrangian does not depend on $\dot{\boldsymbol{u}}$ which vanishes. The particle, located at point $\boldsymbol{r}$, is moving in a straight trajectory at the speed of light and therefore it is not possible to find an inertial rest frame observer. Although we have no center of mass observer, we define the spin by $\boldsymbol{S} = \boldsymbol{J} - \boldsymbol{r} \times \boldsymbol{P} = \boldsymbol{W}$.

If we take in (2.106) the τ-derivative we get $d\boldsymbol{S}/dt = \boldsymbol{P} \times \boldsymbol{u}$. Since $\boldsymbol{P}$ and $\boldsymbol{u}$ are two non-vanishing constant vectors, then the spin has a constant time derivative. It represents a system with a continuously increasing angular momentum. This is not what we understand by

an elementary particle except if this constant $d\boldsymbol{S}/dt = 0$. Therefore for this system the spin is a constant of the motion and $\boldsymbol{P}$ and $\boldsymbol{u}$ are collinear vectors.

Energy and linear momentum are in fact the components of a four-vector and with the spin they transform as

$$H' = \gamma H + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{P}), \tag{2.107}$$

$$\boldsymbol{P}' = R(\boldsymbol{\mu})\boldsymbol{P} + \gamma\boldsymbol{v}H/c^2 + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{P})\boldsymbol{v}, \tag{2.108}$$

$$\boldsymbol{S}' = R(\boldsymbol{\phi})\boldsymbol{S}. \tag{2.109}$$

The relation between $\boldsymbol{P}$ and $\boldsymbol{u}$ can be obtained from (2.105), taking the τ-derivative and the condition that the spin $\boldsymbol{W}$ is constant, $\dot{\boldsymbol{K}} = 0 = -H\dot{\boldsymbol{r}}/c^2 + \boldsymbol{P}\dot{t}$, i.e., $\boldsymbol{P} = H\boldsymbol{u}/c^2$. If we take the scalar product of this expression with $\boldsymbol{u}$ we also get $H = \boldsymbol{P} \cdot \boldsymbol{u}$.

Then, from (2.107) and (2.108), an invariant and constant of the motion, which vanishes, is $(H/c)^2 - \boldsymbol{P}^2$. The mass of this system is zero. It turns out that for this particle both H and $\boldsymbol{P}$ are non-vanishing for every inertial observer. Otherwise, if one of them vanishes for a single observer they vanish for all of them. By (2.109), S^2 is another Poincaré invariant property of the system that is also a constant of the motion.

The first part of the Lagrangian $T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} = -H\dot{t} + \boldsymbol{P} \cdot \dot{\boldsymbol{r}}$, which can be written as $-(H - \boldsymbol{P} \cdot \boldsymbol{u})\dot{t} = 0$, also vanishes. Then the Lagrangian is reduced to the third term $\boldsymbol{S} \cdot \boldsymbol{\omega}$.

We see from (2.94) and (2.109) that the dimensionless magnitude $\epsilon = \boldsymbol{S} \cdot \boldsymbol{u}/Sc$ is another invariant and constant of the motion, and we thus expect that the Lagrangian will be explicitly dependent on both constant parameters S and ϵ. Taking into account the transformation properties under $\mathcal{P}$ of $\boldsymbol{u}$, $\boldsymbol{\omega}$ and $\boldsymbol{S}$, given in (2.94), (2.97) and (2.109) respectively, it turns out that the spin must necessarily be a vector function of $\boldsymbol{u}$ and $\boldsymbol{\omega}$.

If the spin is not transversal, as it happens for real photons, then $\boldsymbol{S} = \epsilon S\boldsymbol{u}/c$ where $\epsilon = \pm 1$, and thus the Lagrangian finally becomes:

$$L = \left(\frac{\epsilon S}{c}\right) \frac{\dot{\boldsymbol{r}} \cdot \boldsymbol{\omega}}{\dot{t}}. \tag{2.110}$$

From this Lagrangian we get that the energy is $H = -\partial L/\partial \dot{t} = \boldsymbol{S} \cdot \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ is the angular velocity of the particle. The linear momentum is $\boldsymbol{P} = \partial L/\partial \dot{\boldsymbol{r}} = \epsilon S\boldsymbol{\Omega}/c$, and, since $\boldsymbol{P}$ and $\boldsymbol{u}$ are parallel vectors, $\boldsymbol{\Omega}$ and $\boldsymbol{u}$ must also be parallel, and if the energy is definite positive, then $\boldsymbol{\Omega} = \epsilon\Omega\boldsymbol{u}/c$.

This means that the energy $H = S\Omega$. For photons we know that $S = \hbar$, and thus $H = \hbar\Omega = h\nu$. In this way the frequency of a photon is the frequency of its rotational motion around the direction of its trajectory. We thus see that the spin and angular velocity for $H > 0$ particles have the same direction, although they are not analytically related, because S is invariant under $\mathcal{P}$ while Ω is not.

We say that the Lagrangian (2.110) represents a photon of spin S and polarization ϵ. A set of photons of this kind, all with the same polarization, corresponds to circularly polarized light, as has been shown by direct measurement of the angular momentum carried by these photons. [5] Left and right polarized photons correspond to $\epsilon = 1$ and $\epsilon = -1$, respectively. Energy is related to the angular frequency $H = \hbar\Omega$, and linear momentum to the wave number $\boldsymbol{P} = \hbar\boldsymbol{k}$, that therefore is related to the angular velocity vector by $\boldsymbol{k} = \epsilon\boldsymbol{\Omega}/c$. If it is possible to talk about the 'wave-length' of a single photon this will be the distance run by the particle during a complete turn.

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[5] R. A. Beth, *Phys. Rev.* **50**, 115 (1936).

2.5.2 Massive particles. (The electron)

If we consider now the other possibility, $\dot{\boldsymbol{u}} \neq 0$ but orthogonal to $\boldsymbol{u}$, then variables $\dot{t}$ and $\dot{\boldsymbol{r}}$ transform as in the previous case (2.77) and (2.78), but for $\dot{\boldsymbol{u}}$ and $\boldsymbol{\omega}$ we have:

$$\dot{\boldsymbol{u}}' = R(\boldsymbol{\phi})\dot{\boldsymbol{u}} + \dot{R}(\boldsymbol{\phi})\boldsymbol{u}, \tag{2.111}$$

$$\boldsymbol{\omega}' = R(\boldsymbol{\phi})\boldsymbol{\omega} + \boldsymbol{\omega}_\phi, \tag{2.112}$$

where the rotation of parameter $\boldsymbol{\phi}$ is again given by (2.95) and vector $\boldsymbol{\omega}_\phi$ is:

$$\boldsymbol{\omega}_\phi = \frac{\gamma R\boldsymbol{u} \times \boldsymbol{v} - (\gamma - 1)R(\boldsymbol{u} \times \dot{\boldsymbol{u}}) + 2\gamma^2(\boldsymbol{v} \cdot R(\boldsymbol{u} \times \dot{\boldsymbol{u}}))\boldsymbol{v}/(1 + \gamma)c^2}{\gamma(c^2 + \boldsymbol{v} \cdot R\boldsymbol{u})}. \tag{2.113}$$

Expression (2.111) is the τ-derivative of (2.94) and can also be written in the form:

$$\dot{\boldsymbol{u}}' = \frac{R(\boldsymbol{\phi})\dot{\boldsymbol{u}}}{\gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)}. \tag{2.114}$$

Expression (2.112) comes from $R(\boldsymbol{\rho}') = R(\boldsymbol{\phi})R(\boldsymbol{\rho})$ and taking the τ-derivative of this expression $\dot{R}(\boldsymbol{\rho}') = \dot{R}(\boldsymbol{\phi})R(\boldsymbol{\rho}) + R(\boldsymbol{\phi})\dot{R}(\boldsymbol{\rho})$, because parameter $\boldsymbol{\phi}$ depends on τ through the velocity $\boldsymbol{u}(\tau)$, and therefore

$$\Omega' = \dot{R}(\boldsymbol{\rho}')R^T(\boldsymbol{\rho}') = R(\boldsymbol{\phi})\Omega R^T(\boldsymbol{\phi}) + \dot{R}(\boldsymbol{\phi})R^T(\boldsymbol{\phi}).$$

$R(\boldsymbol{\phi})\Omega R^T(\boldsymbol{\phi})$ corresponds to $R(\boldsymbol{\phi})\boldsymbol{\omega}$ and the antisymmetric matrix $\Omega_\phi = \dot{R}(\boldsymbol{\phi})R^T(\boldsymbol{\phi})$ has as essential components the $\boldsymbol{\omega}_\phi$ vector, *i.e.*, equation (2.113).

The homogeneity condition of the Lagrangian leads to the general form

$$L = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{U} \cdot \dot{\boldsymbol{u}} + \boldsymbol{W} \cdot \boldsymbol{\omega}, \tag{2.115}$$

where $T = \partial L/\partial \dot{t}$, $R_i = \partial L/\partial \dot{r}^i$, $U_i = \partial L/\partial \dot{u}^i$ and $W_i = \partial L/\partial \omega^i$, and Noether's theorem provides the following constants of the motion:

$$\begin{aligned}
\text{Energy} \quad H &= -T - (d\boldsymbol{U}/dt) \cdot \boldsymbol{u}, & (2.116)\\
\text{linear momentum} \quad \boldsymbol{P} &= \boldsymbol{R} - (d\boldsymbol{U}/dt), & (2.117)\\
\text{kinematical momentum} \quad \boldsymbol{K} &= H\boldsymbol{r}/c^2 - \boldsymbol{P}t - \boldsymbol{S} \times \boldsymbol{u}/c^2, & (2.118)\\
\text{angular momentum} \quad \boldsymbol{J} &= \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}. & (2.119)
\end{aligned}$$

In this case the spin $\boldsymbol{S}$, i.e. the angular momentum with respecto to the point $\boldsymbol{r}$, is defined as in the Galilei case, by

$$\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W}. \tag{2.120}$$

Expressions (2.116, 2.117) imply that H/c and $\boldsymbol{P}$ transform like the components of a four-vector, similarly as in (2.86-2.87), thus defining the invariant and constant of the motion $(H/c)^2 - \boldsymbol{P}^2 = m^2c^2$, in terms of the positive parameter m which is interpreted as the mass of the particle.

Observable $\boldsymbol{S}$ transforms as:

$$\boldsymbol{S}'(\tau) = \gamma R(\boldsymbol{\mu})\boldsymbol{S}(\tau) - \frac{\gamma^2}{(1 + \gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{S}(\tau))\boldsymbol{v} + \frac{\gamma}{c^2}(\boldsymbol{v} \times R(\boldsymbol{\mu})(\boldsymbol{S}(\tau) \times \boldsymbol{u})), \tag{2.121}$$

an expression that corresponds to the transformation of an antisymmetric tensor $S^{\mu\nu}$ with strict components $S^{0i} = (\boldsymbol{S} \times \boldsymbol{u})^i/c$, and $S^{ij} = \epsilon^{ijk}S_k$.

By defining the relative position vector $\boldsymbol{k} = \boldsymbol{S} \times \boldsymbol{u}/H$, the kinematical momentum (2.118) can be cast into the form

$$\boldsymbol{K} = H\boldsymbol{q}/c^2 - \boldsymbol{P}\,t,$$

where $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$, represents the position of the center of mass of the particle.

The spin with respect to the center of mass, is defined as usual

$$\boldsymbol{S}_{CM} = \boldsymbol{J} - \boldsymbol{q} \times \boldsymbol{P} = \boldsymbol{J} - \frac{c^2}{H}\boldsymbol{K} \times \boldsymbol{P}, \tag{2.122}$$

and is a constant of the motion. It takes the form

$$\boldsymbol{S}_{CM} = \boldsymbol{S} + \boldsymbol{k} \times \boldsymbol{P} = \boldsymbol{S} + \frac{1}{H}(\boldsymbol{S} \times \boldsymbol{u}) \times \boldsymbol{P}. \tag{2.123}$$

The helicity $\boldsymbol{S}_{CM} \cdot \boldsymbol{P} = \boldsymbol{S} \cdot \boldsymbol{P} = \boldsymbol{J} \cdot \boldsymbol{P}$, is also a constant of the motion. We can construct the constant Pauli-Lubanski four-vector

$$w^\mu \equiv (\boldsymbol{P} \cdot \boldsymbol{S}_{CM}, H\boldsymbol{S}_{CM}/c), \tag{2.124}$$

with $-w^\mu w_\mu = m^2 c^2 S^2$, in terms of the invariant properties m and S of the particle.

If we take in (2.118) the τ-derivative and the scalar product with the velocity $\boldsymbol{u}$ we get the Poincaré invariant relation:

$$H = \boldsymbol{P} \cdot \boldsymbol{u} + \frac{1}{c^2}\boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u}\right). \tag{2.125}$$

This will give rise to Dirac's Hamiltonian, $H = c\boldsymbol{P} \cdot \boldsymbol{\alpha} + \beta mc^2$ when expressed in the quantum case, in terms of the $\boldsymbol{\alpha}$ and β Dirac matrices. Since $c\boldsymbol{\alpha}$ is usually interpreted as the local velocity operator $\boldsymbol{u}$ of the electron, [6] we have $H = \boldsymbol{P} \cdot \boldsymbol{u} + \beta mc^2$ and this relation suggests the identification

$$\beta = \frac{1}{mc^4}\boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u}\right).$$

Here all magnitudes on the right-hand side are measured in the center of mass frame. We shall come back to this relation after quantization of this system.

The center of mass observer is defined by the conditions $\boldsymbol{P} = \boldsymbol{K} = 0$. For this observer $\boldsymbol{S} = \boldsymbol{S}_{CM}$ is constant, $H = mc^2$ and thus from (2.118) we get

$$\boldsymbol{r} = \frac{1}{mc^2}\,\boldsymbol{S} \times \boldsymbol{u}, \tag{2.126}$$

and the internal motion takes place in a plane orthogonal to the constant spin $\boldsymbol{S}$. The scalar product with $\boldsymbol{u}$ leads to $\boldsymbol{r} \cdot d\boldsymbol{r}/dt = 0$, and thus the zitterbewegung radius is a constant. Taking the time derivative of both sides of (2.126), we obtain $mc^2\boldsymbol{u} = (\boldsymbol{S} \times d\boldsymbol{u}/dt)$, because the spin is constant in this frame, we get that $\boldsymbol{u}$ and $\boldsymbol{S}$ are orthogonal and therefore

$$\boldsymbol{S} = m\boldsymbol{u} \times \boldsymbol{r}. \tag{2.127}$$

Since S and $u = c$ are constant, the motion is a circle of radius $R_0 = S/mc$. For the electron we take $S = \hbar/2$, and the radius is $\hbar/2m_e c = 1.93 \times 10^{-13}$ m., half the Compton wave length

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[6] J.J. Sakurai, *Advanced Quantum Mechanics*, Addison-Wesley Reading, MA (1967).

of the electron. The frequency of this motion in the C.M. frame is $\nu = 2m_e c^2/h = 2.47 \times 10^{20}$ s^{-1}, and $\omega = 2\pi\nu = 1.55 \times 10^{21}$ rad s^{-1}. The ratio of this radius to the so-called classical radius $R_{cl} = e^2/8\pi\varepsilon_0 m_e c^2 = 1.409 \times 10^{-15}$ m, is precisely $R_{cl}/R_0 = e^2/2\varepsilon_0 hc = 1/136.97 = \alpha$, the fine structure constant.

Motions of this sort, in which the particle is moving at the speed of light, can be found in early literature, but the distinction between the motion of center of charge and center of mass is not sufficiently clarified. [7, 8]

Nevertheless, in the model we are analyzing, the idea that the electron has a size of the order of the zitterbewegung radius is a plausible macroscopic vision but is not necessary to maintain any longer, because the only important point from the dynamical point of view is the center of charge position, whose motion completely determines the dynamics of the system. In this form, elementary particles, the kind of objects we are describing, look like extended objects. Nevertheless, although some kind of related length can be defined, they are dealt with as point particles with orientation because the physical attributes are all located at the single point $\boldsymbol{r}$. The dynamics of equation (2.126) can be represented in figure2.2 where we have separated the two contributions to the total spin $\boldsymbol{S} = \boldsymbol{S}_u + \boldsymbol{S}_\omega$, related respectively to the orbital and rotational motion.
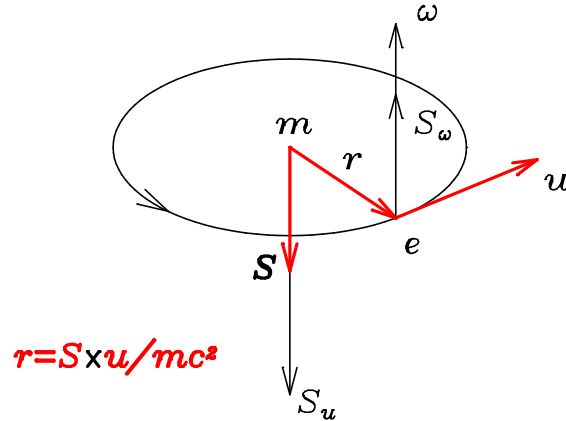

Figure 2.2: **Motion of the charge of the electron in the center of mass frame.**

The transformation equation for the function $\boldsymbol{S}$, (2.121) can also be written as

$$\boldsymbol{S}' = \gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)R(\boldsymbol{\phi})\boldsymbol{S}, \tag{2.128}$$

and therefore $\boldsymbol{S} \cdot \dot{\boldsymbol{u}} = \boldsymbol{S}' \cdot \dot{\boldsymbol{u}}'$ and $\boldsymbol{S}' \cdot \boldsymbol{u}' = \gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}/c^2)\boldsymbol{S} \cdot \boldsymbol{u}$. Since it is orthogonal to $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$, for the center of mass observer, it is also orthogonal to $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ for any other inertial observer.

[7] M. Mathisson, *Acta Phys. Pol.* **6**, 163 (1937); **6**, 218 (1937)

[8] M.H.L. Weyssenhof, *Acta Phys. Pol.* **9**, 46 (1947). M.H.L. Weyssenhof and A. Raabe, *Acta Phys. Pol.* **9**, 7 (1947); **9**, 19 (1947).

An alternative method of verifying this is to take the time derivative in (2.118) and (2.119), and thus

$$H\boldsymbol{u} - c^2\boldsymbol{P} - \frac{d\boldsymbol{S}}{dt}\times\boldsymbol{u} - \boldsymbol{S}\times\frac{d\boldsymbol{u}}{dt} = 0,$$

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P}\times\boldsymbol{u},$$

i.e.,

$$\boldsymbol{S}\times\frac{d\boldsymbol{u}}{dt} = (H - \boldsymbol{u}\cdot\boldsymbol{P})\boldsymbol{u}.$$

and a final scalar product with $\boldsymbol{S}$, leads to $(H - \boldsymbol{u}\cdot\boldsymbol{P})\boldsymbol{u}\cdot\boldsymbol{S} = 0$. The first factor does not vanish since the invariant $H^2/c^2 - P^2 = m^2 c^2$ is positive definite and if $H = \boldsymbol{u}\cdot\boldsymbol{P}$, then $(\boldsymbol{u}\cdot\boldsymbol{P})^2/c^2 - P^2$ with $u \le c$ is always negative, then $\boldsymbol{S}\cdot\boldsymbol{u} = 0$. If we take the time derivative of this last expression, with the condition that $d\boldsymbol{S}/dt$ is orthogonal to $\boldsymbol{u}$, we obtain $\boldsymbol{S}\cdot\dot{\boldsymbol{u}} = 0$. The observable $\boldsymbol{S}$ has always the direction of the non-vanishing vector $\dot{\boldsymbol{u}}\times\boldsymbol{u}$ for positive energy particles and the opposite direction for particles of negative energy.

Equation (2.125) can be recast into the form

$$\frac{H}{c}c\dot{t} - \boldsymbol{P}\cdot\dot{\boldsymbol{r}} - \frac{1}{c^2}\boldsymbol{S}\cdot(\dot{\boldsymbol{u}}\times\boldsymbol{u}) = 0,$$

where the first two terms give rise to the invariant term $P_\mu \dot{x}^\mu = mc^2 \dot{t}_{cm}$, and the third to the invariant relation

$$\boldsymbol{S}\cdot(\dot{\boldsymbol{u}}\times\boldsymbol{u}) = mc^4 \dot{t}_{cm}. \tag{2.129}$$

Here t_{cm} is the time observable measured in the center of mass frame, and the right-hand side, which is positive definite for particles, implies that $\boldsymbol{S}$ has precisely the direction of $\dot{\boldsymbol{u}}\times\boldsymbol{u}$. In the case of antiparticles it has the opposite direction.

If we take the time derivative of the kinematical momentum (2.118) for the free particle, we get

$$H\boldsymbol{u} - c^2\boldsymbol{P} + \frac{d\boldsymbol{S}}{dt}\times\boldsymbol{u} + \boldsymbol{S}\times\frac{d\boldsymbol{u}}{dt} = 0.$$

Taking into account that $d\boldsymbol{S}/dt = \boldsymbol{P}\times\boldsymbol{u}$ and making a cross product with $d\boldsymbol{u}/dt$ we get

$$\boldsymbol{S} = \left(\frac{H - \boldsymbol{u}\cdot\boldsymbol{P}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}\times\boldsymbol{u}, \tag{2.130}$$

and $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{S}\times\boldsymbol{u}/H$ leads for the center of mass position to

$$\boldsymbol{q} = \boldsymbol{r} + \frac{c^2}{H}\left(\frac{H - \boldsymbol{u}\cdot\boldsymbol{P}}{(d\boldsymbol{u}/dt)^2}\right)\frac{d\boldsymbol{u}}{dt}. \tag{2.131}$$

We see that the particle has mass and spin, and the center of charge moves in circles at the speed of light in a plane orthogonal to the spin, for the center of mass observer. All these features are independent of the particular Lagrangian of the type (2.115) we can consider. All that remains is to describe the evolution of the orientation and therefore its angular velocity. The analysis developed until now is compatible with many different possibilities for the angular velocity. The behaviour of the angular velocity depends on the particular model we work with.

To end this section and with the above model of the electron in mind, it is convenient to remember some of the features that Dirac [9] obtained for the motion of a free electron. Let

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[9] P.A.M. Dirac, *The Principles of Quantum mechanics*, Oxford Univ. Press, 4th ed. Oxford (1967).

point $\boldsymbol{r}$ be the position vector on which Dirac's spinor $\psi(t, \boldsymbol{r})$ is defined. When computing the velocity of point $\boldsymbol{r}$, Dirac arrives at:

a) The velocity $\boldsymbol{u} = i/\hbar[H, \boldsymbol{r}] = c\boldsymbol{\alpha}$, is expressed in terms of $\boldsymbol{\alpha}$ matrices and writes, '. . . a *measurement of a component of the velocity of a free electron is certain to lead to the result* $\pm c$'.

b) The linear momentum does not have the direction of this velocity $\boldsymbol{u}$, but must be related to some average value of it: . . . '*the x_1 component of the velocity, $c\alpha_1$, consists of two parts, a constant part $c^2 p_1 H^{-1}$, connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least $2mc^2/h$, . . .*'.

c) About the position $\boldsymbol{r}$: '*The oscillatory part of x_1 is small, . . . , which is of order of magnitude $\hbar/mc$, . . .*'.

And when analyzing, in his original 1928 paper, [10] the interaction of the electron with an external electromagnetic field, after performing the square of Dirac's operator, he obtains two new interaction terms:

$$\frac{e\hbar}{2mc}\boldsymbol{\Sigma} \cdot \boldsymbol{B} + \frac{ie\hbar}{2mc}\boldsymbol{\alpha} \cdot \boldsymbol{E}, \tag{2.132}$$

where the electron spin is written as $\boldsymbol{S} = \hbar\boldsymbol{\Sigma}/2$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

in terms of σ-Pauli matrices and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the external electric and magnetic fields, respectively. He says, '*The electron will therefore behave as though it has a magnetic moment $(e\hbar/2mc)\boldsymbol{\Sigma}$ and an electric moment $(ie\hbar/2mc)\boldsymbol{\alpha}$. The magnetic moment is just that assumed in the spinning electron model*' (Pauli model). '*The electric moment, being a pure imaginary, we should not expect to appear in the model.*'

However, if we look at our classical model, we see that for the center of mass observer, there is a non-vanishing electric and magnetic dipole moment

$$\boldsymbol{d} = e\boldsymbol{k} = \frac{e}{mc^2}\boldsymbol{S} \times \boldsymbol{u}, \quad \boldsymbol{\mu} = \frac{e}{2}\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{Z}, \tag{2.133}$$

where $\boldsymbol{S}$ is the total spin and $\boldsymbol{Z} = -m\boldsymbol{k} \times d\boldsymbol{k}/dt$ is the zitterbewegung part of spin. The time average value of $\boldsymbol{d}$ is zero, and the average value of $\boldsymbol{\mu}$ is the constant vector $\boldsymbol{\mu}$.

This classical model gives rise to the same kinematical prediction as the nonrelativistic model described in Sec.2.2.3. If the charge of the particle is negative, the current of Fig.2.2 produces a magnetic moment that necessarily has the same direction as the spin. If the electron spin and magnetic moments are antiparallel, then we need another contribution to the total spin, different from the zitterbewegung. All real experiments to determine very accurately the gyromagnetic ratio are based on the determination of precession frequencies, but these precession frequencies are independent of the spin orientation. However, the difficulty to separate electrons in a Stern-Gerlach type experiment, suggests to perform polarization experiments in order to determine in a direct way whether spin and magnetic moment for elementary particles are either parallel or antiparallel. We have suggested a couple of plausible experiments to determine the relative orientation between the spin and magnetic moment of free electrons and also for electrons in the outer shell of atoms[11]

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[10] P.A.M. Dirac, *Proc. Roy. Soc. Lon.* **A117**, 610 (1928).

[11] M. Rivas, *Are the electron spin parallel or antiparallel vectors?*, ArXiv:physics/0112057.

Another consequence of the classical model is that it enhances the role of the so-called minimal coupling interaction $j_\mu A^\mu$. The magnetic properties of the electron are produced by the current of its internal motion and not by some possible distribution of magnetic dipoles, so that the only possible interaction of a point charge at $\boldsymbol{r}$ with the external electromagnetic field is that of the current j^μ, associated to the motion of point $\boldsymbol{r}$, with the external potentials.

2.6   The dynamical equation of the spinning electron

We have seen that for relativistic particles with $u = c$ and $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ orthogonal vectors, the position vector $\boldsymbol{r}$ moves in circles according to the dynamical equation (2.126) in the center of mass frame, as depicted in figure 2.2. But this solution is independent of the particular Lagrangian we choose as an invariant function of the kinematical variables and their derivatives, which accomplish with this orthogonality $\boldsymbol{u} \cdot \dot{\boldsymbol{u}} = 0$, requirement. We are going to analyze this dynamical equation for any arbitrary inertial observer.

As mentioned in the Preamble, let us consider the trajectory $\boldsymbol{r}(t)$, $t \in [t_1, t_2]$ followed by a point of a system for an arbitrary inertial observer O. Any other inertial observer O' is related to the previous one by a transformation of a kinematical group such that their relative space-time measurements of any space-time event are given by

$$t' = T(t, \boldsymbol{r}; g_1, \ldots, g_r), \quad \boldsymbol{r}' = \boldsymbol{R}(t, \boldsymbol{r}; g_1, \ldots, g_r),$$

where the functions T and $\boldsymbol{R}$ define the action of the kinematical group G, of parameters $(g_1, \ldots, g_r)$, on space-time. Then the description of the trajectory of that point for observer O' is obtained from

$$t'(t) = T(t, \boldsymbol{r}(t); g_1, \ldots, g_r), \quad \boldsymbol{r}'(t) = \boldsymbol{R}(t, \boldsymbol{r}(t); g_1, \ldots, g_r), \quad \forall t \in [t_1, t_2].$$

If we eliminate t as a function of t' from the first equation and substitute into the second we shall get

$$\boldsymbol{r}'(t') = \boldsymbol{r}'(t'; g_1, \ldots, g_r). \tag{2.134}$$

Since observer O' is arbitrary, equation (2.134) represents the complete family of trajectories of the point for all inertial observers. Elimination of the r group parameters among the function $\boldsymbol{r}'(t')$ and their time derivatives will give us the differential equation satisfied by the trajectory of the point. This differential equation is invariant by construction because it is independent of the group parameters and therefore independent of the inertial observer. If G is either the Galilei or Poincaré group it is a ten-parameter group so that we have to work out in general up to the fourth derivative to obtain sufficient equations to eliminate the ten group parameters. Therefore the order of the differential equation is dictated by the number of parameters and the structure of the kinematical group.

2.6.1   The relativistic spinning electron

Let us assume the above electron model. Since the charge is moving at the speed of light for the center of mass observer O^* it is moving at this speed for every other inertial observer O. Now, the relationship of space-time measurements between the center of mass observer and

any arbitrary inertial observer is given by:

$$
\begin{aligned}
t(t^*; g) &= \gamma \left(t^* + \boldsymbol{v} \cdot R(\boldsymbol{\alpha}) \boldsymbol{r}^*(t^*)\right) + b, \\
\boldsymbol{r}(t^*; g) &= R(\boldsymbol{\alpha}) \boldsymbol{r}^*(t^*) + \gamma \boldsymbol{v} t^* + \frac{\gamma^2}{1+\gamma} \left(\boldsymbol{v} \cdot R(\boldsymbol{\alpha}) \boldsymbol{r}^*(t^*)\right) \boldsymbol{v} + \boldsymbol{a}.
\end{aligned}
$$

With the shorthand notation for the following expressions:

$$
\boldsymbol{K}(t^*) = R(\boldsymbol{\alpha}) \boldsymbol{r}^*(t^*), \quad \boldsymbol{V}(t^*) = R(\boldsymbol{\alpha}) \frac{d\boldsymbol{r}^*(t^*)}{dt^*} = \frac{d\boldsymbol{K}}{dt^*}, \quad \frac{d\boldsymbol{V}}{dt^*} = -\boldsymbol{K},
$$

$$
B(t^*) = \boldsymbol{v} \cdot \boldsymbol{K}, \quad A(t^*) = \boldsymbol{v} \cdot \boldsymbol{V} = \frac{dB}{dt^*}, \quad \frac{dA}{dt^*} = -B
$$

we obtain

$$
\boldsymbol{r}^{(1)} = \frac{1}{\gamma(1+A)} \left(\boldsymbol{V} + \frac{\gamma}{1+\gamma}(1+\gamma+\gamma A)\boldsymbol{v}\right), \tag{2.135}
$$

$$
\boldsymbol{r}^{(2)} = \frac{1}{\gamma^2(1+A)^3} \left(-(1+A)\boldsymbol{K} + B\boldsymbol{V} + \frac{\gamma}{1+\gamma}B\boldsymbol{v}\right), \tag{2.136}
$$

$$
\boldsymbol{r}^{(3)} = \frac{1}{\gamma^3(1+A)^5} \left(-3B(1+A)\boldsymbol{K} - (1+A-3B^2)\boldsymbol{V} + \right.
$$
$$
\left. \frac{\gamma}{1+\gamma} (A(1+A)+3B^2)\boldsymbol{v}\right) \tag{2.137}
$$

$$
\boldsymbol{r}^{(4)} = \frac{1}{\gamma^4(1+A)^7} \left((1+A)(1-2A-3A^2-15B^2)\boldsymbol{K} - \right.
$$
$$
B(7+4A-3A^2-15B^2)\boldsymbol{V} -
$$
$$
\left. \frac{\gamma}{1+\gamma}(1-8A-9A^2-15B^2)B\boldsymbol{v}\right). \tag{2.138}
$$

From this we get

$$
\left(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(1)}\right)^2 = 1, \quad \left(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(2)}\right) = 0, \tag{2.139}
$$

$$
\left(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)}\right) = -\left(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(3)}\right) = \frac{1}{\gamma^4(1+A)^4}, \tag{2.140}
$$

$$
\left(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)}\right) = -\frac{1}{3}\left(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(4)}\right) = \frac{2B}{\gamma^5(1+A)^6}, \tag{2.141}
$$

$$
\left(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)}\right) = \frac{1}{\gamma^6(1+A)^8}\left(1-A^2+3B^2\right), \tag{2.142}
$$

$$
\left(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(4)}\right) = \frac{1}{\gamma^6(1+A)^8}\left(-1+2A+3A^2+9B^2\right), \tag{2.143}
$$

$$
\left(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(4)}\right) = \frac{1}{\gamma^7(1+A)^{10}}\left(1+A+3B^2\right)4B. \tag{2.144}
$$

From equations (2.140)-(2.142) we can express the magnitudes A, B and γ in terms of these scalar products between the different time derivatives $(\boldsymbol{r}^{(i)} \cdot \boldsymbol{r}^{(j)})$. The constraint that the velocity is 1 implies that all these and further scalar products for higher derivatives can be

expressed in terms of only three of them. If the three equations (2.135)-(2.137) are solved in terms of the unknowns $\boldsymbol{v}$, $\boldsymbol{V}$ and $\boldsymbol{K}$ and substituded into (3.24), we obtain the differential equation satisfied by the charge position, for any arbitrary inertial observer

$$\boldsymbol{r}^{(4)} - \frac{3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})} \, \boldsymbol{r}^{(3)} +$$

$$\left(\frac{2(\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)})}{(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})} - \frac{3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^2} - (\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{1/2} \right) \boldsymbol{r}^{(2)} = 0. \tag{2.145}$$

It is a fourth order ordinary differential equation which contains as solutions motions at the speed of light. In fact, if $(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(1)}) = 1$, then by derivation we have $(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(2)}) = 0$ and the next derivative leads to $(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)}) + (\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(3)}) = 0$. If we take this into account and make the scalar product of (2.145) with $\boldsymbol{r}^{(1)}$, we get $(\boldsymbol{r}^{(1)} \cdot \boldsymbol{r}^{(4)}) + 3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)}) = 0$, which is another relationship between the derivatives as a consequence of $|\boldsymbol{r}^{(1)}| = 1$. It corresponds to a helical motion since the term in the first derivative $\boldsymbol{r}^{(1)}$ is lacking.

2.6.2   The center of mass

The center of mass position is defined by

$$\boldsymbol{q} = \boldsymbol{r} + \frac{2(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)}) \, \boldsymbol{r}^{(2)}}{(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})^{3/2} + (\boldsymbol{r}^{(3)} \cdot \boldsymbol{r}^{(3)}) - \dfrac{3(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(3)})^2}{4(\boldsymbol{r}^{(2)} \cdot \boldsymbol{r}^{(2)})}}. \tag{2.146}$$

We can check that both $\boldsymbol{q}$ and $\boldsymbol{q}^{(1)}$ vanish for the center of mass observer. Then, the fourth order dynamical equation for the position of the charge can also be rewritten here as a system of two second order differential equations for the positions $\boldsymbol{q}$ and $\boldsymbol{r}$

$$\boldsymbol{q}^{(2)} = 0, \quad \boldsymbol{r}^{(2)} = \frac{1 - \boldsymbol{q}^{(1)} \cdot \boldsymbol{r}^{(1)}}{(\boldsymbol{q} - \boldsymbol{r})^2} \, (\boldsymbol{q} - \boldsymbol{r}), \tag{2.147}$$

a free motion for the center of mass and a kind of central motion for the charge around the center of mass.

For the non-relativistic electron we get in the low velocity case $\boldsymbol{q}^{(1)} \to 0$ and $|\boldsymbol{q} - \boldsymbol{r}| = 1$, the equations of the Galilei case

$$\boldsymbol{q}^{(2)} = 0, \quad \boldsymbol{r}^{(2)} = \boldsymbol{q} - \boldsymbol{r}. \tag{2.148}$$

a free motion for the center of mass and a harmonic motion around $\boldsymbol{q}$ for the position of the charge.

2.6.3   Interaction with some external field

The free equation for the center of mass motion $\boldsymbol{q}^{(2)} = 0$, represents the conservation of the linear momentum $d\boldsymbol{P}/dt = 0$. But the linear momentum is written in terms of the center of mass velocity as $\boldsymbol{P} = m\gamma(q^{(1)})\boldsymbol{q}^{(1)}$, so that the free dynamical equations (2.147) in the presence of an external field should be replaced by

$$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F}, \quad \boldsymbol{r}^{(2)} = \frac{1 - \boldsymbol{q}^{(1)} \cdot \boldsymbol{r}^{(1)}}{(\boldsymbol{q} - \boldsymbol{r})^2} \, (\boldsymbol{q} - \boldsymbol{r}), \tag{2.149}$$

where $\boldsymbol{F}$ is the external force and the second equation is left unchanged because we consider, even with interaction, the same definition of the center of mass position.

$$\frac{d\boldsymbol{P}}{dt} = m\gamma(q^{(1)})\boldsymbol{q}^{(2)} + m\gamma(q^{(1)})^3(\boldsymbol{q}^{(1)} \cdot \boldsymbol{q}^{(2)})\boldsymbol{q}^{(1)}$$

we get

$$m\gamma(q^{(1)})^3(\boldsymbol{q}^{(1)} \cdot \boldsymbol{q}^{(2)}) = \boldsymbol{F} \cdot \boldsymbol{q}^{(1)}$$

and by leaving the highest derivative $\boldsymbol{q}^{(2)}$ on the left hand side we finally get the differential equations which describe the evolution of a relativistic spinning electron in the presence of an external electromagnetic field:

$$m\boldsymbol{q}^{(2)} = \frac{e}{\gamma(q^{(1)})}\left[\boldsymbol{E} + \boldsymbol{r}^{(1)} \times \boldsymbol{B} - \boldsymbol{q}^{(1)}\left(\left[\boldsymbol{E} + \boldsymbol{r}^{(1)} \times \boldsymbol{B}\right] \cdot \boldsymbol{q}^{(1)}\right)\right], \tag{2.150}$$

$$\boldsymbol{r}^{(2)} = \frac{1 - \boldsymbol{q}^{(1)} \cdot \boldsymbol{r}^{(1)}}{(\boldsymbol{q} - \boldsymbol{r})^2}\left(\boldsymbol{q} - \boldsymbol{r}\right). \tag{2.151}$$

2.7 Appendix: Galilei group

The Galilei group is a group of space-time transformations characterized by ten parameters $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$. The action of g on a space-time point $x \equiv (t, \boldsymbol{r})$ is given by $x' = gx$, and is considered in the form

$$x' = \exp(bH)\exp(\boldsymbol{a} \cdot \boldsymbol{P})\exp(\boldsymbol{v} \cdot \boldsymbol{K})\exp(\boldsymbol{\alpha} \cdot \boldsymbol{J})\,x$$

as the action of a rotation followed by a pure Galilei transformation and finally a space and time translation. In this way all parameters that define each one-parameter subgroup are normal, because the exponential mapping works. Explicitly

$$t' = t + b, \tag{2.152}$$

$$\boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}, \tag{2.153}$$

and the composition law of the group $g'' = g'g$ is:

$$b'' = b' + b, \tag{2.154}$$

$$\boldsymbol{a}'' = R(\boldsymbol{\alpha}')\boldsymbol{a} + \boldsymbol{v}'b + \boldsymbol{a}', \tag{2.155}$$

$$\boldsymbol{v}'' = R(\boldsymbol{\alpha}')\boldsymbol{v} + \boldsymbol{v}', \tag{2.156}$$

$$R(\boldsymbol{\alpha}'') = R(\boldsymbol{\alpha}')R(\boldsymbol{\alpha}). \tag{2.157}$$

For rotations we shall alternatively use two different parametrizations. One is the normal or canonical parametrization in terms of a three vector $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector along the rotation axis, and $\alpha \in [0, \pi]$ is the clockwise rotation angle in radians, when looking along $\boldsymbol{n}$. Another, in terms of a three vector $\boldsymbol{\mu} = \boldsymbol{n}\tan(\alpha/2)$, which is more suitable to represent algebraically the composition of rotations.

The rotation matrix $R(\boldsymbol{\alpha}) = \exp(\boldsymbol{\alpha} \cdot \boldsymbol{J})$ is expressed in terms of the normal parameters α_i and in terms of the antisymmetric matrix generators J_i which have the usual matrix representation

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and satisfy the commutation relations $[J_i, J_k] = \epsilon_{ikl} J_l$, such that if we write the normal parameters $\boldsymbol{\alpha} = \alpha \boldsymbol{n}$ in terms of the rotation angle α and the unit vector $\boldsymbol{n}$ along the rotation axis, it is written as

$$R(\boldsymbol{\alpha})_{ij} = \delta_{ij} \cos\alpha + n_i n_j (1 - \cos\alpha) - \epsilon_{ijk} n_k \sin\alpha, \quad i, j, k = 1, 2, 3. \tag{2.158}$$

In the parametrization $\boldsymbol{\mu} = \boldsymbol{n} \tan(\alpha/2)$, the rotation matrix is

$$R(\boldsymbol{\mu})_{ij} = \frac{1}{1 + \mu^2} \left((1 - \mu^2)\delta_{ij} + 2\mu_i \mu_j - 2\epsilon_{ijk}\mu_k \right), \quad i, j, k = 1, 2, 3. \tag{2.159}$$

In terms of these variables, $R(\boldsymbol{\mu}'') = R(\boldsymbol{\mu}')R(\boldsymbol{\mu})$ is equivalent to

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu}}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu}}. \tag{2.160}$$

This can be seen in a simple manner by using the homomorphism between the rotation group and the group $SU(2)$. The matrix generators of $SU(2)$ are $\boldsymbol{J} = -i\boldsymbol{\sigma}/2$ in terms of Pauli matrices $\boldsymbol{\sigma}$. In the normal parametrization the rotation matrix $\exp(\boldsymbol{\alpha} \cdot \boldsymbol{J}) = \exp(-i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2)$ is written in the form

$$R(\boldsymbol{\alpha}) = \cos(\alpha/2)\mathbb{I} - i(\boldsymbol{n} \cdot \boldsymbol{\sigma})\sin(\alpha/2).$$

By defining $\boldsymbol{\mu} = \boldsymbol{n} \tan(\alpha/2)$, this rotation matrix is expressed as

$$R(\boldsymbol{\mu}) = \frac{1}{\sqrt{1 + \mu^2}} \left(\mathbb{I} - i\boldsymbol{\mu} \cdot \boldsymbol{\sigma} \right), \tag{2.161}$$

where $\mathbb{I}$ is the 2×2 unit matrix and in this form we can get the composition law (2.160). [12]

> If the rotation is of value π, then eqs. (2.158) or (2.159) lead to
>
> $$R(\boldsymbol{n}, \pi)_{ij} = -\delta_{ij} + 2n_i n_j.$$
>
> Even if the two rotations $R(\boldsymbol{\mu})$ and $R(\boldsymbol{\mu}')$ involved in (2.160) are of value π, although $\tan(\pi/2) = \infty$, this expression is defined and gives:
>
> $$\boldsymbol{n}'' \tan(\alpha''/2) = \frac{\boldsymbol{n} \times \boldsymbol{n}'}{\boldsymbol{n} \cdot \boldsymbol{n}'}.$$
>
> The absolute value of this relation leads to $\tan(\alpha''/2) = \tan\theta$, *i.e.*, $\alpha'' = 2\theta$, where θ is the angle between the two unit vectors $\boldsymbol{n}$ and $\boldsymbol{n}'$. We obtain the known result that every rotation of value α around an axis $\boldsymbol{n}$ can be obtained as the composition of two rotations of value π around two axes orthogonal to $\boldsymbol{n}$ and separated by an angle $\alpha/2$.

For the orientation variables we shall use throughout the book the early Greek variables $\boldsymbol{\alpha}, \boldsymbol{\beta}, \dots$ whenever we consider the normal parametrization, while for the $\tan(\alpha/2)$ parametrization we will express rotations in terms of the intermediate Greek variables $\boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\rho}, \dots$. In this last notation, transformation equations (2.154-2.157) should be replaced by

$$b'' = b' + b, \tag{2.162}$$
$$\boldsymbol{a}'' = R(\boldsymbol{\mu}')\boldsymbol{a} + \boldsymbol{v}'b + \boldsymbol{a}', \tag{2.163}$$
$$\boldsymbol{v}'' = R(\boldsymbol{\mu}')\boldsymbol{v} + \boldsymbol{v}', \tag{2.164}$$
$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu}}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu}}. \tag{2.165}$$

---

[12] D. Hestenes, *Space-time algebra*, Gordon and Breach, NY (1966).

The neutral element of the Galilei group is $(0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ and the inverse of every element is

$$(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})^{-1} = (-b, -R(-\boldsymbol{\alpha})(\boldsymbol{a} - b\boldsymbol{v}), -R(-\boldsymbol{\alpha})\boldsymbol{v}, -\boldsymbol{\alpha}).$$

The generators of the group in the realization (2.152, 2.153) are the differential operators

$$H = \partial/\partial t, \quad P_i = \partial/\partial r_i, \quad K_i = t\partial/\partial r_i, \quad J_k = \varepsilon_{kli} r_l \partial/\partial r_i \tag{2.166}$$

and the commutation rules of the Galilei Lie algebra are

$$[\boldsymbol{J}, \boldsymbol{J}] = -\boldsymbol{J}, \quad [\boldsymbol{J}, \boldsymbol{P}] = -\boldsymbol{P}, \quad [\boldsymbol{J}, \boldsymbol{K}] = -\boldsymbol{K}, \quad [\boldsymbol{J}, H] = 0, \tag{2.167}$$

$$[H, \boldsymbol{P}] = 0, \; [H, \boldsymbol{K}] = \boldsymbol{P}, \; [\boldsymbol{P}, \boldsymbol{P}] = 0, \; [\boldsymbol{K}, \boldsymbol{K}] = 0, \; [\boldsymbol{K}, \boldsymbol{P}] = 0. \tag{2.168}$$

All throughout this book, except when explicitly stated, we shall use the following shorthand notation for commutators of scalar and 3-vector operators, that as usual, are represented by bold face characters:

$$\begin{aligned}
[\boldsymbol{A}, \boldsymbol{B}] &= \boldsymbol{C}, &\implies& \quad [A_i, B_j] = \epsilon_{ijk} C_k, \\
[\boldsymbol{A}, \boldsymbol{B}] &= C, &\implies& \quad [A_i, B_j] = \delta_{ij} C, \\
[\boldsymbol{A}, B] &= \boldsymbol{C}, &\implies& \quad [A_i, B] = C_i, \\
[B, \boldsymbol{A}] &= \boldsymbol{C}, &\implies& \quad [B, A_i] = C_i,
\end{aligned}$$

where $\delta_{ij} = \delta_{ji}$ is Kronecker's delta and ϵ_{ijk} is the completely antisymmetric symbol, so that Latin indexes match on both sides of commutators.

The group action (2.152)-(2.153) represents the relationship between the coordinates $(t, \boldsymbol{r})$ of a space-time event as measured by the inertial observer O and the corresponding coordinates $(t', \boldsymbol{r}')$ of the same space-time event as measured by another inertial observer O'. The ten group parameters have the following meaning. If we consider the event $(0, \boldsymbol{0})$ measured by O, for instance the flashing of a light beam from its origin at time $t = 0$, it takes the values $(b, \boldsymbol{a})$ in O', where b is the time parameter that represents the time translation and $\boldsymbol{a}$ is the space translation. The parameter $\boldsymbol{v}$ of dimensions of velocity represents the velocity of the origin of the Cartesian frame of O as measured by O', and finally the parameters $\boldsymbol{\alpha}$, or $R(\boldsymbol{\alpha})$, represent the orientation of the Cartesian frame of O as measured by O'. In a certain sense the ten parameters $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ with dimensions respectively of time, position, velocity and orientation describe the relative motion of the Cartesian frame of O by O'.

The Galilei group has non-trivial exponents given by [13]

$$\xi(g, g') = m \left(\frac{1}{2} \boldsymbol{v}^2 b' + \boldsymbol{v} \cdot R(\boldsymbol{\alpha}) \boldsymbol{a}' \right). \tag{2.169}$$

They are characterized by the non-vanishing parameter m.

The central extension of the Galilei group [14] is an 11-parameter group with an additional generator I which commutes with the other ten,

$$[I, H] = [I, \boldsymbol{P}] = [I, \boldsymbol{K}] = [I, \boldsymbol{J}] = 0, \tag{2.170}$$

---

[13] V. Bargmann, *Ann. Math.* **5**, 1 (1954).

[14] J.M. Levy-Leblond, *Galilei Group and Galilean Invariance*, in E.M. Loebl, *Group Theory and its applications*, Acad. Press, NY (1971), vol. 2, p. 221.

and the remaining commutation relations are the same as above (2.167, 2.168), except the last one which appears as

$$[K_i, P_j] = -m\delta_{ij}I, \quad \text{or} \quad [\boldsymbol{K}, \boldsymbol{P}] = -mI, \tag{2.171}$$

using our shorthand notation, in terms of a non-vanishing parameter m. If we define the following polynomial operators on the group algebra

$$\boldsymbol{W} = I\boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P}, \quad U = IH - \frac{1}{2m}\boldsymbol{P}^2, \tag{2.172}$$

U commutes with all generators of the extended Galilei group and $\boldsymbol{W}$ satisfies the commutation relations:

$$[\boldsymbol{W}, \boldsymbol{W}] = -I\boldsymbol{W}, \quad [\boldsymbol{J}, \boldsymbol{W}] = -\boldsymbol{W}, \quad [\boldsymbol{W}, \boldsymbol{P}] = [\boldsymbol{W}, \boldsymbol{K}] = [\boldsymbol{W}, H] = 0,$$

so that $\boldsymbol{W}^2$ also commutes with all generators. It turns out that the extended Galilei group has three functionally independent Casimir operators which, in those representations in which the operator I becomes the unit operator, for instance in irreducible representations, are interpreted as the mass, $M = mI$, the internal energy $H_0 = H - P^2/2m$, and the absolute value of spin

$$S^2 = \left(\boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P}\right)^2. \tag{2.173}$$

The spin operator $\boldsymbol{S}$ in those representations in which $I = \mathbb{I}$, satisfy the commutation relations:

$$[\boldsymbol{S}, \boldsymbol{S}] = -\boldsymbol{S}, \quad [\boldsymbol{J}, \boldsymbol{S}] = -\boldsymbol{S}, \quad [\boldsymbol{S}, \boldsymbol{P}] = [\boldsymbol{S}, H] = [\boldsymbol{S}, \boldsymbol{K}] = 0,$$

i.e., it is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations and under Galilei boosts, respectively. It reduces to the total angular momentum operator $\boldsymbol{J}$ in those frames in which $\boldsymbol{P} = \boldsymbol{K} = 0$.

2.8   Appendix: Poincaré group

The Poincaré group is the group of transformations of Minkowski's space-time that leave invariant the separation between any two close space-time events $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. We shall consider the contravariant components $x^\mu \equiv (ct, \boldsymbol{r})$, and $x' = gx$ is expressed as $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, in terms of a constant matrix Λ and a constant translation four-vector $a^\mu \equiv (cb, \boldsymbol{a})$. We take for the covariant components of Minkowski's metric tensor $\eta_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$. Then $dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$ and $ds^2 = \eta_{\mu\nu}dx'^\mu dx'^\nu = \eta_{\sigma\rho}dx^\sigma dx^\rho$ implies for the matrix Λ

$$\eta_{\mu\nu}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho = \eta_{\sigma\rho}. \tag{2.174}$$

Relations (2.174) represent ten conditions among the 16 components of the matrix Λ, so that each matrix depends on six essential parameters, which can be chosen in many ways. Throughout this book we shall take three of them as the components of the relative velocity $\boldsymbol{v}$ between inertial observers and the remaining three as the orientation $\boldsymbol{\alpha}$ of their Cartesian frames, expressed in a suitable parametrization of the rotation group.

Therefore, every element of the Poincaré group $\mathcal{P}$ will be represented, as in the previous case of the Galilei group, by the ten parameters $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ and the group action on a space-time point $x \equiv (t, \boldsymbol{r})$ will be interpreted in the same way, *i.e.*, $x' = gx$:

$$x' = \exp(bH)\exp(\boldsymbol{a} \cdot \boldsymbol{P})\exp(\boldsymbol{\beta} \cdot \boldsymbol{K})\exp(\boldsymbol{\alpha} \cdot \boldsymbol{J})x, \tag{2.175}$$

as the action of a rotation followed by a boost or pure Lorentz transformation and finally a space and time translation. It is explicitly given on the space-time variables by

$$t' = \gamma t + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r})/c^2 + b, \tag{2.176}$$

$$\boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \gamma \boldsymbol{v}t + \gamma^2(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r})\boldsymbol{v}/(1+\gamma)c^2 + \boldsymbol{a}. \tag{2.177}$$

Parameter $\boldsymbol{\beta}$ in (2.175) is the normal parameter for the pure Lorentz transformations, that in terms of the relative velocity among observers $\boldsymbol{v}$ is expressed as $\boldsymbol{\beta}/\beta \tanh \beta = \boldsymbol{v}/c$ as we shall see below. The dimensions and domains of the parameters b, $\boldsymbol{a}$ and $\boldsymbol{\mu}$ are the same as those of the Galilei group, and the parameter $\boldsymbol{v} \in \mathbb{R}^3$, with the upper bound $v < c$, has also dimensions of velocity. The physical meaning of these ten parameters, that relate any two inertial observers, is the same as in the Galilei case. The parameter $\boldsymbol{v}$ is the velocity of observer O, as measured by O', and $R(\boldsymbol{\mu})$ represents the orientation of Cartesian frame O relative to O', once O' is boosted with velocity $\boldsymbol{v}$. The factor $\gamma(v) = (1 - v^2/c^2)^{-1/2}$.

The composition law of the group is obtained from $x'' = \Lambda'x' + a' = \Lambda'(\Lambda x + a) + a'$ that by identification with $x'' = \Lambda''x + a''$ reduces to $\Lambda'' = \Lambda'\Lambda$ and $a'' = \Lambda'a + a'$, *i.e.*, the composition law of the Lorentz transformations, that we will find in the next Section 2.8.1, and a Poincaré transformation (Λ', a') of the four-vector a^μ. In this parametrization $g'' = g'g$, is: [15]

$$b'' = \gamma'b + \gamma'(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})/c^2 + b', \tag{2.178}$$

$$\boldsymbol{a}'' = R(\boldsymbol{\mu}')\boldsymbol{a} + \gamma'\boldsymbol{v}'b + \frac{\gamma'^2}{(1+\gamma')c^2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})\boldsymbol{v}' + \boldsymbol{a}', \tag{2.179}$$

$$\boldsymbol{v}'' = \frac{R(\boldsymbol{\mu}')\boldsymbol{v} + \gamma'\boldsymbol{v}' + \dfrac{\gamma'^2}{(1+\gamma')c^2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{v})\boldsymbol{v}'}{\gamma'(1 + \boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\,\boldsymbol{v}/c^2)}, \tag{2.180}$$

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu} + \boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu} + G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})}, \tag{2.181}$$

where $\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})$ and $G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})$ are the real analytic functions:

$$\begin{aligned}
\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu}) = {} & \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}[\boldsymbol{v} \times \boldsymbol{v}' + \boldsymbol{v}(\boldsymbol{v}' \cdot \boldsymbol{\mu}') + \boldsymbol{v}'(\boldsymbol{v} \cdot \boldsymbol{\mu}) \\
& + \boldsymbol{v} \times (\boldsymbol{v}' \times \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu}) \times \boldsymbol{v}' + (\boldsymbol{v} \cdot \boldsymbol{\mu})(\boldsymbol{v}' \times \boldsymbol{\mu}') \\
& + (\boldsymbol{v} \times \boldsymbol{\mu})(\boldsymbol{v}' \cdot \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu}) \times (\boldsymbol{v}' \times \boldsymbol{\mu}')]\,,
\end{aligned} \tag{2.182}$$

$$\begin{aligned}
G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu}) = {} & \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}[\boldsymbol{v} \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot (\boldsymbol{v}' \times \boldsymbol{\mu}') + \boldsymbol{v}' \cdot (\boldsymbol{v} \times \boldsymbol{\mu}) \\
& - (\boldsymbol{v} \cdot \boldsymbol{\mu})(\boldsymbol{v}' \cdot \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu}) \cdot (\boldsymbol{v}' \times \boldsymbol{\mu}')]\,.
\end{aligned} \tag{2.183}$$

The unit element of the group is $(0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ and the inverse of any arbitrary element $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ is

$$\left(-\gamma b + \gamma\boldsymbol{v} \cdot \boldsymbol{a}/c^2, -R(-\boldsymbol{\mu})(\boldsymbol{a} - \gamma\boldsymbol{v}b + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot \boldsymbol{a})\boldsymbol{v}), -R(-\boldsymbol{\mu})\boldsymbol{v}, -\boldsymbol{\mu}\right).$$

---

[15] M.Rivas, M.Valle and J.M.Aguirregabiria, *Eur. J. Phys.* **6**, 128 (1986).

The group generators in the realization (2.176, 2.177), and in terms of the normal parameters $(b, \boldsymbol{a}, \boldsymbol{\beta}, \boldsymbol{\alpha})$, are

$$H = \partial/\partial t, \; P_i = \partial/\partial r_i, \; K_i = ct\partial/\partial r_i + (r_i/c)\partial/\partial t, \; J_k = \varepsilon_{kl}{}^i r_l \partial/\partial r_i.$$

Thus, $\boldsymbol{K}$ and $\boldsymbol{J}$ are dimensionless and the commutation relations become

$$[\boldsymbol{J}, \boldsymbol{J}] = -\boldsymbol{J}, \; [\boldsymbol{J}, \boldsymbol{P}] = -\boldsymbol{P}, \; [\boldsymbol{J}, \boldsymbol{K}] = -\boldsymbol{K}, \; [\boldsymbol{J}, H] = 0, \; [H, \boldsymbol{P}] = 0, \tag{2.184}$$

$$[H, \boldsymbol{K}] = c\boldsymbol{P}, \; [\boldsymbol{P}, \boldsymbol{P}] = 0, \; [\boldsymbol{K}, \boldsymbol{K}] = \boldsymbol{J}, \; [\boldsymbol{K}, \boldsymbol{P}] = -H/c. \tag{2.185}$$

If, as usual, we call $x^0 = ct$, $P_0 = H/c$, and $K_i = J_{0i} = -J_{i0}$ and $J_k = \frac{1}{2}\epsilon_{klr}J_{lr}$, $x_\mu = \eta_{\mu\nu}x^\nu$, $\mu = 0, 1, 2, 3$ and $\partial_\nu \equiv \partial/\partial x^\nu$, then,

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = -J_{\nu\mu} = x_\mu \partial_\nu - x_\nu \partial_\mu.$$

In covariant notation the commutation relations appear:

$$
\begin{aligned}
{[P_\mu, P_\nu]} &= 0, \\
[J_{\mu\nu}, P_\sigma] &= -\eta_{\mu\sigma}P_\nu + \eta_{\nu\sigma}P_\mu, \\
[J_{\mu\nu}, J_{\rho\sigma}] &= -\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho}.
\end{aligned}
$$

The Poincaré group has two functionally independent Casimir invariants. One is interpreted as the squared mass of the system,

$$P^\mu P_\mu = (H/c)^2 - \boldsymbol{P}^2 = m^2 c^2, \tag{2.186}$$

and the other is the square of the Pauli-Lubanski four-vector W^μ. The Pauli-Lubanski four-vector is defined as

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda} P_\nu J_{\sigma\lambda} \equiv (\boldsymbol{P} \cdot \boldsymbol{J}, H\boldsymbol{J}/c - \boldsymbol{K} \times \boldsymbol{P}) \equiv (\boldsymbol{P} \cdot \boldsymbol{S}, H\boldsymbol{S}/c), \tag{2.187}$$

which is by construction orthogonal to P_μ, i.e., $W^\mu P_\mu = 0$.

It is related to the spin of the system $\boldsymbol{S}$, defined through the relation

$$H\boldsymbol{S}/c = H\boldsymbol{J}/c - \boldsymbol{K} \times \boldsymbol{P}, \tag{2.188}$$

so that its time component $W^0 = \boldsymbol{P} \cdot \boldsymbol{S} = \boldsymbol{P} \cdot \boldsymbol{J}$ is the helicity of the particle, and the spatial part is the vector (2.188).

The other Casimir operator is thus

$$W^\mu W_\mu = (\boldsymbol{P} \cdot \boldsymbol{J})^2 - (H\boldsymbol{J}/c - \boldsymbol{K} \times \boldsymbol{P})^2 = -m^2 c^2 S^2, \tag{2.189}$$

where it depends on S^2, the absolute value squared of the spin. We see in the relativistic case that the two parameters m and S characterize the two Casimir invariants and therefore they are the intrinsic properties of the elementary particle the formalism provides. In the quantum case, since the representation must be irreducible $S^2 = s(s+1)\hbar^2$, for any $s = 0, 1/2, 1, \ldots$, depending on the value of the quantized spin of the particle, but in the classical case S^2 can take any continuous value.

These W^μ operators satisfy the commutation relations:

$$[W^\mu, W^\nu] = \epsilon^{\mu\nu\sigma\rho}W_\sigma P_\rho, \tag{2.190}$$

where we take $\epsilon^{0123} = +1$, and

$$[P^\mu, W^\nu] = 0, \qquad [M_{\mu\nu}, W_\sigma] = -\eta_{\mu\sigma}W_\nu + \eta_{\nu\sigma}W_\mu. \tag{2.191}$$

The Poincaré group has no non-trivial exponents, so that gauge functions when restricted to homogeneous spaces of $\mathcal{P}$ vanish.

2.8.1 Lorentz group

The Lorentz group $\mathcal{L}$ is the subgroup of transformations of the form $(0, \mathbf{0}, \boldsymbol{v}, \boldsymbol{\mu})$, and every Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$ will be interpreted as $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, as mentioned before where $L(\boldsymbol{v})$ is a boost or pure Lorentz transformation and $R(\boldsymbol{\mu})$ a spatial rotation. Expressions (2.180, 2.181) come from $\Lambda(\boldsymbol{v}'', \boldsymbol{\mu}'') = \Lambda(\boldsymbol{v}', \boldsymbol{\mu}')\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$. Expression (2.180) is the relativistic composition of velocities since

$$
\begin{aligned}
L(\boldsymbol{v}'')R(\boldsymbol{\mu}'') &= L(\boldsymbol{v}')R(\boldsymbol{\mu}')L(\boldsymbol{v})R(\boldsymbol{\mu}) \\
&= L(\boldsymbol{v}')R(\boldsymbol{\mu}')L(\boldsymbol{v})R(-\boldsymbol{\mu}')R(\boldsymbol{\mu}')R(\boldsymbol{\mu}),
\end{aligned}
$$

but the conjugate of the boost $R(\boldsymbol{\mu}')L(\boldsymbol{v})R(-\boldsymbol{\mu}') = L(R(\boldsymbol{\mu}')\boldsymbol{v})$ is another boost and thus

$$
L(\boldsymbol{v}'')R(\boldsymbol{\mu}'') = L(\boldsymbol{v}')L(R(\boldsymbol{\mu}')\boldsymbol{v})R(\boldsymbol{\mu}')R(\boldsymbol{\mu}).
$$

The product $L(\boldsymbol{v}')L(R(\boldsymbol{\mu}')\boldsymbol{v}) = L(\boldsymbol{v}'')R(\boldsymbol{w})$ where $\boldsymbol{v}''$ is the relativistic composition of the velocities $\boldsymbol{v}'$ and $R(\boldsymbol{\mu}')\boldsymbol{v}$, and $R(\boldsymbol{w})$ is the Thomas-Wigner rotation associated to the boosts $L(\boldsymbol{v}')$ and $L(R(\boldsymbol{\mu}')\boldsymbol{v})$.

Therefore, expression (2.180) is equivalent to

$$
L(\boldsymbol{v}'') = L(\boldsymbol{v}')L(R(\boldsymbol{\mu}')\boldsymbol{v})R(-\boldsymbol{w}), \tag{2.192}
$$

and (2.181) is

$$
R(\boldsymbol{\mu}'') = R(\boldsymbol{w})R(\boldsymbol{\mu}')R(\boldsymbol{\mu}) \equiv R(\boldsymbol{\phi})R(\boldsymbol{\mu}). \tag{2.193}
$$

The Thomas-Wigner rotation matrix $R(\boldsymbol{w})$ is:

$$
R(\boldsymbol{w})_{ij} = \delta_{ij} + \frac{1}{1+\gamma''}\left(\frac{\gamma'^2}{c^2}\left(\frac{1-\gamma}{1+\gamma'}\right)v_i'v_j' + \frac{\gamma^2}{c^2}\left(\frac{1-\gamma'}{1+\gamma}\right)R_{ik}'v_k R_{jl}'v_l \right.
$$

$$
\left. + \frac{\gamma'\gamma}{c^2}(v_i'R_{jk}'v_k - v_j'R_{ik}'v_k) + \frac{2\gamma'^2\gamma^2(v_k'R_{kl}'v_l)}{(1+\gamma')(1+\gamma)c^2}v_i'R_{jk}'v_k \right),
$$

and the factor

$$
\gamma'' = \gamma'\gamma\left(1 + \frac{\boldsymbol{v}' \cdot R(\boldsymbol{\mu})\boldsymbol{v}}{c^2}\right).
$$

Matrix $R(\boldsymbol{w})$ is written in terms of the vector parameter $\boldsymbol{w}$, which is a function of $\boldsymbol{v}'$, $\boldsymbol{\mu}'$ and $\boldsymbol{v}$, given by

$$
\boldsymbol{w} = \frac{\boldsymbol{F}(\boldsymbol{v}', \mathbf{0}, R(\boldsymbol{\mu}')\boldsymbol{v}, \mathbf{0})}{1 + G(\boldsymbol{v}', \mathbf{0}, R(\boldsymbol{\mu}')\boldsymbol{v}, \mathbf{0})}, \tag{2.194}
$$

and the parameter $\boldsymbol{\phi}$, such that $R(\boldsymbol{\phi}) = R(\boldsymbol{w})R(\boldsymbol{\mu}')$ is

$$
\boldsymbol{\phi} = \frac{\boldsymbol{\mu}' + \boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \mathbf{0})}{1 + G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \mathbf{0})}. \tag{2.195}
$$

If any one of the two velocities $\boldsymbol{v}$ or $\boldsymbol{v}'$ vanishes, $R(\boldsymbol{w})_{ij} = \delta_{ij}$.

The composition law is obtained by the homomorphism between the Lorentz group $\mathcal{L}$ and the group $SL(2, \mathbb{C})$ of 2×2 complex matrices of determinant $+1$. The Lie algebra of this group has as generators $\boldsymbol{J} = -i\boldsymbol{\sigma}/2$ and $\boldsymbol{K} = \boldsymbol{\sigma}/2$, where σ_i are Pauli spin matrices. A rotation of

angle α around a rotation axis given by the unit vector $\boldsymbol{n}$ is given by the 2×2 unitary matrix $\exp(\boldsymbol{\alpha} \cdot \boldsymbol{J})$,

$$R(\boldsymbol{\alpha}) = \cos(\alpha/2)\sigma_0 - i\boldsymbol{n} \cdot \boldsymbol{\sigma} \sin(\alpha/2). \tag{2.196}$$

In terms of the vector $\boldsymbol{\mu} = \tan(\alpha/2)\boldsymbol{n}$,

$$R(\boldsymbol{\mu}) = \frac{1}{\sqrt{1 + \mu^2}}(\sigma_0 - i\boldsymbol{\mu} \cdot \boldsymbol{\sigma}), \tag{2.197}$$

where σ_0 is the 2×2 unit matrix. A pure Lorentz transformation of normal parameters β_i is represented by the hermitian matrix $\exp(\boldsymbol{\beta} \cdot \boldsymbol{K})$. This matrix is:

$$L(\boldsymbol{\beta}) = \cosh(\beta/2)\sigma_0 + \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\beta}}{\beta} \sinh(\beta/2). \tag{2.198}$$

In terms of the relative velocity parameters, taking into account the functions $\cosh\beta = \gamma(v)$, $\sinh\beta = \gamma v/c$ and the trigonometric relations $\cosh(\beta/2) = \sqrt{(\cosh\beta + 1)/2}$ and $\tanh(\beta/2) = \sinh\beta/(1 + \cosh\beta)$, the matrix can be written as

$$L(\boldsymbol{v}) = \sqrt{\frac{1+\gamma}{2}} \left(\sigma_0 + \frac{\gamma}{1+\gamma} \frac{\boldsymbol{\sigma} \cdot \boldsymbol{v}}{c} \right). \tag{2.199}$$

Then, every element of $SL(2, \mathbb{C})$ is parametrized by the six real numbers $(\boldsymbol{v}, \boldsymbol{\mu})$, and interpreted as

$$A(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu}). \tag{2.200}$$

We thus see that every 2×2 matrix $A \in SL(2, \mathbb{C})$ can be written in terms of a complex four-vector a^μ and the four Pauli matrices σ_μ. As $A = a^\mu \sigma_\mu$, and $\det A = 1$ leads to $a^\mu a_\mu = 1$ or $(a^0)^2 - \boldsymbol{a}^2 = 1$. The general form of (2.200) is

$$A(\boldsymbol{v}, \boldsymbol{\mu}) = \sqrt{\frac{1+\gamma}{2(1+\mu^2)}} \left[\sigma_0 \left(1 - i\frac{\boldsymbol{\mu} \cdot \boldsymbol{u}}{1+\gamma} \right) + \boldsymbol{\sigma} \cdot \left(\frac{\boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{\mu}}{1+\gamma} - i\boldsymbol{\mu} \right) \right], \tag{2.201}$$

here the dimensionless vector $\boldsymbol{u} = \gamma(v)\boldsymbol{v}/c$.

Conversely, since $\mathrm{Tr}\,(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}$, we obtain $a^\mu = (1/2)\mathrm{Tr}\,(A\sigma_\mu)$. If we express (2.201) in the form $A(\boldsymbol{v}, \boldsymbol{\mu}) = a^\mu \sigma_\mu$ we can determine $\boldsymbol{\mu}$ and $\boldsymbol{v}$, and thus $\boldsymbol{u}$, from the components of the complex four-vector a^μ as:

$$\boldsymbol{\mu} = -\frac{\mathrm{Im}\,(\boldsymbol{a})}{\mathrm{Re}\,(a^0)}, \tag{2.202}$$

$$\boldsymbol{u} = 2 \left[\mathrm{Re}\,(a^0)\mathrm{Re}\,(\boldsymbol{a}) + \mathrm{Im}\,(a^0)\mathrm{Im}\,(\boldsymbol{a}) + \mathrm{Re}\,(\boldsymbol{a}) \times \mathrm{Im}\,(\boldsymbol{a}) \right], \tag{2.203}$$

where $\mathrm{Re}\,(a^\mu)$ and $\mathrm{Im}\,(a^\mu)$ are the real and imaginary parts of the corresponding components of the four-vector a^μ. When $\mathrm{Re}\,(a^0) = 0$ expression (2.202) is defined and represents a rotation of value π along the axis in the direction of vector $\mathrm{Im}\,(\boldsymbol{a})$.

If we represent every Lorentz transformation in terms of a rotation and a boost, i.e., in the reverse order, $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v})$, then the general expression of A is the same as (2.201) with a change of sign in the cross product term $\boldsymbol{u} \times \boldsymbol{\mu}$. Therefore, the decomposition is also unique, the rotation $R(\boldsymbol{\mu})$ is the same as before but the Lorentz boost is given in terms of the variables a^μ by

$$\boldsymbol{u} = 2 \left[\mathrm{Re}\,(a^0)\mathrm{Re}\,(\boldsymbol{a}) + \mathrm{Im}\,(a^0)\mathrm{Im}\,(\boldsymbol{a}) + \mathrm{Im}\,(\boldsymbol{a}) \times \mathrm{Re}\,(\boldsymbol{a}) \right].$$

Note the difference in the third term which is reversed when compared with (2.203).

In the four-dimensional representation of the Lorentz group on Minkowski space-time, a boost is expressed as $L(\boldsymbol{\beta}) = \exp(\boldsymbol{\beta} \cdot \boldsymbol{K})$ in terms of the dimensionless normal parameters β_i and the 4×4 boost generators K_i given by

$$
K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
$$

If we call $B = \boldsymbol{\beta} \cdot \boldsymbol{K} \equiv \sum_i \beta_i K_i$, we have

$$
B = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 \\ \beta_3 & 0 & 0 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} \beta^2 & 0 & 0 & 0 \\ 0 & \beta_1\beta_1 & \beta_1\beta_2 & \beta_1\beta_3 \\ 0 & \beta_2\beta_1 & \beta_2\beta_2 & \beta_2\beta_3 \\ 0 & \beta_3\beta_1 & \beta_3\beta_2 & \beta_3\beta_3 \end{pmatrix},
$$

with $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$ and $B^3 = \beta^2 B$, and so on for the remaining powers of B, so that the final expression for $L(\boldsymbol{\beta}) = \exp(\boldsymbol{\beta} \cdot \boldsymbol{K})$ is

$$
\begin{pmatrix} C & (\beta_1/\beta)S & (\beta_2/\beta)S & (\beta_3/\beta)S \\ (\beta_1/\beta)S & 1 + \dfrac{\beta_1\beta_1}{\beta^2}(C-1) & \dfrac{\beta_1\beta_2}{\beta^2}(C-1) & \dfrac{\beta_1\beta_3}{\beta^2}(C-1) \\ (\beta_2/\beta)S & \dfrac{\beta_2\beta_1}{\beta^2}(C-1) & 1 + \dfrac{\beta_2\beta_2}{\beta^2}(C-1) & \dfrac{\beta_2\beta_3}{\beta^2}(C-1) \\ (\beta_3/\beta)S & \dfrac{\beta_3\beta_1}{\beta^2}(C-1) & \dfrac{\beta_3\beta_2}{\beta^2}(C-1) & 1 + \dfrac{\beta_3\beta_3}{\beta^2}(C-1) \end{pmatrix}
$$

where $S = \sinh\beta$ and $C = \cosh\beta$. What is the physical interpretation of β_i? Let us assume that observers O and O' relate their space-time measurements x and x' by $x'^\mu = L(\boldsymbol{\beta})^\mu{}_\nu x^\nu$. Observer O sends at time t and at a later time $t + dt$ two light signals from a source placed at the origin of its Cartesian frame. These two signals when measured by O' take place at points $\boldsymbol{r}'$ and $\boldsymbol{r}' + d\boldsymbol{r}'$ and at instants t' and $t' + dt'$, respectively. Then they are related by

$$
cdt' = L^0{}_0 cdt, \qquad dx'^i = L^i{}_0 cdt
$$

because $dx^i = 0$. The quotient dx'^i/dt' is just the velocity of the light source v^i, i.e., of the origin of the O frame as measured by observer O', and then this velocity $v^i = cL^i{}_0/L^0{}_0 = c(\beta_i/\beta)S/C$, such that the relation between the normal parameters and the relative velocity between observers is

$$
\frac{\boldsymbol{v}}{c} = \frac{\boldsymbol{\beta}}{\beta} \tanh\beta
$$

and therefore $\tanh\beta = v/c$. Function $\cosh\beta \equiv \gamma(v) = (1 - v^2/c^2)^{-1/2}$ and when the transformation is expressed in terms of the relative velocity it takes the form of the symmetric matrix:

$$
L(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma v_x/c & \gamma v_y/c & \gamma v_z/c \\ \gamma v_x/c & 1 + \dfrac{v_x^2}{c^2}\dfrac{\gamma^2}{\gamma+1} & \dfrac{v_x v_y}{c^2}\dfrac{\gamma^2}{\gamma+1} & \dfrac{v_x v_z}{c^2}\dfrac{\gamma^2}{\gamma+1} \\ \gamma v_y/c & \dfrac{v_y v_x}{c^2}\dfrac{\gamma^2}{\gamma+1} & 1 + \dfrac{v_y^2}{c^2}\dfrac{\gamma^2}{\gamma+1} & \dfrac{v_y v_z}{c^2}\dfrac{\gamma^2}{\gamma+1} \\ \gamma v_z/c & \dfrac{v_z v_x}{c^2}\dfrac{\gamma^2}{\gamma+1} & \dfrac{v_z v_y}{c^2}\dfrac{\gamma^2}{\gamma+1} & 1 + \dfrac{v_z^2}{c^2}\dfrac{\gamma^2}{\gamma+1} \end{pmatrix}. \tag{2.204}
$$

The inverse transformation $L^{-1}(\boldsymbol{v}) = L(-\boldsymbol{v})$. The orthogonal 4×4 rotation matrix takes the block form

$$R(\boldsymbol{\mu}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{R}(\boldsymbol{\mu}) \end{pmatrix}, \tag{2.205}$$

where $\widetilde{R}(\boldsymbol{\mu})$ is the 3×3 orthogonal matrix (2.158). When a Lorentz transformation is expressed in the form $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, then by construction the first column of $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$ is just the first column of (2.204) where the velocity parameters $\boldsymbol{v}$ are defined. Therefore, given the general Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$, from its first column we determine the parameters $\boldsymbol{v}$ and thus the complete $L(\boldsymbol{v})$ can be worked out. The rotation involved can be easily calculated as $L(-\boldsymbol{v})\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})$. If expressed in the reverse order $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v})$, then it is the first row of Λ that coincides with the first row of (2.204). It turns out that, given any general Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$, then $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v}')$ with the same rotation in both sides as derived in (2.202) and $L(\boldsymbol{v}') = R(-\boldsymbol{\mu})L(\boldsymbol{v})R(\boldsymbol{\mu}) = L(R(-\boldsymbol{\mu})\boldsymbol{v})$, *i.e*, the velocity $\boldsymbol{v}' = R(-\boldsymbol{\mu})\boldsymbol{v}$. In any case, the decomposition of a general Lorentz transformation as a product of a rotation and a boost is a unique one, in terms of the same rotation $R(\boldsymbol{\mu})$ and a boost to be determined, depending on the order in which we take these two operations.

Matrix Λ can be considered as a tetrad (*i.e.*, a set of four orthonormal four-vectors, one time-like and the other three space-like) attached by observer O' to the origin of observer O. In fact, if the matrix is considered in the form $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, then the first column of Λ is the four-velocity of the origin of the O Cartesian frame and the other three columns are just the three unit vectors of the O reference frame, rotated with rotation $R(\boldsymbol{\mu})$ and afterwards boosted with $L(\boldsymbol{v})$.

Chapter 3

Quantization of the models

Quantization of generalized Lagrangian systems will suggest that wave functions for elementary particles must be squared integrable functions defined on the kinematical space.

We shall use Feynman's quantization method to show the structure of the wave function and the way it transforms under the kinematical or symmetry group of the theory. Once the Hilbert space structure of the state space is determined, this leads to a specific representation of the generators of the group as self-adjoint operators and the remaining analysis is done within the usual quantum mechanical context, *i.e.*, by choosing the complete commuting set of operators to properly determine a set of orthogonal basis vectors of the Hilbert space. Special emphasis is devoted to the analysis of the different angular momentum operators the formalism supplies. They have a similar structure to the classical ones, and this will help us to properly obtain the identification of the spin observable.

The structure of the spin operator depends on the kind of translation invariant kinematical variables we use to describe the particle, and the way these variables transform under the rotation group. Since in the Galilei and Poincaré case, as we have seen previously, these variables are the velocity $\boldsymbol{u}$ and orientation $\boldsymbol{\alpha}$ and they transform in the same way under rotations in both approaches, then the structure of the spin operator is exactly the same in both relativistic and nonrelativistic formalisms.

As we have seen in the classical description the position of the charge of the particle and its center of mass are different points, and spin is related to the rotation and internal motion (zitterbewegung) of the charge around the center of mass of the particle. The magnetic properties of the particle are connected only with the motion of the charge and therefore to the zitterbewegung part of spin. It is this double spin structure that gives rise to the concept of gyromagnetic ratio when expressing the magnetic moment in terms of the total spin. If the Lagrangian shows no dependence on the acceleration, the spin is only of rotational nature, and the position and center of mass position define the same point. Spin $1/2$ particles arise if the corresponding classical model rotates but no half integer spins are obtained for systems with spin of orbital nature related only to the zitterbewegung. On the manifold spanned by non-compact variables $\boldsymbol{u}$ no half-integer spins can be found, because the spin operator has the form of an orbital angular momentum and eigenvectors are but spherical harmonics.

Dirac's equation will be obtained when quantizing the classical relativistic spinning particles whose center of charge is circling around its center of mass at the speed c. In that case, the internal orientation of the electron completely characterizes its Dirac algebra.

3.1   Feynman's quantization of Lagrangian systems

Let us consider a generalized Lagrangian system as described in previous chapters and whose evolution is considered on the kinematical space between points x_1 and x_2.

For quantizing these generalized Lagrangian systems we shall follow Feynman's path integral method. [1] The Uncertainty Principle is introduced in Feynman's approach by the condition that if no measurement is performed to determine the trajectory followed by the system from x_1 to x_2, then all paths $x(\tau)$ are allowed with the same probability. Therefore a probability definition $P[x(\tau)]$, must be given for every path.

But instead of defining the probability associated to each possible path $P[x(\tau)]$, this is calculated in terms of a **probability amplitude**, $\phi[x(\tau)]$ for that path such that $P[x(\tau)] = |\phi[x(\tau)]|^2$, where $0 \le P \le 1$. But in general ϕ does not need to be a positive real number; in fact it is a complex number. Thus, to every possible trajectory followed by the system, $x(\tau)$ in X space, Feynman associates a complex number $\phi[x(\tau)]$ called the probability amplitude of this alternative, given by

$$\phi[x(\tau)] = N \exp\left\{ \frac{i}{\hbar} \int_{\tau_1}^{\tau_2} L(x(\tau), \dot{x}(\tau)) d\tau \right\} = N \exp\left\{ \frac{i}{\hbar} A_{[x]}(x_1, x_2) \right\}, \qquad (3.1)$$

where N is a path independent normalization factor, and where the phase of this complex number in units of $\hbar$ is the classical action of the system $A_{[x]}(x_1, x_2)$ along the path $x(\tau)$. Once we perform the integration along the path, this probability amplitude becomes clearly a function of the initial and final points in X space, x_1 and x_2, respectively.

In this Feynman statistical procedure, the probability amplitude of the occurrence of any alternative of a set of independent alternatives is the sum of the corresponding probability amplitudes of the different independent events. The probability of the whole process is the square of the absolute value of the total probability amplitude. This produces the effect that the probability of the whole process can be less than the probability of any single alternative of the set. This is what Feynman calls **interfering statistics**.

Then, the total probability amplitude that the system arrives at point x_2 coming from x_1, *i.e.*, Feynman's kernel $K(x_1, x_2)$, is obtained as the sum or integration over all paths, of terms of the form of Eq. (3.1). Feynman's kernel $K(x_1, x_2)$, will be in general a function, or more precisely a distribution, on the $X \times X$ manifold. If information concerning the initial point is lost, and the final point is left arbitrary, say x, the kernel reduces to the probability amplitude for finding the system at point x, *i.e.*, the usual interpretation of the quantum mechanical wave function $\Phi(x)$. By the above discussion we see that wave functions must be complex functions of the kinematical variables.

We thus see that Feynman's quantization method enhances the role of the kinematical variables to describe the quantum state of an arbitrary system, in spite of the independent degrees of freedom. We consider that this is one of the reasons why the kinematical variables have to play a leading role also in the classical approach.

We are used to consider in quantum mechanics, instead of a single function $\Phi(x)$, multicomponent wave functions, *i.e*, a set of linearly independent functions $\psi_i(t, \boldsymbol{r})$ defined on space-time and labeled with a discrete subindex that runs over a finite range, such that it can be considered as a vector valued function in a finite dimensional complex vector space. In general this finite space carries some irreducible representation of the rotation group and each

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[1] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, MacGraw Hill, NY (1965), p. 36.

component ψ_i represents a definite spin state of the system. Nevertheless, our wave function $\Phi(x)$ depends on more variables than space-time variables. Once we define later the complete commuting set of observables to obtain, in terms of their simultaneous eigenvectors, an orthonormal basis for the Hilbert space of states, we shall find that $\Phi(x)$ can be separated in two parts. One part $\phi(t, \boldsymbol{r})$ depending on space-time variables and another part χ that depends on the remaining translation invariant kinematical variables, that in our case will reduce to the velocity $\boldsymbol{u}$ and orientation $\boldsymbol{\alpha}$. It is this possible separation of our wave function that will produce the emergence of the different components of the usual formalism.

3.1.1 Transformation of the wave function

To see how the wave function transforms between inertial observers, and therefore to obtain its transformation equations under the kinematical groups, let us consider that O and O' are two inertial observers related by means of a transformation $g \in G$, such that the kinematical variables transform as:

$$x'^i = f^i(x, g). \tag{3.2}$$

If observer O considers that the system follows the path $\bar{x}(\tau)$, then it follows for O' the path $\bar{x}'(\tau) = f(\bar{x}(\tau), g)$ and because the action along classical paths transforms according to Eq. (1.28), the probability amplitude for observer O' is just

$$\phi'[\bar{x}'(\tau)] = N \exp\left\{\frac{i}{\hbar} \int_{\tau_1}^{\tau_2} L(\bar{x}'(\tau), \dot{\bar{x}}'(\tau)) d\tau\right\}$$

$$= N \exp\left\{\frac{i}{\hbar} \int_{\tau_1}^{\tau_2} L(\bar{x}(\tau), \dot{\bar{x}}(\tau)) d\tau\right\} \exp\left\{\frac{i}{\hbar} \int_{\tau_1}^{\tau_2} \frac{d\alpha(g; \bar{x}(\tau))}{d\tau} d\tau\right\},$$

i.e.,

$$\phi'[\bar{x}'(\tau)] = \phi[\bar{x}(\tau)] \exp\left\{\frac{i}{\hbar}\left(\alpha(g; x_2) - \alpha(g; x_1)\right)\right\},$$

where the last phase factor is independent of the integration path. If we add all probability amplitudes of this form, it turns out that Feynman's kernel transforms as:

$$K'(x_1', x_2') = K(x_1, x_2) \exp\left\{\frac{i}{\hbar}\left(\alpha(g; x_2) - \alpha(g; x_1)\right)\right\}. \tag{3.3}$$

If information concerning the initial point x_1 is lost, the wave function transforms as the part related to the variables x_2, up to an arbitrary function on G,

$$\Phi'(x'(x)) = \Phi'(gx) = \Phi(x) \exp\left\{\frac{i}{\hbar}\left(\alpha(g; x) + \theta(g)\right)\right\}, \tag{3.4}$$

or in terms of unprimed x variables

$$\Phi'(x) = \Phi(g^{-1}x) \exp\left\{\frac{i}{\hbar}\left(\alpha(g; g^{-1}x) + \theta(g)\right)\right\}, \tag{3.5}$$

where $\theta(g)$ is some function defined on G but independent of x.

Since our system is somewhere in X space, the probability of finding the system anywhere is 1. Then we have to define the way of adding probabilities at different points $x \in X$. If we define

a measure on X, $\mu(x)$, such that $d\mu(x)$ is the volume element in X space and $|\Phi(x)|^2 d\mu(x)$ is interpreted as the probability of finding the system inside the volume element $d\mu(x)$ around point x, the probability of finding it anywhere in X must be unity, so that

$$\int_X |\Phi(x)|^2 d\mu(x) = 1. \tag{3.6}$$

Since from (3.5)

$$|\Phi'(x')|^2 = |\Phi(x)|^2, \tag{3.7}$$

it is sufficient for the conservation of probability to assume that the measure to be defined $\mu(x)$ is group invariant. In that case, equation (3.7) implies also that inertial observers measure locally the same probability. This will have strong consequences about the possibility of invariance of the formalism under arbitrary changes of phase of the wave function. But the phase can be changed in a different manner at different points x. We can use this fact to further impose the local gauge invariance of the theory. It must be remarked that this arbitrary change of phase $\beta(x)$ is not only a phase on space-time, but rather on the whole kinematical space of the system and this enlarges the possibilities of analyzing different transformation groups that can be more general than the original kinematical groups, because they act on a larger manifold.

3.1.2 Hilbert space structure of the probability amplitudes

The complex function $\Phi(x)$, if interpreted as the probability amplitude of finding the system at point $x \in X$, coming from anywhere, satisfies (3.6). It means that $\Phi(x)$ is a complex, squared integrable function defined on the kinematical space. Because probability amplitudes add to form new probability amplitudes when properly normalized, the set of possible functions $\Phi(x)$ forms a complex vector space because we can add and multiply them by arbitray complex numbers to produce new complex functions which will describe this probability amplitude.

Consequently, the Hilbert space $\mathcal{H}$ whose unit rays represent the pure states of the system is the space of squared-integrable functions $\mathbb{L}^2(X, \mu)$ defined on the kinematical space X, $\mu(x)$ being an invariant measure such that the scalar product on $\mathcal{H}$ is defined as

$$< \Phi|\Psi > = \int_X \Phi^*(x)\Psi(x)d\mu(x), \tag{3.8}$$

$\Phi^*(x)$ being the complex conjugate function of $\Phi(x)$. There is an arbitrariness in the election of the invariant measure $\mu(x)$ but this will be guided by physical arguments. Nevertheless, the invariance condition will restrict the possible measures to be used.

The absolute value of the above (3.8) $| < \Phi|\Psi > |$ represents the probability that preparing the system in the state given by $\Psi(x)$ we find the system in the state $\Phi(x)$, and conversely, because $| < \Phi|\Psi > | = | < \Psi|\Phi > |$.

3.1.3 Representation of Observables

Wigner's theorem, [2,3] implies that to every symmetry $g \in G$ of a continuous group, there exists a one to one mapping of unit rays into unit rays that is induced on $\mathcal{H}$ by a unitary

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[2] E.P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, Acad. Press, NY (1959).

[3] V. Bargmann, *J. Math. Phys.* **5**, 862 (1964).

operator $U(g)$ defined up to a phase that maps a wave function defined on x into an arbitrary wave function of the image unit ray in x'. The Relativity Principle is a strong symmetry of physical systems that defines the equivalence between the set of inertial observers whose space-time measurements are related by means of a transformation of a kinematical group G. Now, if we interpret $\Phi(x)$ as the wave function that describes the state of the system for the observer O and $\Phi'(x)$ for O', then we have

$$U(g)\Phi(x) = \Phi'(x) = \Phi(g^{-1}x) \exp\left\{\frac{i}{\hbar}\alpha(g; g^{-1}x) + \theta(g)\right\}. \qquad (3.9)$$

Since the $\theta(g)$ function gives rise to a constant phase we can neglect it and then take as the definition of the unitary representation of the group G on Hilbert space $\mathcal{H}$

$$\Phi'(x) = U(g)\Phi(x) = \Phi(g^{-1}x) \exp\left\{\frac{i}{\hbar}\alpha(g; g^{-1}x)\right\}. \qquad (3.10)$$

Gauge functions satisfy (1.30), and therefore the phase term can be replaced by

$$\alpha(g; g^{-1}x) = -\alpha(g^{-1}; x) + \alpha(0; x) + \xi(g, g^{-1}) = -\alpha(g^{-1}; x) + \zeta(g), \qquad (3.11)$$

because gauge functions can always be chosen such that $\alpha(0; x) = 0$ and the group function $\zeta(g) = \xi(g, g^{-1})$ giving rise also to a constant phase, can be suppressed. We thus define the transformation of the wave function by

$$\Phi'(x) = U(g)\Phi(x) = \Phi(g^{-1}x) \exp\left\{-\frac{i}{\hbar}\alpha(g^{-1}; x)\right\}. \qquad (3.12)$$

If the unitary operator is represented in terms of the corresponding self-adjoint generators of the Lie algebra in the form

$$U(g) = \exp\left\{-\frac{i}{\hbar} g^\sigma X_\sigma\right\}, \qquad (3.13)$$

then, for an infinitesimal transformation of parameters δg^σ its inverse transformation has infinitesimal parameters $-\delta g^\sigma$, we obtain at first order in δg^σ

$$U(\delta g)\Phi(x) = \left(\mathbb{I} - \frac{i}{\hbar}\delta g^\sigma X_\sigma\right)\Phi(x) = \Phi(x) - \frac{i}{\hbar}\delta g^\sigma X_\sigma\,\Phi(x),$$

while

$$\Phi(\delta g^{-1}x) \equiv \Phi(f(x, \delta g^{-1})) = \Phi(x) - \delta g^\sigma u_\sigma^i(x)\frac{\partial\Phi(x)}{\partial x^i},$$

and

$$\exp\left\{-\frac{i}{\hbar}\alpha(\delta g^{-1}; x)\right\} = 1 - \frac{i}{\hbar}\alpha(\delta g^{-1}; x).$$

But because $\alpha(0; x) = 0$,

$$\alpha(\delta g^{-1}; x) = \left.\frac{\partial\alpha(g; x)}{\partial g^\sigma}\right|_{g=0}(-\delta g^\sigma),$$

and the substitution of the above terms in (3.12) and further identification of the first order terms in δg^σ imply that the self-adjoint operators X_σ when acting on the wave functions have the differential representation

$$X_\sigma = \frac{\hbar}{i} u_\sigma^j(x)\frac{\partial}{\partial x^j} - v_\sigma(x), \qquad (3.14)$$

where

$$u_\sigma^j(x) = \left.\frac{\partial f^j(x,g)}{\partial g^\sigma}\right|_{g=0}, \qquad v_\sigma(x) = \left.\frac{\partial \alpha(g;x)}{\partial g^\sigma}\right|_{g=0}. \tag{3.15}$$

If we restrict ourselves to transformations of the enlarged configuration space (t, q_i) that can be extended to the whole kinematical space $x \equiv (t, q_i, \ldots, q_i^{(k-1)})$, then, using the same notation as in (1.12)-(1.15), if the infinitesimal transformation is of the form

$$t' = t + M_\sigma \delta g^\sigma, \ q_i' = q_i + M_{i\sigma}\delta g^\sigma, \ \ldots, q_i'^{(k-1)} = q_i^{(k-1)} + M_{i\sigma}^{(k-1)}\delta g^\sigma,$$

these generators take the form

$$X_\sigma = \frac{\hbar}{i}\left(M_\sigma\frac{\partial}{\partial t} + M_{i\sigma}\frac{\partial}{\partial q_i} + \ldots + M_{i\sigma}^{(k-1)}\frac{\partial}{\partial q_i^{(k-1)}}\right) - v_\sigma(x). \tag{3.16}$$

When compared with the Noether constants of the motion (1.22) written in the form

$$-N_\sigma = -H\,M_\sigma + p_{(s+1)}^i M_{i\sigma}^{(s)} - B_\sigma(x), \tag{3.17}$$

we see a certain kind of **'correspondence recipe'**. When restricted to kinematical groups, the functions $B_\sigma(x)$ of (1.22), are obtained from the Lagrangian gauge functions $\alpha(g;x)$, by (1.29), which is exactly the same derivation as the functions $v_\sigma(x)$ above in (3.15). Now, by identifying the different classical observables and generalized momenta that appear here in (3.17) with the corresponding differential operators of (3.16) that multiply the corresponding $M_{i\sigma}^{(s)}$ function, we get: the generalized Hamiltonian $H = p_{(s)}^i q_i^{(s)} - L$, which is multiplied in (3.17) by the function M_σ, is identified with the operator $i\hbar\partial/\partial t$ which is also in front of the function M_σ in (3.16), and similarly, the generalized momentum $p_{(s+1)}^i$, the factor that multiplies the function $M_{i\sigma}^{(s)}$, with the differential operator $-i\hbar\partial/\partial q_i^{(s)}$, for $s = 0, \ldots, k-1$, because the functions $v_\sigma(x) = B_\sigma(x)$, are the same.

Remember that $p_{(s+1)}^i$ and $q_i^{(s)}$ are canonical conjugate variables. Then, each generalized momentum is replaced by $(\hbar/i)$ times the differential operator that differentiates with respect to its conjugate generalized coordinate and the generalized Hamiltonian by $i\hbar\partial/\partial t$.

The Heisenberg representation is that representation in which the time dependence has been withdrawn from the wave function by means of a time dependent unitary transformation. Then the wave function in this representation depends on the kinematical variables with the time excluded, *i.e.*, it depends only on the generalized coordinates $q_i^{(r)}$. Therefore, when acting on the wave function in the Heisenberg representation $\psi(q_i, q_i^{(1)}, \ldots, q_i^{(k-1)})$, the observables $q_i^{(r)}$ and $p_{(s)}^j$ satisfy the canonical commutation relations

$$[q_i^{(r)}, p_{(s+1)}^j] = i\hbar\delta_i^j\delta_s^r.$$

If functions $v_\sigma(x)$ in (3.14) vanish, the X_σ generators satisfy the commutation relations of the group G. But if some $v_\sigma(x) \neq 0$ the X_σ generators do not satisfy in general the commutation relations of the initial group G where they come from, but rather the commutation relations of a central extension of G. The group representation is not a true representation but a projective representation of G as shown by Bargmann. [4]

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[4] V. Bargmann, *Ann. Math.* **59**, 1 (1954).

In fact, from (3.10) we get

$$U(g_1)\Phi(x) = \Phi(g_1^{-1}x) \exp\{\frac{i}{\hbar}\alpha(g_1; g_1^{-1}x)\},$$

acting now on the left with $U(g_2)$,

$$U(g_2)U(g_1)\Phi(x) = U(g_2)\Phi(g_1^{-1}x) \exp\{\frac{i}{\hbar}\alpha(g_1; g_1^{-1}x)\}$$

$$= \Phi((g_2g_1)^{-1}x) \exp\{\frac{i}{\hbar}\alpha(g_2; g_2^{-1}x)\} \exp\{\frac{i}{\hbar}\alpha(g_1; (g_2g_1)^{-1}x)\}, \tag{3.18}$$

while acting on $\Phi(x)$ with $U(g_2g_1)$,

$$U(g_2g_1)\Phi(x) = \Phi((g_2g_1)^{-1}x) \exp\{\frac{i}{\hbar}\alpha(g_2g_1; (g_2g_1)^{-1}x)\}. \tag{3.19}$$

If we define $(g_2g_1)^{-1}x = g_1^{-1}g_2^{-1}x = z$, then $g_1 z = g_2^{-1}x$ and because gauge functions satisfy (1.30), we write

$$\alpha(g_2; g_1z) + \alpha(g_1; z) = \alpha(g_2g_1; z) + \xi(g_2, g_1), \tag{3.20}$$

and by comparing (3.18) with (3.19), taking into account (3.20), we obtain

$$U(g_2)U(g_1)\Phi(x) = U(g_2g_1)\Phi(x) \exp\{\frac{i}{\hbar}\xi(g_2; g_1)\}. \tag{3.21}$$

Since $\Phi(x)$ is arbitrary, we have a projective unitary representation of the group G characterized by the non-trivial exponent $\xi(g, g')$.

For both Galilei and Poincaré particles the kinematical space is the ten-dimensional manifold spanned by the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$, t being the time, $\boldsymbol{r}$ the charge position, $\boldsymbol{u}$ the velocity and $\boldsymbol{\alpha}$ the orientation of the particle. Thus in the quantum formalism the wave function of the most general elementary particle is a squared-integrable function $\Phi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$ of these kinematical variables. For point particles, the kinematical space is just the four-dimensional space-time, so that wave functions are only functions of time and position, but spinning particles will have to depend on the additional variables like velocity and orientation. The spin structure will thus be related to these additional variables.

3.1.4 Nonrelativistic spinning particles. Bosons

Now let us apply the formalism to the most interesting case of spinning particles. Let us consider next Galilei particles with (anti)orbital spin. This corresponds for example to systems for which $X = \mathcal{G}/SO(3)$ and thus the kinematical variables are time, position and velocity. A particular classical example is given in Chapter 2, Section 2.2 by the free Lagrangian

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d\boldsymbol{u}}{dt}\right)^2, \tag{3.22}$$

with $\boldsymbol{u} = d\boldsymbol{r}/dt$. For the free particle, the center of mass $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$ has a straight motion while the relative position vector $\boldsymbol{k}$ follows an elliptic trajectory with frequency ω around its center of mass, being the spin related to this internal motion. It is expressed as $\boldsymbol{S} = -m\boldsymbol{k} \times d\boldsymbol{k}/dt$.

The kinematical variables transform under $\mathcal{G}$ in the form

$$
\begin{aligned}
t'(\tau) &= t(\tau) + b, & \text{(3.23)} \\
\boldsymbol{r}'(\tau) &= R(\boldsymbol{\alpha})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}, & \text{(3.24)} \\
\boldsymbol{u}'(\tau) &= R(\boldsymbol{\alpha})\boldsymbol{u}(\tau) + \boldsymbol{v}. & \text{(3.25)}
\end{aligned}
$$

The wave functions are functions on X and thus functions of the variables $(t, \boldsymbol{r}, \boldsymbol{u})$. On this kinematical space the gauge function is the same as in (2.33), where m defines again the mass of the system. Taking into account as in the previous example the correspondence recipe for the Hamiltonian $H \to i\hbar\partial/\partial t$, the first generalized momentum $\boldsymbol{p}_1 \equiv \boldsymbol{P} \to -i\hbar\nabla$ and the other generalized momentum $\boldsymbol{p}_2 \equiv \boldsymbol{U} \to -i\hbar\nabla_u$, the generators of the projective representation are given by

$$
H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \quad \boldsymbol{K} = m\boldsymbol{r} - t\frac{\hbar}{i}\nabla - \frac{\hbar}{i}\nabla_u, \tag{3.26}
$$

$$
\boldsymbol{J} = \boldsymbol{r} \times \frac{\hbar}{i}\nabla + \boldsymbol{u} \times \frac{\hbar}{i}\nabla_u = \boldsymbol{L} + \boldsymbol{Z}, \tag{3.27}
$$

where ∇ is the gradient operator with respect to $\boldsymbol{q}_1 \equiv \boldsymbol{r}$ variables and ∇_u the gradient operator with respect to the $\boldsymbol{q}_2 \equiv \boldsymbol{u}$ variables. It is important to stress that this representation of the generators is independent of the particular Lagrangian that describes the system. It depends only on the kinematical variables $(t, \boldsymbol{r}, \boldsymbol{u})$ and the usual Galilei gauge function. We write the symbol $\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U}$ for the angular momentum related to the **Zitterbewegung** motion of the particle and we shall see that quantizes with integer values.

If we define $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k} = (\boldsymbol{K} + \boldsymbol{P}t)/m$, it satisfies the commutation relations with $\boldsymbol{P}$,

$$
[q_i, P_j] = i\hbar\,\delta_{ij},
$$

which are the canonical commutation relations between the linear momentum and position for a point particle and therefore these canonical commutation relations between the total linear momentum and the center of mass position for a spinning particle are already contained in the commutation relations of the extended Lie algebra of the kinematical group. Therefore the quantum mechanical operator

$$
\boldsymbol{q} = \boldsymbol{r} - \frac{\hbar}{im}\nabla_u, \tag{3.28}
$$

can be interpreted as the center of mass position operator. Discussion of other possibilities for the center of mass position operator can be found in the book by the author.

In this representation, one Casimir operator is the internal energy $H - \boldsymbol{P}^2/2m$. We see that the spin operator is defined as usual

$$
\boldsymbol{S} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K} \times \boldsymbol{P} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{k} \times \boldsymbol{P} = \boldsymbol{u} \times \frac{\hbar}{i}\nabla_u + \frac{\hbar}{im}\nabla_u \times \frac{\hbar}{i}\nabla;
$$

written in terms of two non-commuting terms, it satisfies

$$
[\boldsymbol{S}, \boldsymbol{S}] = i\hbar\boldsymbol{S}, \quad [\boldsymbol{J}, \boldsymbol{S}] = i\hbar\boldsymbol{S}, \quad [\boldsymbol{S}, \boldsymbol{P}] = [\boldsymbol{S}, H] = [\boldsymbol{S}, \boldsymbol{K}] = 0,
$$

i.e., it is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations and under Galilei boosts, respectively. The second part of the spin operator is of order $\hbar^2$ so that it produces a very small correction to the first $\boldsymbol{Z}$ part.

Operators $\boldsymbol{Z}$ satisfy the commutation relations

$$[\boldsymbol{Z}, \boldsymbol{Z}] = i\hbar\boldsymbol{Z}, \quad [\boldsymbol{J}, \boldsymbol{Z}] = i\hbar\boldsymbol{Z}, \quad [\boldsymbol{Z}, \boldsymbol{P}] = [\boldsymbol{Z}, H] = 0,$$

$$[\boldsymbol{Z}, \boldsymbol{K}] = -i\hbar\boldsymbol{U} = -\hbar^2\nabla_u,$$

i.e., $\boldsymbol{Z}$ is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations but not under Galilei boosts. It is usually considered as the quantum mechanical spin operator.

We see however, that the angular momentum operator $\boldsymbol{J}$ is split into two commuting terms $\boldsymbol{r} \times \boldsymbol{P}$ and $\boldsymbol{Z}$. They both commute with H, but the first one is not invariant under space translations. The $\boldsymbol{Z}$ operators are angular momentum operators that only differentiate the wave function with respect to the velocity variables, and consequently commute with H and $\boldsymbol{P}$, and although it is not the true Galilei invariant spin operator, we can find simultaneous eigenstates of the three commuting operators $H - \boldsymbol{P}^2/2m$, Z^2 and Z_3. Because the $\boldsymbol{Z}$ operators only affect the wave function in its dependence on $\boldsymbol{u}$ variables, we can choose functions with the variables separated in the form $\Phi(t, \boldsymbol{r}, \boldsymbol{u}) = \sum_i \psi_i(t, \boldsymbol{r})\chi_i(\boldsymbol{u})$ so that

$$(H - \boldsymbol{P}^2/2m)\psi_i(t, \boldsymbol{r}) = E\psi_i(t, \boldsymbol{r}), \tag{3.29}$$

$$Z^2\chi_i(\boldsymbol{u}) = z(z+1)\hbar^2\chi_i(\boldsymbol{u}), \tag{3.30}$$

$$Z_3\chi_i(\boldsymbol{u}) = m_z\hbar\chi_i(\boldsymbol{u}). \tag{3.31}$$

The space-time dependent wave function $\psi_i(t, \boldsymbol{r})$, satisfies Schroedinger's equation and is uncoupled with the spin part $\chi(\boldsymbol{u})$.

Due to the structure of Z^2 in terms of the $\boldsymbol{u}$ variables, which is that of an orbital angular momentum, the spin part of the wave function is of the form

$$\chi(\boldsymbol{u}) = f(u)Y_z^{m_z}(\theta, \phi), \tag{3.32}$$

$f(u)$ being an arbitrary function of the modulus of $\boldsymbol{u}$ and $Y_z^{m_z}(\theta, \phi)$ the spherical harmonics on the direction of $\boldsymbol{u}$.

For the center of mass observer, $\boldsymbol{S} = \boldsymbol{Z}$ and both angular momentum operators are the same. But for an arbitrary observer, $\boldsymbol{Z}$ operators do not commute with the boosts generators so that its absolute value is not Galilei invariant, while $\boldsymbol{S}$ is. But the splitting of the wave function into a multiple-component function that reflects its spin structure is an intrinsic property that can be done in any frame.

It turns out that if for an arbitrary observer $\boldsymbol{Z}$ is not the spin of the system, $\boldsymbol{r} \times \boldsymbol{P}$ is not the conserved orbital angular momentum, because $\boldsymbol{r}$ does not represent the position of the center of mass of the particle.

When there is an interaction with an external electromagnetic field, equation (3.29) is satisfied for the mechanical parts $H_m = H - e\phi$ and $\boldsymbol{P}_m = \boldsymbol{P} - e\boldsymbol{A}$ and we thus obtain the usual equation

$$\left(H - e\phi - \frac{(\boldsymbol{P} - e\boldsymbol{A})^2}{2m}\right)\psi_i(t, \boldsymbol{r}) = E\psi_i(t, \boldsymbol{r}). \tag{3.33}$$

This formalism, when the classical spin is of orbital nature, does not lead to half integer spin values, and therefore, from the quantum mechanical point of view these particles can be used only as models for representing bosons.

3.1.5   Nonrelativistic spinning particles. Fermions

Other examples of nonrelativistic spinning particles are those which have orientation and thus angular velocity. For instance, if $X = \mathcal{G}/\mathbb{R}_v^3$, $\mathbb{R}_v^3$ being the subgroup $\{\mathbb{R}^3, +\}$ of pure Galilei transformations, then the kinematical space is spanned by the variables $(t, \boldsymbol{r}, \boldsymbol{\alpha})$. This corresponds for instance to the Lagrangian system described by

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{I}{2}\,\boldsymbol{\omega}^2. \tag{3.34}$$

The particle travels freely at constant velocity while it rotates with constant angular velocity $\boldsymbol{\omega}$. The classical spin is just $\boldsymbol{S} = I\boldsymbol{\omega}$, and the center of charge and center of mass represent the same point.

To describe orientation we can think of the three orthogonal unit vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$ linked to the body, similarly as in a rigid rotator. If initially they are taken parallel to the spatial Cartesian axis of the laboratory inertial frame, then their nine components considered by columns define an orthogonal rotation matrix $R_{ij}(\boldsymbol{\alpha})$ that describes the triad evolution with the initial condition $R_{ij}(t = 0) = \delta_{ij}$.

Now, kinematical variables t, $\boldsymbol{r}$ and $\boldsymbol{\rho}$ transform under $\mathcal{G}$ in the form

$$
\begin{aligned}
t'(\tau) &= t(\tau) + b, & (3.35) \\
\boldsymbol{r}'(\tau) &= R(\boldsymbol{\alpha})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a}, & (3.36) \\
\boldsymbol{\rho}'(\tau) &= \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu} \times \boldsymbol{\rho}(\tau)}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau)}. & (3.37)
\end{aligned}
$$

On the corresponding Hilbert space, the Galilei generators are given by:

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \quad \boldsymbol{K} = m\boldsymbol{r} - t\,\frac{\hbar}{i}\,\nabla, \tag{3.38}$$

$$\boldsymbol{J} = \frac{\hbar}{i}\,\boldsymbol{r} \times \nabla + \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho)\right\} = \boldsymbol{L} + \boldsymbol{W}, \tag{3.39}$$

∇_ρ being the gradient operator with respect to the $\boldsymbol{\rho}$ variables and in the $\boldsymbol{\rho}$ parameterization of the rotation group.

The $\boldsymbol{W}$ part comes from the general group analysis. The group generators in this parametrization X_i will be obtained from (3.37) and according to (1.38) and (1.40). They are obtained as

$$X_i = \left.\left(\frac{\partial \rho'^k}{\partial \mu^i}\right)\right|_{\mu=0} \frac{\partial}{\partial \rho^k},$$

that can be written in vector notation as

$$\boldsymbol{X} = \nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho)$$

They satisfy the commutation relations

$$[X_i, X_k] = -2\epsilon_{ikl}X_l$$

and therefore operators $W_k = \frac{\hbar}{2i}X_k$, or in vector notation

$$\boldsymbol{W} = \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho)\right\}, \tag{3.40}$$

will satisfy the angular momentum commutation relations

$$[\boldsymbol{W}, \boldsymbol{W}] = i\hbar \boldsymbol{W}. \tag{3.41}$$

In this way since $\boldsymbol{L}$ and $\boldsymbol{W}$ commute among each other, we also get $[\boldsymbol{J}, \boldsymbol{J}] = i\hbar \boldsymbol{J}$.

In this example the center of mass and center of charge are the same point, $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}$ is the orbital angular momentum associated to the center of mass motion and $\boldsymbol{W} \equiv \boldsymbol{S}$ is the spin operator. The spin operator commutes with H, $\boldsymbol{P}$ and $\boldsymbol{K}$ and the wave function can be separated as $\Phi(t, \boldsymbol{r}, \boldsymbol{\rho}) = \sum_i \psi_i(t, \boldsymbol{r})\chi_i(\boldsymbol{\rho})$ leading to the equations

$$(H - \boldsymbol{P}^2/2m)\psi_i(t, \boldsymbol{r}) = E\psi_i(t, \boldsymbol{r}), \tag{3.42}$$

$$S^2\chi_i(\boldsymbol{\rho}) = s(s+1)\hbar^2\chi_i(\boldsymbol{\rho}), \tag{3.43}$$

$$S_3\chi_i(\boldsymbol{\rho}) = m_s\hbar\chi_i(\boldsymbol{\rho}). \tag{3.44}$$

Bopp and Haag [5] succeeded in finding $s = 1/2$ solutions for the system of equations (3.43) and (3.44). They are called Wigner's functions. [6] Solutions of (3.43) for arbitrary spin s are but a linear combination of the matrix elements of a $(2s + 1) \times (2s + 1)$ irreducible matrix representation of the rotation group as can be derived from the Peter-Weyl theorem on finite representations of compact groups. [7,8,9] We shall deal with the $s = 1/2$ functions in the Appendix Section 3.2, where explicit expressions and a short introduction to the Peter-Weyl theorem, will be given.

To describe fermions, the classical particles must necessarily have compact orientation variables as kinematical variables, otherwise no spin 1/2 values can be obtained when the classical spin is related only to the zitterbewegung.

3.2   Appendix: Spinors

In this section of mathematical content we shall review the main properties of spinors, in particular those connected with the possible representation of the wave function to describe spin 1/2 particles. We shall describe the representations in terms of eigenfunctions of the different commuting spin operators. But it must be remarked that in addition to the spin operators in the laboratory frame we also have spin operators in the body frame, because our general spinning particle has orientation, and therefore, a local Cartesian frame linked to its motion. This produces the result that for a spin 1/2 particle the wave function necessarily is a four-component object.

The general wave function is a function of the ten kinematical variables, $\Phi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$, and the spin part of the system related to the translation invariant kinematical variables $\boldsymbol{u}$ and $\boldsymbol{\rho}$ is

$$\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W}, \tag{3.45}$$

[5] F. Bopp and R. Haag, *Z. Naturforschg.* **5**a, 644 (1950).

[6] L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics. Theory and Application*, Cambridge U. P., Cambridge, England (1989).

[7] A.R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton U. P., Princeton NJ (1957).

[8] N. Ja. Vilenkin, *Fonctions spéciales et Théorie de la représentation des groups*, Dunod, Paris (1969).

[9] A.O. Barut and R. Raczka, *Theory of group representations and applications*, PWN, Warszawa (1980).

where $\boldsymbol{Z}$ and $\boldsymbol{W}$ are given by

$$\boldsymbol{Z} = \boldsymbol{u} \times \frac{\hbar}{i} \nabla_u, \quad \boldsymbol{W} = \frac{\hbar}{2i} \left\{ \nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho) \right\}, \tag{3.46}$$

in the $\tan(\alpha/2)$ representation of the rotation group, as has been deduced in previous sections. ∇_u and ∇_ρ are respectively the gradient operators with respect to $\boldsymbol{u}$ and $\boldsymbol{\rho}$ variables. These operators always commute with the $H = i\hbar\partial/\partial t$ and $\boldsymbol{P} = -i\hbar\nabla$ operators, and therefore they are translation invariant. This feature allows the separation of the general wave function in terms of space-time variables and velocity-orientation variables to describe the translation invariant properties of the system.

The above spin operators satisfy the commutation relations

$$[\boldsymbol{Z}, \boldsymbol{Z}] = i\hbar\boldsymbol{Z}, \quad [\boldsymbol{W}, \boldsymbol{W}] = i\hbar\boldsymbol{W}, \quad [\boldsymbol{Z}, \boldsymbol{W}] = 0, \tag{3.47}$$

and thus

$$[\boldsymbol{S}, \boldsymbol{S}] = i\hbar\boldsymbol{S}.$$

Because we are describing the orientation of the particle by attaching to it a system of three unit vectors $\boldsymbol{e}_i$, whose orientation in space is described by variables $\boldsymbol{\rho}$ or $\boldsymbol{\alpha}$, then, if at initial instant $\tau = 0$ we choose the body axes coincident with the laboratory axes, the components of the unit vectors $\boldsymbol{e}_i$ at any time are

$$(\boldsymbol{e}_i)_j = R_{ji}(\boldsymbol{\alpha}) = \delta_{ji}\cos\alpha + n_j n_i (1 - \cos\alpha) - \epsilon_{jik} n_k \sin\alpha, \tag{3.48}$$

in the normal parametrization and also in the $\boldsymbol{\rho}$ parametrization by

$$(\boldsymbol{e}_i)_j = R_{ji}(\boldsymbol{\rho}) = \frac{1}{1 + \rho^2}((1 - \rho^2)\delta_{ji} + 2\rho_j\rho_i - 2\epsilon_{jik}\rho_k), \tag{3.49}$$

where the Cartesian components of the rotation axis unit vector $\boldsymbol{n}$ are:

$$n_1 = \sin\theta\cos\phi, \qquad n_2 = \sin\theta\sin\phi, \qquad n_3 = \cos\theta, \tag{3.50}$$

where θ is the polar angle and ϕ the usual azimuth angle. Explicitly:

$$
\begin{aligned}
e_{11} &= \cos\alpha + \sin^2\theta\cos^2\phi(1 - \cos\alpha), \\
e_{12} &= \cos\theta\sin\alpha + \sin^2\theta\sin\phi\cos\phi(1 - \cos\alpha), \\
e_{13} &= -\sin\theta\sin\phi\sin\alpha + \sin\theta\cos\theta\cos\phi(1 - \cos\alpha),
\end{aligned}
$$

$$
\begin{aligned}
e_{21} &= -\cos\theta\sin\alpha + \sin^2\theta\sin\phi\cos\phi(1 - \cos\alpha), \\
e_{22} &= \cos\alpha + \sin^2\theta\sin^2\phi(1 - \cos\alpha), \\
e_{23} &= \sin\theta\cos\phi\sin\alpha + \sin\theta\cos\theta\sin\phi(1 - \cos\alpha),
\end{aligned}
$$

$$
\begin{aligned}
e_{31} &= \sin\theta\sin\phi\sin\alpha + \sin\theta\cos\theta\cos\phi(1 - \cos\alpha), \\
e_{32} &= -\sin\theta\cos\phi\sin\alpha + \sin\theta\cos\theta\sin\phi(1 - \cos\alpha), \\
e_{33} &= \cos\alpha + \cos^2\theta(1 - \cos\alpha),
\end{aligned}
$$

in the $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, or normal parametrization of the rotation group. In the $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{n}$ parametrization the body frame is

$$
\begin{aligned}
e_{11} &= (1 + \rho_1^2 - \rho_2^2 - \rho_3^2)/(1 + \rho^2), \\
e_{12} &= (2\rho_1\rho_2 + 2\rho_3)/(1 + \rho^2), \\
e_{13} &= (2\rho_1\rho_3 - 2\rho_2)/(1 + \rho^2), \\[8pt]
e_{21} &= (2\rho_2\rho_1 - 2\rho_3)/(1 + \rho^2), \\
e_{22} &= (1 - \rho_1^2 + \rho_2^2 - \rho_3^2)/(1 + \rho^2), \\
e_{23} &= (2\rho_2\rho_3 + 2\rho_1)/(1 + \rho^2), \\[8pt]
e_{31} &= (2\rho_1\rho_3 + 2\rho_2)/(1 + \rho^2), \\
e_{32} &= (2\rho_3\rho_2 - 2\rho_1)/(1 + \rho^2), \\
e_{33} &= (1 - \rho_1^2 - \rho_2^2 + \rho_3^2)/(1 + \rho^2),
\end{aligned}
$$

where $\rho^2 \equiv \rho_1^2 + \rho_2^2 + \rho_3^2 = \tan^2(\alpha/2)$.

In addition to the different components of the spin operators S_i, Z_i and W_i in the laboratory frame, we also have another set of spin operators. They are the spin projections on the body axes $\boldsymbol{e}_i$, i.e., the operators $R_i = \boldsymbol{e}_i \cdot \boldsymbol{S}$, $M_i = \boldsymbol{e}_i \cdot \boldsymbol{Z}$ and $T_i = \boldsymbol{e}_i \cdot \boldsymbol{W}$, respectively. In particular, spin operators T_i, collecting terms from (3.49) and (3.46), take the expression

$$
\begin{aligned}
T_i = \sum_{k=1}^{k=3} (e_i)_k W_k &= \frac{\hbar}{2i(1 + \rho^2)} \sum_{k=1}^{k=3} \left((1 - \rho^2)\delta_{ik} + 2\rho_i\rho_k - 2\epsilon_{kij}\rho_j \right) \\
&\times \left(\frac{\partial}{\partial \rho_k} + \epsilon_{klr}\rho_l \frac{\partial}{\partial \rho_r} + \rho_k(\boldsymbol{\rho} \cdot \nabla_\rho) \right),
\end{aligned}
$$

and after some tedious manipulations we reach the final result, written in vector notation as

$$
\boldsymbol{T} = \frac{\hbar}{2i} \left\{ \nabla_\rho - \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho) \right\}. \tag{3.51}
$$

We see, by inspection, that this result can also be obtained from the expression of $\boldsymbol{W}$ in (3.46), just by replacing $\boldsymbol{\rho}$ by $-\boldsymbol{\rho}$, followed by a global change of sign. This is because we describe the orientation of the particle by vector $\boldsymbol{\rho}$ in the laboratory frame from the active viewpoint, i.e., with the laboratory reference frame fixed. However, its orientation with respect to the body frame is described by the motion of the laboratory frame, whose orientation for the body is $-\boldsymbol{\rho}$, and the global change of sign comes from the change from the active point of view to the passive one. This is the difference in the spin description in one frame or another.

It satisfies the following commutation relations

$$
[\boldsymbol{T}, \boldsymbol{T}] = -i\hbar\boldsymbol{T}, \quad [\boldsymbol{T}, \boldsymbol{W}] = 0.
$$

and in general all spin projections on the body frame R_i, M_i and T_i, commute with all the spin projections on the laboratory frame S_i, Z_i and W_i. This is in agreement with the quantum mechanical uncertainty principle, because spin components with respect to different frames are compatible observables.

To find eigenstates of the spin operator we have to solve equations of the form:

$$S^2 \chi(\boldsymbol{u}, \boldsymbol{\rho}) = s(s+1)\hbar^2 \chi(\boldsymbol{u}, \boldsymbol{\rho}), \quad S_3 \chi(\boldsymbol{u}, \boldsymbol{\rho}) = m\hbar \chi(\boldsymbol{u}, \boldsymbol{\rho}).$$

But we also have the orientation of the particle, and therefore the spin projections on the body axes. These projections commute with S^2 and S_3, and it is possible to choose another commuting spin operator, like the T_3 operator, and therefore our wave function can be taken also as an eigenvector of T_3,

$$T_3 \chi(\boldsymbol{u}, \boldsymbol{\rho}) = n\hbar \chi(\boldsymbol{u}, \boldsymbol{\rho}),$$

so that the complete commuting set of operators that describe the spin structure must also include spin projections on the body axes.

The spin squared operator is

$$S^2 = \boldsymbol{Z}^2 + \boldsymbol{W}^2 + 2\boldsymbol{Z} \cdot \boldsymbol{W}, \tag{3.52}$$

and we see from (3.47) that is expressed as the sum of three commuting terms and its eigenvectors can be obtained as the simultaneous eigenvectors of the three commuting operators on the right-hand side of (3.52). Operators $\boldsymbol{Z}$ and $\boldsymbol{W}$ produce derivatives of the wave function with respect to $\boldsymbol{u}$ and $\boldsymbol{\rho}$ variables, separately. Thus, each $\chi(\boldsymbol{u}, \boldsymbol{\rho})$ can again be separated as

$$\chi(\boldsymbol{u}, \boldsymbol{\rho}) = \sum_j U_j(\boldsymbol{u}) \, V_j(\boldsymbol{\rho}), \tag{3.53}$$

where the sum runs over a finite range, and where $U_j(\boldsymbol{u})$ will be eigenfunctions of $\boldsymbol{Z}^2$ and $V_j(\boldsymbol{\rho})$ of $\boldsymbol{W}^2$, respectively.

Functions $U_j(\boldsymbol{u})$ are multiples of spherical harmonics defined on the orientation of the velocity vector $\boldsymbol{u}$, because the $\boldsymbol{Z}$ operator has the structure of an orbital angular momentum in terms of the $\boldsymbol{u}$ variables, and thus its eigenvalues are integer numbers. The global factor left out is an arbitrary function depending on the absolute value of the velocity u.

It turns out that to find the most general spinor is necessary to seek also solutions of the $V_j(\boldsymbol{\rho})$ part, depending on the orientation variables. This goal will be achieved in the next section, where we consider the action of the rotation group on itself as a transformation group.

3.2.1   Spinor representation on SU(2)

We shall describe now in detail the orientation part of the general wave function, $V(\boldsymbol{\rho})$. If there is no contribution to spin from the zitterbewegung part $\boldsymbol{Z}$, the spin operator (3.45) reduces to the $\boldsymbol{W}$ operator given in (3.46). To solve the corresponding eigenvalue equations we shall first represent the spin operators in spherical coordinates.

If we represent vector $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{n} = r\boldsymbol{n}$ in spherical coordinates (r, θ, ϕ), with $r = |\boldsymbol{\rho}| = \tan(\alpha/2)$ and θ and ϕ the usual polar and azimuth angles, respectively, then unit vector $\boldsymbol{n}$ has the Cartesian components given in (3.50). If from now on we take $\hbar = 1$, the spin operators (3.46) are represented by the differential operators

$$W_1 = \frac{1}{2i} \left[(1+r^2) \sin\theta \, \cos\phi \, \frac{\partial}{\partial r} + \left(\frac{1}{r} \cos\theta \, \cos\phi - \sin\phi \right) \frac{\partial}{\partial \theta} - \left(\frac{\sin\phi}{r \sin\theta} + \frac{\cos\theta \, \cos\phi}{\sin\theta} \right) \frac{\partial}{\partial \phi} \right],$$

$$W_2 = \frac{1}{2i} \left[(1+r^2) \sin\theta \, \sin\phi \, \frac{\partial}{\partial r} + \left(\frac{1}{r} \cos\theta \, \sin\phi + \cos\phi \right) \frac{\partial}{\partial \theta} - \left(\frac{\cos\theta \, \sin\phi}{\sin\theta} - \frac{\cos\phi}{r \sin\theta} \right) \frac{\partial}{\partial \phi} \right],$$

$$W_3 = \frac{1}{2i} \left[(1+r^2) \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi} \right].$$

The Casimir operator of the rotation group W^2 is:

$$W^2 = -\frac{1+r^2}{4} \left[(1+r^2) \frac{\partial^2}{\partial r^2} + \frac{2(1+r^2)}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left\{ \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\} \right].$$

The up and down spin operators defined as usual by $W_{\pm} = W_1 \pm iW_2$ are

$$W_+ = \frac{e^{i\phi}}{2i} \left[(1+r^2) \sin \theta \frac{\partial}{\partial r} + \left(\frac{\cos \theta + ir}{r} \right) \frac{\partial}{\partial \theta} - \left(\frac{r \cos \theta - i}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right],$$

$$W_- = \frac{e^{-i\phi}}{2i} \left[(1+r^2) \sin \theta \frac{\partial}{\partial r} + \left(\frac{\cos \theta - ir}{r} \right) \frac{\partial}{\partial \theta} - \left(\frac{r \cos \theta + i}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right].$$

They satisfy the commutation relations

$$[W_3, W_+] = W_+, \quad [W_3, W_-] = -W_-, \quad [W_+, W_-] = 2W_3.$$

We can check that $(W_i)^* = -W_i$ and $W_+ = -(W_-)^*$, where * means to take the complex conjugate of the corresponding operator.

If $F_s^m(r, \theta, \phi)$ is an eigenfunction of W^2 and W_3, it satisfies the differential equations:

$$W^2 F_s^m(r, \theta, \phi) = s(s+1) F_s^m(r, \theta, \phi), \quad W_3 F_s^m(r, \theta, \phi) = m F_s^m(r, \theta, \phi).$$

To find solutions of the above system we know that we can proceed in the following way. Let us compute first the eigenfunctions of the form F_s^s. Then operator W_+ annihilates this state $W_+ F_s^s = 0$ and by acting on this function with operator W_- we can obtain the remaining eigenstates F_s^m of the same irreducible representation characterized by parameter s and for $-s \leq m \leq s$. Then our task will be to obtain first the F_s^s functions.

Now, let us consider eigenfunctions F_s^s that can be written in separate variables as $F_s^s(r, \theta, \phi) = A(r)B(\theta)C(\phi)$. Then

$$W_3 A(r) B(\theta) C(\phi) = s A(r) B(\theta) C(\phi)$$

gives rise to

$$(1+r^2) \cos \theta A' BC - \frac{\sin \theta}{r} AB'C + ABC' = 2is ABC$$

where A' is the derivative of A and so on, and by dividing both sides by ABC we have

$$(1+r^2) \cos \theta \frac{A'(r)}{A(r)} - \frac{\sin \theta}{r} \frac{B'(\theta)}{B(\theta)} + \frac{C'(\phi)}{C(\phi)} = 2is.$$

Now, the third term on the left-hand side must be a constant, because the remaining terms are functions independent of ϕ. Therefore, this term is written as $C'(\phi)/C(\phi) = ik$ and thus $C(\phi) = e^{ik\phi}$ up to an arbitrary constant factor. Since $C(\phi + 2\pi) = C(\phi)$ this implies that the constant k must be an integer. The other two functions satisfy

$$r(1+r^2) \cos \theta A'B - \sin \theta AB' + ir(k - 2s)AB = 0. \tag{3.54}$$

If there exist solutions with real functions A and B, then necessarily $k = 2s$ so that the eigenvalue s can be any integer or half integer, and equation (3.54) can be separated in the form:

$$r(1+r^2) \frac{A'(r)}{A(r)} = \frac{\sin \theta}{\cos \theta} \frac{B'(\theta)}{B(\theta)} = p = \text{constant}, \tag{3.55}$$

where, up to constant factors, the general solution is

$$A(r) = \left(\frac{r^2}{1+r^2}\right)^{p/2}, \quad B(\theta) = (\sin\theta)^p.$$

By acting on this solution $F_s^s \equiv A(r)B(\theta)C(\phi)$, with W_+, since $W_+F_s^s = 0$, it gives:

$$r(1+r^2)\sin^2\theta A'B + (\sin\theta\cos\theta + ir\sin\theta)AB' - 2s(ir\cos\theta + 1)AB = 0.$$

By dividing all terms by AB, taking into account (3.55), we get the condition $(p - 2s)(1 + ir\cos\theta) = 0$. Then there exist real solutions in separate variables whenever $p = 2s = k$. They are given, up to a constant factor, by

$$F_s^s(r,\theta,\phi) = \left(\frac{r^2}{1+r^2}\right)^s (\sin\theta)^{2s}e^{i2s\phi}. \tag{3.56}$$

For $s = 1/2$ and after the action of W_- we obtain the two orthogonal spinors

$$\Psi_{1/2}^{1/2} = \frac{r}{\sqrt{1+r^2}}\sin\theta\, e^{i\phi}, \qquad W_-\Psi_{1/2}^{1/2} = \Psi_{1/2}^{-1/2} = \frac{r\cos\theta + i}{\sqrt{1+r^2}},$$

that produce a two-dimensional representation of the rotation group. We can similarly check that $W_-\Psi_{1/2}^{-1/2} = 0$.

By inspection of the structure of $W_\pm$ operators, if we take the complex conjugate of expression $W_+F_s^s = 0$ we get $-W_-(F_s^s)^* = 0$ and therefore $(F_s^s)^* \sim G_s^{-s}$ so that taking the complex conjugate spinors of the above representation we obtain another pair of orthogonal $s = 1/2$ spinors,

$$\widetilde{\Psi}_{1/2}^{1/2} = \frac{r\cos\theta - i}{\sqrt{1+r^2}}, \qquad \widetilde{\Psi}_{1/2}^{-1/2} = \frac{r}{\sqrt{1+r^2}}\sin\theta\, e^{-i\phi}.$$

The remaining representations for higher spins can thus be obtained by the same method, or by taking tensor products of the above two-dimensional representations. For instance, for $s = 1$ we can obtain the following three orthogonal representations. From (3.56) with $s = 1$ and acting with the W_- operator we get

$$\begin{aligned}
\Psi_1^1 = (\Psi_{1/2}^{1/2})^2 &= \frac{r^2}{1+r^2}\sin^2\theta\, e^{i2\phi}, \\
\Psi_1^0 = (\Psi_{1/2}^{1/2})(\Psi_{1/2}^{-1/2}) &= \frac{r}{1+r^2}\sin\theta\,(i + r\cos\theta)\, e^{i\phi}, \\
\Psi_1^{-1} = (\Psi_{1/2}^{-1/2})^2 &= \frac{(i + r\cos\theta)^2}{1+r^2},
\end{aligned}$$

that can also be obtained as the tensor product $\Psi \otimes \Psi$.

If we work in the normal or canonical representation of the rotation group, where the parameters are $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, this amounts to replacing the variable $r = \tan(\alpha/2)$ in terms of parameter α and expressing the differential operator $\partial/\partial r$ in terms of $\partial/\partial\alpha$, and then the spin operators are given by

$$W_1 = \frac{1}{2i}\left[2\sin\theta\,\cos\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\cos\phi}{\tan(\alpha/2)} - \sin\phi\right)\frac{\partial}{\partial\theta} - \left(\frac{\sin\phi}{\tan(\alpha/2)\sin\theta} + \frac{\cos\theta\,\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi}\right],$$

$$W_2 = \frac{1}{2i} \left[2\sin\theta\,\sin\phi\,\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\,\sin\phi}{\tan(\alpha/2)} + \cos\phi \right) \frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\,\sin\phi}{\sin\theta} - \frac{\cos\phi}{\tan(\alpha/2)\sin\theta} \right) \frac{\partial}{\partial\phi} \right],$$

$$W_3 = \frac{1}{2i} \left[2\cos\theta\,\frac{\partial}{\partial\alpha} - \frac{\sin\theta}{\tan(\alpha/2)}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi} \right],$$

$$W^2 = - \left[\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\tan(\alpha/2)}\frac{\partial}{\partial\alpha} + \frac{1}{4\sin^2(\alpha/2)} \left\{ \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right\} \right],$$

$$W_+ = \frac{e^{i\phi}}{2i} \left[2\sin\theta\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta}{\tan(\alpha/2)} + i \right) \frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\tan(\alpha/2) - i}{\tan(\alpha/2)\sin\theta} \right) \frac{\partial}{\partial\phi} \right],$$

$$W_- = \frac{e^{-i\phi}}{2i} \left[2\sin\theta\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta}{\tan(\alpha/2)} - i \right) \frac{\partial}{\partial\theta} - \left(\frac{\cos\theta\tan(\alpha/2) + i}{\tan(\alpha/2)\sin\theta} \right) \frac{\partial}{\partial\phi} \right]$$

and the orthogonal spinors of the two two-dimensional representations can be written as

$$\Psi^{1/2}_{1/2} = i\sin\frac{\alpha}{2}\sin\theta\,e^{i\phi}, \qquad \Psi^{-1/2}_{1/2} = \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\cos\theta \tag{3.57}$$

and

$$\widetilde{\Psi}^{1/2}_{1/2} = \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\theta, \qquad \widetilde{\Psi}^{-1/2}_{1/2} = -i\sin\frac{\alpha}{2}\sin\theta\,e^{-i\phi}. \tag{3.58}$$

We have mentioned that the different spinors are orthogonal. To endow the group manifold with a Hilbert space structure it is necessary to define a hermitian, definite positive, scalar product. The Jacobian matrix of variables $\boldsymbol{\rho}'$ in terms of variables $\boldsymbol{\rho}$ given in (3.37), has the determinant

$$\det\left(\frac{\partial\rho'^i}{\partial\rho^j} \right) = \frac{(1+\mu^2)^2}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^4},$$

and thus the transformation of the volume element

$$d^3\rho' = \frac{(1+\mu^2)^2}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^4}\,d^3\rho.$$

We also get from (3.37) that

$$1 + \rho'^2 = \frac{(1+\mu^2)}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^2}(1+\rho^2)$$

and then the measure

$$\frac{d^3\rho'}{(1+\rho'^2)^2} = \left(\frac{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^2}{(1+\mu^2)(1+\rho^2)} \right)^2 \frac{(1+\mu^2)^2}{(1-\boldsymbol{\mu}\cdot\boldsymbol{\rho})^4}d^3\rho = \frac{d^3\rho}{(1+\rho^2)^2}$$

is in fact an invariant measure.

In spherical coordinates it is written as

$$\frac{r^2\sin\theta}{(1+r^2)^2}\,dr d\theta d\phi$$

and in the normal representation is

$$\sin^2(\alpha/2)\sin\theta d\alpha d\theta d\phi.$$

Since the rotation group is a double-connected group, the above measure must be defined on a simply connected manifold, *i.e.*, on the universal covering group of $SO(3)$, which is $SU(2)$. The $SU(2)$ group manifold in the normal representation is given by the three-dimensional sphere of radius 2π and where points on the surface of this sphere represent a unique $SU(2)$ element, namely the 2×2 unitary matrix $-\mathbb{I}$. The normalized invariant measure becomes

$$d\mu_N(\alpha, \theta, \phi) \equiv \frac{1}{4\pi^2} \sin^2(\alpha/2) \sin\theta \, d\alpha \, d\theta \, d\phi. \tag{3.59}$$

Therefore, the hermitian scalar product will be defined as

$$< f|g >= \frac{1}{4\pi^2} \int_0^{2\pi} d\alpha \int_0^\pi d\theta \int_0^{2\pi} d\phi \, f^*(\alpha, \theta, \phi) g(\alpha, \theta, \phi) \sin^2(\alpha/2) \sin\theta, \tag{3.60}$$

where f^* is the complex conjugate function of f.

All the previous computed spinors are orthogonal vectors with respect to the group invariant measure (3.59). In particular, the normalized $s = 1/2$ spinors are those given in (3.57)-(3.58), multiplied by $\sqrt{2}$.

The spin projection operators on the body axis e_i linked to the particle, are given in (3.51) in the ρ parametrization, and we have seen that they differ from the spin operators W only in the change of $\rho \to -\rho$, and a global change of sign. In the normal parametrization this corresponds to the change $\alpha \to -\alpha$, followed by a global change of sign.

It can be checked as mentioned before, that

$$[T_i, T_k] = -i\epsilon_{ikl} T_l, \tag{3.61}$$

$$[W_i, T_k] = 0. \tag{3.62}$$

Since $W^2 = T^2$ we can find simultaneous eigenvectors of the operators W^2, W_3 and T_3, which will be denoted by $D_{mn}^{(s)}(\boldsymbol{\alpha})$ in such a way that

$$\begin{aligned}
W^2 D_{mn}^{(s)}(\boldsymbol{\alpha}) &= s(s+1) D_{mn}^{(s)}(\boldsymbol{\alpha}), \\
W_3 D_{mn}^{(s)}(\boldsymbol{\alpha}) &= m D_{mn}^{(s)}(\boldsymbol{\alpha}), \\
T_3 D_{mn}^{(s)}(\boldsymbol{\alpha}) &= n D_{mn}^{(s)}(\boldsymbol{\alpha}).
\end{aligned}$$

Since $W_3(\alpha) D_{mn}^{(s)}(\alpha) = m D_{mn}^{(s)}(\alpha)$, by producing the change $\alpha \to -\alpha$ we get $W_3(-\alpha) D_{mn}^{(s)}(-\alpha) = m D_{mn}^{(s)}(-\alpha)$ and the subsequent global change of sign it reduces to

$$-W_3(-\alpha) D_{mn}^{(s)}(-\alpha) = T_3(\alpha) D_{mn}^{(s)}(-\alpha) = -m D_{mn}^{(s)}(-\alpha),$$

so that the above spinors (3.57)-(3.58) are also eigenvectors of T_3.

With this notation, the four normalized spinors, denoted by the corresponding eigenvalues $|s, m, n >$, are

$$\begin{aligned}
\Phi_1 &= |1/2, 1/2, 1/2 > &&= \sqrt{2}(\cos(\alpha/2) + i\cos\theta\sin(\alpha/2)), &\tag{3.63}\\
\Phi_2 &= |1/2, -1/2, 1/2 > &&= i\sqrt{2}\sin(\alpha/2)\sin\theta e^{-i\phi}, &\tag{3.64}\\
\Phi_3 &= |1/2, 1/2, -1/2 > &&= i\sqrt{2}\sin(\alpha/2)\sin\theta e^{i\phi}. &\tag{3.65}\\
\Phi_4 &= |1/2, -1/2, -1/2 > &&= \sqrt{2}(\cos(\alpha/2) - i\cos\theta\sin(\alpha/2)), &\tag{3.66}
\end{aligned}$$

$$\tag{3.67}$$

They form an orthonormal set with respect to the normalized invariant measure (3.59) and with the scalar product defined in (3.60). We can check that the lowering operators $W_-\Phi_1 = \Phi_2$, $W_-\Phi_2 = 0$, $W_-\Phi_3 = \Phi_4$, $W_-\Phi_4 = 0$, and simmilarly $T_-\Phi_1 = 0$, $T_-\Phi_3 = \Phi_1$, $T_-\Phi_2 = 0$, and $T_-\Phi_4 = \Phi_2$, and the corresponding up relations when acting with the rising operators W_+ and T_+, respectively. Remark that because the opposite sign in the commutation relations of the T_i operators, here the $T_\pm$ operate in the reverse direction.

The important feature is that if the system has spin 1/2, although the $s = 1/2$ irreducible representations of the rotation group are two-dimensional, to describe the spin part of the wave function we need a function defined in the above four-dimensional complex Hilbert space, because to describe orientation we attach some local frame to the particle, and therefore in addition to the spin values in the laboratory frame we also have as additional observables the spin projections in the body axes, which can be included within the set of commuting operators.

3.2.2 Matrix representation of internal observables

The matrix representation of any observable A that acts on the orientation variables or in this internal four-dimensional space spanned by these spin 1/2 wave functions Φ_i, is obtained as $A_{ij} = <\Phi_i|A\Phi_j>$, $i, j = 1, 2, 3, 4$. Once these four normalized basis vectors are fixed, when acting on the subspace they span, the differential operators W_i and T_i have the 4×4 block matrix representation

$$S \equiv W = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{3.68}$$

$$T_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad T_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \quad T_3 = \frac{\hbar}{2} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{3.69}$$

where $\boldsymbol{\sigma}$ are the three Pauli matrices and $\mathbb{I}$ represents the 2×2 unit matrix. We have included Planck's constant into the angular momentum operators.

If we similarly compute the matrix elements of the nine components of the unit vectors $(\boldsymbol{e}_i)_j$, $i, j = 1, 2, 3$ we obtain the nine traceless hermitian matrices

$$\boldsymbol{e}_1 = \frac{1}{3} \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \frac{1}{3} \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{3} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}. \tag{3.70}$$

We can check that the $T_i = \boldsymbol{S} \cdot \boldsymbol{e}_i = \boldsymbol{e}_i \cdot \boldsymbol{S}$. We see that the different components of the unit vectors $\boldsymbol{e}_i$, in general do not commute. The eigenvalues of every component e_{ij}, in this matrix representation of definite spin, are $\pm 1/3$. However, the matrix representation of the square of any component is $(e_{ij})^2 = \mathbb{I}/3$, so that the magnitude squared of each vector $\boldsymbol{e}_i^2 = \sum_j (e_{ij})^2 = \mathbb{I}$ when acting on these wave functions. The eigenvalues of the squared operator $(e_{ij})^2$ are not the squared eigenvalues of e_{ij}. This is because the function $e_{ij}\Phi_k$ does not belong in general to the same space spanned by the Φ_k, $k = 1, \ldots, 4$ although this space is invariant space for operators W_i and T_j. In fact, each function $e_{ij}\Phi_k$ is a linear combination of a spin 1/2 and a spin 3/2 wave function.

We do not understand why any component of a classical unit vector e_{ij} of a Cartessian frame, can have as eigenvalues $\pm 1/3$ in the quantum case and its square $(e_{jj})^2 = \mathbb{I}/3$ instead of $\mathbb{I}/9$.

3.2.3   Peter-Weyl theorem for compact groups

The above spinors can also be obtained by making use of an important theorem for representations of compact groups, known as the Peter-Weyl theorem, [10] which is stated without proof that can be read in any of the mentioned references.

Theorem.-   Let $D^{(s)}(g)$ be a complete system of non-equivalent, unitary, irreducible representations of a compact group G, labeled by the parameter s. Let d_s be the dimension of each representation and $D_{ij}^{(s)}(g)$, $1 \leq i, j \leq d_s$ the corresponding matrix elements. Then, the functions

$$\sqrt{d_s}\, D_{ij}^{(s)}(g), \quad 1 \leq i, j \leq d_s$$

form a complete orthonormal system on G, with respect to some normalized invariant measure $\mu_N(g)$ defined on this group, *i.e.*,

$$\int_G \sqrt{d_s}\, D_{ij}^{(s)*}(g)\, \sqrt{d_r}\, D_{kl}^{(r)}(g)\, d\mu_N(g) = \delta^{sr}\delta_{ik}\delta_{jl}. \tag{3.71}$$

That the set is complete means that every square integrable function defined on G, $f(g)$, admits a series expansion, convergent in norm, in terms of the above orthogonal functions $D_{ij}^{(s)}(g)$, in the form

$$f(g) = \sum_{s,i,j} a_{ij}^{(s)}\, \sqrt{d_s}\, D_{ij}^{(s)}(g),$$

where the coefficients, in general complex numbers $a_{ij}^{(s)}$, are obtained by

$$a_{ij}^{(s)} = \int_G \sqrt{d_s}\, D_{ij}^{(s)*}(g)\, f(g) d\mu_N(g).$$

In our case $SU(2)$, as a group manifold, is the simply connected three-dimensional sphere of radius 2π, with the normalized measure as seen before (3.59),

$$d\mu_N(\alpha, \theta, \phi) = \frac{1}{4\pi^2} \sin\theta \sin(\alpha/2)^2\, d\alpha d\theta d\phi.$$

In the normal parametrization, the two-dimensional representation of $SU(2)$ corresponds to the eigenvalue $s = 1/2$ of S^2 and the matrix representation is given by

$$D^{(1/2)}(\boldsymbol{\alpha}) = \cos(\alpha/2)\mathbb{I} - i\sin(\alpha/2)(\boldsymbol{u} \cdot \boldsymbol{\sigma}),$$

i.e.,

$$D^{(1/2)}(\boldsymbol{\alpha}) = \begin{pmatrix} \cos(\alpha/2) - i\cos\theta \sin(\alpha/2) & -i\sin\theta \sin(\alpha/2)\, e^{-i\phi} \\ -i\sin\theta \sin(\alpha/2)\, e^{i\phi} & \cos(\alpha/2) + i\cos\theta \sin(\alpha/2) \end{pmatrix}.$$

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[10] N. Ja. Vilenkin, *Fonctions spéciales et Théorie de la représentation des groupes*, Dunod, Paris (1969), p. 39.
A.O. Barut and R. Raczka, *Theory of group representations and applications*, PWN-Polish Scientific Publishers, Warszawa (1980), p. 174.
F. Peter and H. Weyl, *Math. Ann.* **7**, 735 (1927).

If we compare these four matrix components with the four orthogonal spinors given in (4.81)-(4.84) we see that

$$D^{(1/2)}(\boldsymbol{\alpha}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_4 & -\Phi_2 \\ -\Phi_3 & \Phi_1 \end{pmatrix} \qquad (3.72)$$

In the three-dimensional representation of $SO(3)$, considered as a representation of SU(2)

$$D^{(1)}_{ij}(\boldsymbol{\alpha}) = \delta_{ij} \cos\alpha + u_i u_j (1 - \cos\alpha) + \epsilon_{ikj} u_k \sin\alpha \equiv e_{j_i}$$

we get another set of nine orthogonal functions. Multiplied by $\sqrt{3}$ they form another orthonormal set orthogonal to the previous four spinors. It is a good exercise to check this orthogonality among these functions.

3.2.4 General spinors

In the case that the zitterbewegung content of the spin is not vanishing we can also obtain spin $1/2$ wave-functions as the irreducible representations contained in the tensor product of integer and half-integer spin states coming from the $U(\boldsymbol{u})$ and $V(\boldsymbol{\rho})$ part of the general wave function (3.53).

The total spin operator of the system is of the form

$$\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W} = \boldsymbol{Z} + \boldsymbol{W},$$

where $\boldsymbol{Z} = -i\hbar\nabla_u$ and $\boldsymbol{W}$ is given in (3.46). Spin projections on the body axes, *i.e.*, operators $T_i = \boldsymbol{e}_i \cdot \boldsymbol{W}$, are described in (3.51). They satisfy the commutation relations

$$[\boldsymbol{Z}, \boldsymbol{Z}] = i\boldsymbol{Z}, \quad [\boldsymbol{W}, \boldsymbol{W}] = i\boldsymbol{W}, \quad [\boldsymbol{T}, \boldsymbol{T}] = i\boldsymbol{T},$$

$$[\boldsymbol{Z}, \boldsymbol{W}] = 0, \quad [\boldsymbol{Z}, \boldsymbol{T}] = 0, \quad [\boldsymbol{W}, \boldsymbol{T}] = 0.$$

These commutation relations are invariant under the change $\boldsymbol{\rho}$ by $-\boldsymbol{\rho}$ in the definition of the operators $\boldsymbol{W}$ and $\boldsymbol{T}$, because they are changed into each other. The expression of the body frame unit vectors $\boldsymbol{e}_i$ is given in (3.48) and (3.49).

We can see that these unit vector components and spin operators W_i and T_j satisfy the following properties:
    1) $e_{ij}(-\alpha, \theta, \phi) = -e_{j_i}(\alpha, \theta, \phi)$.
    2) $\boldsymbol{e}_i \cdot \boldsymbol{W} \equiv \sum_j e_{ij} W_j = T_i$.
    3) $\sum_j \boldsymbol{e}_j T_j = \boldsymbol{W}$.
    4) For all i, j, the action $W_i e_{j_i} = 0$, with no addition on index i.
    5) For all i, j, the action $T_i e_{ij} = 0$, with no addition on index i.
    6) For all i, j, k, with $i \neq j$, we have that $W_i e_{kj} + W_j e_{ki} = 0$, and in the case that $i = j$, it leads to property 4.
    7) For all i, j, k, with $i \neq j$, we have that $T_i e_{j_k} + T_j e_{ik} = 0$, and similarly as before in the case $i = j$ it leads to property 4.

This implies that $\boldsymbol{e}_i \cdot \boldsymbol{W} = \boldsymbol{W} \cdot \boldsymbol{e}_i = T_i$, because of property 4, since when acting on an arbitray function f,

$$(\boldsymbol{W} \cdot \boldsymbol{e}_i)f \equiv \sum_j W_j(e_{ij}f) = f \sum_j W_j(e_{ij}) + \sum_j e_{ij} W_j(f) = T_i(f),$$

because $\sum_j W_j e_{ij} = 0$.

In the same way $\sum_j \boldsymbol{e}_j T_j \equiv \sum_j T_j \boldsymbol{e}_j = \boldsymbol{W}$.

Now we fix the value of spin. Particles of different values of spin can be described. Let us consider systems that take the lowest admissible spin values. For spin $1/2$ particles, if we take first for simplicity eigenfunctions $V(\boldsymbol{\rho})$ of W^2 with eigenvalue $1/2$, and then since the total spin has to be $1/2$, the orbital $\boldsymbol{Z}$ part can only contribute with spherical harmonics of value $z = 0$ and $z = 1$.

If there is no zitterbewegung spin, $z = 0$, and Wigner's functions can be taken as simultaneous eigenfunctions of the three commuting W^2, W_3, and T_3 operators, and the normalized eigenvectors $|w, w_3, t_3 >$ are explicitly given by the functions (4.81-4.84).

If we have a zitterbewegung spin of value $z = 1$, then the $U(\boldsymbol{u})$ part contributes with the spherical harmonics

$$Y_1^1(\tilde{\theta}, \tilde{\phi}) \equiv |1, 1 >= -\sin(\tilde{\theta})e^{i\tilde{\phi}}\sqrt{\frac{3}{8\pi}}, \qquad (3.73)$$

$$Y_1^0(\tilde{\theta}, \tilde{\phi}) \equiv |1, 0 >= \cos(\tilde{\theta})\sqrt{\frac{3}{4\pi}}, \qquad (3.74)$$

$$Y_1^{-1}(\tilde{\theta}, \tilde{\phi}) \equiv |1, -1 >= \sin(\tilde{\theta})e^{-i\tilde{\phi}}\sqrt{\frac{3}{8\pi}}, \qquad (3.75)$$

normalized with respect to the measure

$$\int_0^\pi \int_0^{2\pi} \sin(\tilde{\theta})d\tilde{\theta}d\tilde{\phi},$$

which are the indicated eigenfunctions $|z, z_3 >$ of $\boldsymbol{Z}^2$ and Z_3, and where the variables $\tilde{\theta}$ and $\tilde{\phi}$ determine the orientation of the velocity $\boldsymbol{u}$.

The tensor product representation of the rotation group constructed from the two irreducible representations $\mathbf{1}$ associated to the spherical harmonics (3.73)-(3.75) and $\mathbf{1/2}$ given in (4.81)-(4.84) is split into the direct sum $\mathbf{1} \otimes \mathbf{1/2} = \mathbf{3/2} \oplus \mathbf{1/2}$.

The following functions of five variables $\tilde{\theta}$, $\tilde{\phi}$, α, θ and ϕ, where variables $\tilde{\theta}$ and $\tilde{\phi}$ correspond to the ones of the spherical harmonics Y_l^m, and the remaining α, θ and ϕ, to the previous spinors Φ_i, are normalized spin $1/2$ functions $|s, s_3, t_3 >$ that are eigenvectors of total spin S^2, and S_3 and T_3 operators

$$\Psi_1 \equiv |1/2, 1/2, 1/2 > \quad = \frac{1}{\sqrt{3}}\left(Y_1^0\Phi_1 - \sqrt{2}Y_1^1\Phi_2\right), \qquad (3.76)$$

$$\Psi_2 \equiv |1/2, -1/2, 1/2 > \quad = \frac{1}{\sqrt{3}}\left(-Y_1^0\Phi_2 + \sqrt{2}Y_1^{-1}\Phi_1\right), \qquad (3.77)$$

$$\Psi_3 \equiv |1/2, 1/2, -1/2 > \quad = \frac{1}{\sqrt{3}}\left(Y_1^0\Phi_3 - \sqrt{2}Y_1^1\Phi_4\right), \qquad (3.78)$$

$$\Psi_4 \equiv |1/2, -1/2, -1/2 > \quad = \frac{1}{\sqrt{3}}\left(-Y_1^0\Phi_4 + \sqrt{2}Y_1^{-1}\Phi_3\right), \qquad (3.79)$$

such that $\Psi_2 = S_-\Psi_1$ and similarly $\Psi_4 = S_-\Psi_3$, and also that $\Psi_3 = T_-\Psi_1$, and $\Psi_4 = T_-\Psi_2$. They are no longer eigenfunctions of the W_3 operator, although they span an invariant vector space for S^2, S_3 and T_3 operators. In the above basis (3.76)-(3.79) formed by orthonormal

vectors Ψ_i, the matrix representation of the spin is

$$S = Z + W = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{3.80}$$

while the matrix representation of the Z and W part is

$$Z = \frac{2\hbar}{3} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad W = \frac{-\hbar}{6} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{3.81}$$

which do not satisfy commutation relations of angular momentum operators because the vector space spanned by the above basis is not an invariant space for these operators Z and W.

The spin projection of the W part on the body axis, *i.e.*, the T operator, takes the same form as before (3.69)

$$T_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad T_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \quad T_3 = \frac{\hbar}{2} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{3.82}$$

because Ψ_1 and Ψ_2 functions are eigenfunctions of T_3 with eigenvalue $1/2$, while Ψ_3 and Ψ_4 are of eigenvalue $-1/2$, and thus the spinors Ψ_i span an invariant space for S_i and T_j operators. In fact the basis is formed by simultaneous eigenfunctions of total spin S^2, S_3 and T_3, and the ket representation is the same as in the case of the Φ_i given in (4.81)-(4.84).

The expression in this basis of the components of the unit vectors $\boldsymbol{e}_i$ are represented by

$$\boldsymbol{e}_1 = -\frac{1}{9} \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \ \boldsymbol{e}_2 = -\frac{1}{9} \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix}, \ \boldsymbol{e}_3 = -\frac{1}{9} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}. \tag{3.83}$$

Chapter 4

Dirac particle

4.1 Quantization of the $u = c$ model

For Luxons we have the nine-dimensional homogeneous space of the Poincaré group, spanned by the ten variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha})$ similarly as before, but now $\boldsymbol{u}$ is restricted to $u = c$. For this system, since $\boldsymbol{u} \cdot \dot{\boldsymbol{u}} = 0$ and $\dot{\boldsymbol{u}} \neq 0$, we are describing particles with a circular internal orbital motion at the constant speed c.

In the center of mass frame, (see Fig.4.1) the center of charge describes a circle of radius $R_0 = S/mc$ at the constant speed c, the spin being orthogonal to the charge trajectory plane and a constant of the motion in this frame. Let us consider the quantization of this $u = c$ model whose dynamical equation is given by (2.126).
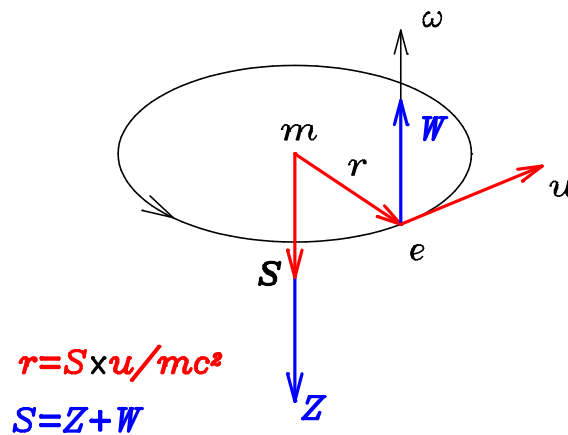
Figure 4.1: **Motion of the charge in the C.M. frame.**

If we analyse this particle in the centre of mass frame it becomes a system of three degrees of freedom. These are the x and y coordinates of the point charge on the plane and the phase α of the rotation of the body axis with angular velocity ω. But this phase is the same as the phase of the orbital motion, as we shall see later, and because this motion is a circle of constant

radius only one degree of freedom is left, for instance the x coordinate. In the centre of mass frame the system is equivalent to a one-dimensional harmonic oscillator of angular frequency $\omega = mc^2/S$ in its ground state.

Identification of the ground energy of the one-dimensional harmonic oscillator $\hbar\omega/2$ with the rest energy of the system in the center of mass frame $+mc^2$, for $H > 0$ particles, implies that the classical constant parameter $S = \hbar/2$. All Lagrangian systems defined with this kinematical space, irrespective of the particular Lagrangian we choose, have this behaviour and represent spin 1/2 particles when quantized.

4.2   Dirac's equation

The kinematical variables of this system transform under $\mathcal{P}$ according to

$$t'(\tau) = \gamma t(\tau) + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))/c^2 + b, \tag{4.1}$$

$$\boldsymbol{r}'(\tau) = R(\boldsymbol{\mu})\boldsymbol{r}(\tau) + \gamma\boldsymbol{v}t(\tau) + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{r}(\tau))\boldsymbol{v} + \boldsymbol{a}, \tag{4.2}$$

$$\boldsymbol{u}'(\tau) = \frac{R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \gamma\boldsymbol{v} + (\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau))\boldsymbol{v}\gamma^2/(1+\gamma)c^2}{\gamma(1 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{u}(\tau)/c^2)}, \tag{4.3}$$

$$\boldsymbol{\rho}'(\tau) = \frac{\boldsymbol{\mu} + \boldsymbol{\rho}(\tau) + \boldsymbol{\mu} \times \boldsymbol{\rho}(\tau) + \boldsymbol{F}_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}(\tau), \boldsymbol{\rho}(\tau))}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}(\tau) + G_c(\boldsymbol{v}, \boldsymbol{\mu}; \boldsymbol{u}(\tau), \boldsymbol{\rho}(\tau))}, \tag{4.4}$$

where the functions $\boldsymbol{F}_c$ and G_c are given in (2.92) and (2.93), respectively. When quantized, the wave function of the system is a function $\Phi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$ of these kinematical variables. For the Poincaré group all exponents and thus all gauge functions on homogeneous spaces are equivalent to zero, and the Lagrangians for free particles can thus be taken strictly invariant. Projective representations reduce to true representations so that the ten generators on the Hilbert space, taking into account (4.1)-(4.4) and (3.15) are given by:

$$H = i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{P} = \frac{\hbar}{i}\nabla, \quad \boldsymbol{K} = \boldsymbol{r}\frac{i\hbar}{c^2}\frac{\partial}{\partial t} - t\frac{\hbar}{i}\nabla - \frac{1}{c^2}\boldsymbol{S} \times \boldsymbol{u}, \tag{4.5}$$

$$\boldsymbol{J} = \boldsymbol{r} \times \frac{\hbar}{i}\nabla + \boldsymbol{S}, \tag{4.6}$$

where as we shall see, the angular momentum operator $\boldsymbol{S}$ represents Dirac's spin operator and is given by the differential operator

$$\boldsymbol{S} = \boldsymbol{u} \times \frac{\hbar}{i}\nabla_u + \frac{\hbar}{2i}\left\{\nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho)\right\} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W}, \tag{4.7}$$

and where the differential operators ∇_u and ∇_ρ are the corresponding gradient operators with respect to the $\boldsymbol{u}$ and $\boldsymbol{\rho}$ variables as in the Galilei case. The operator $\boldsymbol{S}$ is not a constant of the motion even for the free particle, and although it is not the angular momentum of the system with respect to its center of mass we keep this notation because it is the equivalent to Dirac's spin operator. Of course, it reduces to the true spin $\boldsymbol{S}$ in the center of mass frame.

To obtain the complete commuting set of observables we start with the Casimir invariant operator, or Klein-Gordon operator

$$H^2 - c^2\boldsymbol{P}^2 = m^2c^4. \tag{4.8}$$

In the above representation, H and $\boldsymbol{P}$ only differentiate the wave function with respect to time t and position $\boldsymbol{r}$, respectively. Since the spin operator $\boldsymbol{S}$ operates only on the velocity and orientation variables, it commutes with the Klein-Gordon operator (4.8). Thus, we can find simultaneous eigenfunctions of (4.8), S^2, and S_3. This allows us to try solutions in separate variables so that the wave function can be written

$$\Phi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho}) = \sum_i \psi_i(t, \boldsymbol{r}) \chi_i(\boldsymbol{u}, \boldsymbol{\rho}), \tag{4.9}$$

where $\psi_i(t, \boldsymbol{r})$ are the space-time components and the $\chi_i(\boldsymbol{u}, \boldsymbol{\rho})$ represent the internal spin structure. Consequently

$$\left(H^2 - c^2 \boldsymbol{P}^2 - m^2 c^4 \right) \psi_i(t, \boldsymbol{r}) = 0, \tag{4.10}$$

i.e., space-time components satisfy the Klein-Gordon equation, while the internal structure part satisfies

$$S^2 \chi_i(\boldsymbol{u}, \boldsymbol{\rho}) = s(s+1) \hbar^2 \chi_i(\boldsymbol{u}, \boldsymbol{\rho}), \tag{4.11}$$

$$S_3 \chi_i(\boldsymbol{u}, \boldsymbol{\rho}) = m_s \hbar \chi_i(\boldsymbol{u}, \boldsymbol{\rho}). \tag{4.12}$$

Eigenfunctions of the above type have been found in Section 3.2, in particular we are interested in solutions that give rise to spin $1/2$ particles. These solutions, which are also eigenvectors of the spin projection on the body axis T_3, become a four-component wave function.

For spin $1/2$ particles, if we take first for simplicity eigenfunctions $\chi(\boldsymbol{\rho})$ of S^2 with eigenvalue $1/2$, then since the total spin has to be $1/2$, the orbital zitterbewegung part $\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U}$ can only contribute with spherical harmonics of value $z = 0$ and $z = 1$. This means that we can find at least two different kinds of elementary particles of spin $1/2$, one characterized by the singlet $z = 0$ (lepton?) and another by $z = 1$ (quark?) in three possible states according to the component z_3. If we call to the spin part Z the **colour**, we can have colourless and coloured systems of spin $1/2$. The three different colours z_3 are unobservable because the Ψ_i states (3.76-3.79) are eigenstates of S_3 and T_3 but not eigenstates of z_3. Nevertheless this interpretation of this spin part as representing the colour, as in the standard model, is still unclear and will be discussed elsewhere.

For $z = 0$, the spin $1/2$ functions $\chi_i(\boldsymbol{\rho})$ are linear combinations of the four Φ_i functions (4.81)-(4.84) and in the case $z = 1$ they are linear combinations of the four Ψ_i of (3.76)-(3.79), such that the factor function in front of the spherical harmonics is 1 because for this model $u = c$ is a constant. It turns out that the Hilbert space that describes the internal structure of this particle is isomorphic to the four-dimensional Hilbert space $\mathbb{C}^4$.

If we have two arbitrary directions in space characterized by the unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively, and $S_{\boldsymbol{u}}$ and $S_{\boldsymbol{v}}$ are the corresponding angular momentum projections $S_{\boldsymbol{u}} = \boldsymbol{u} \cdot \boldsymbol{S}$ and $S_{\boldsymbol{v}} = \boldsymbol{v} \cdot \boldsymbol{S}$, then $S_{-\boldsymbol{u}} = -S_{\boldsymbol{u}}$, and $[S_{\boldsymbol{u}}, S_{\boldsymbol{v}}] = i\hbar S_{\boldsymbol{u} \times \boldsymbol{v}}$. In the case of the opposite sign commutation relations of operators T_i, we have for instance for the spin projections $[T_1, T_2] = -i\hbar T_3$, thus suggesting that $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = -\boldsymbol{e}_3$, and any cyclic permutation $1 \to 2 \to 3$, and thus $\boldsymbol{e}_i$ vectors linked to the body, not only have as eigenvalues $\pm 1/3$, but also behave in the quantum case as a left-handed system. In this case $\boldsymbol{e}_i$ vectors are not arbitrary vectors in space, but rather vectors linked to the rotating body and thus they are not compatible observables, so that any measurement to determine, say the components of $\boldsymbol{e}_i$, will produce some interaction with the body that will mask the measurement of the others. We shall use this interpretation of a left-handed system for particles later, when we analyse the chirality in section 4.2.3. For antiparticles it will behave as a right handed one.

Operators S_i and T_i have the matrix representation obtained before which is just

$$S \equiv W = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{4.13}$$

$$T_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad T_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & i\mathbb{I} \\ -i\mathbb{I} & 0 \end{pmatrix}, \quad T_3 = \frac{\hbar}{2} \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{4.14}$$

where we represent by $\boldsymbol{\sigma}$ the three Pauli matrices and $\mathbb{I}$ is the 2×2 unit matrix.

Similarly, the matrix elements of the nine components of the unit vectors $(\boldsymbol{e}_i)_j$, $i, j = 1, 2, 3$ give rise to the two alternative sets of representations depending on whether the zitterbewegung contribution is $y = 0$ or $y = 1$. In the first case we get

$$\boldsymbol{e}_1 = \frac{1}{3} \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \, \boldsymbol{e}_2 = \frac{1}{3} \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix}, \, \boldsymbol{e}_3 = \frac{1}{3} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \tag{4.15}$$

while in the $y = 1$ case the representation is

$$\boldsymbol{e}_1 = -\frac{1}{9} \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \, \boldsymbol{e}_2 = -\frac{1}{9} \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix}, \, \boldsymbol{e}_3 = -\frac{1}{9} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}. \tag{4.16}$$

It must be remarked that the different components of the observables $\boldsymbol{e}_i$ are not compatible in general, because they are represented by non-commuting operators.

We finally write the wave function for spin $1/2$ particles in the following form for $y = 0$

$$\Phi_{(0)}(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}) = \sum_{i=1}^{i=4} \psi_i(t, \boldsymbol{r}) \Phi_i(\alpha, \theta, \phi), \tag{4.17}$$

independent of the $\boldsymbol{u}$ variables, and in the case $y = 1$ by

$$\Phi_{(1)}(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}) = \sum_{i=1}^{i=4} \psi_i(t, \boldsymbol{r}) \Psi_i(\widetilde{\theta}, \widetilde{\phi}; \alpha, \theta, \phi). \tag{4.18}$$

where $\widetilde{\theta}$ and $\widetilde{\phi}$ represent the direction of vector $\boldsymbol{u}$. Then, once the Φ_i or Ψ_j functions that describe the internal structure are identified with the four orthogonal unit vectors of the internal Hilbert space $\mathbb{C}^4$, the wave function becomes a four-component space-time wave function, and the six spin components S_i and T_j and the nine vector components $(\boldsymbol{e}_i)_j$, together the 4×4 unit matrix, completely exhaust the 16 linearly independent 4×4 hermitian matrices. They form a vector basis of Dirac's algebra, such that any other translation invariant internal observable that describes internal structure, for instance internal velocity and acceleration, angular velocity, etc., must necessarily be expressed as a real linear combination of the mentioned 16 hermitian matrices. We shall see in Sec. 4.3 that the internal orientation completely characterizes its internal structure.

The spin operator $\boldsymbol{S} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W}$ which, as seen in (3.80) and (4.13), coincides with the usual matrix representation of Dirac's spin operator.

If we consider the expression of the kinematical momentum for $u = c$ particles

$$\boldsymbol{K} = \frac{H}{c^2} \boldsymbol{r} - t\boldsymbol{P} - \frac{1}{c^2} \boldsymbol{S} \times \boldsymbol{u}$$

and we take the time derivative of this expression followed by the scalar product with $\boldsymbol{u}$, it leads to the Poincaré invariant operator (Dirac's operator):

$$H - \boldsymbol{P} \cdot \boldsymbol{u} - \frac{1}{c^2} \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u} \right) \cdot \boldsymbol{S} = 0. \tag{4.19}$$

When Dirac's operator acts on a general wave function $\Phi_{(0)}$ or $\Phi_{(1)}$, we know that H and $\boldsymbol{P}$ have the differential representation given by (3.38) and the spin the differential representation (3.46), or the equivalent matrix representation (4.13), but we do not know how to represent the action of the velocity $\boldsymbol{u}$ and the $(d\boldsymbol{u}/dt) \times \boldsymbol{u}$ observable. However, we know that for this particle $\boldsymbol{u}$ and $d\boldsymbol{u}/dt$ are orthogonal vectors and together with vector $\boldsymbol{u} \times d\boldsymbol{u}/dt$ they form an orthogonal right-handed system, and in the center of mass frame the particle describes a circle of radius $R_0 = \hbar/2mc$ in the plane spanned by $\boldsymbol{u}$ and $d\boldsymbol{u}/dt$.
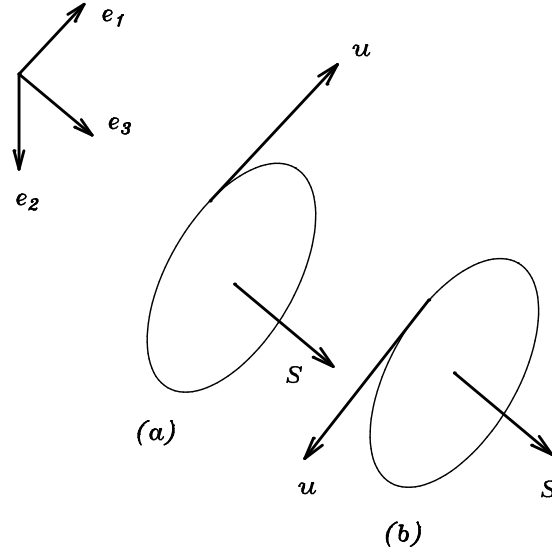
Figure 4.2: **Representation of the local body frame and the different observables for the (a) $H > 0$ solution and (b) $H < 0$ solution. This orientation produces Dirac equation in the Pauli-Dirac representation**

Let us consider first the case $z = 0$. Since $\boldsymbol{u}$ and $d\boldsymbol{u}/dt$ are translation invariant observables they will be elements of Dirac's algebra, and it turns out that we can relate these three vectors with the left-handed orthogonal system formed by vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ with representation (4.15). Then, as shown in part (a) of Figure 4.2 for the $H > 0$ system, we have $\boldsymbol{u} = a\boldsymbol{e}_1$ and $d\boldsymbol{u}/dt \times \boldsymbol{u} = b\boldsymbol{e}_3$, where a and b are constant positive real numbers. Then the third term in Dirac's operator is $(b/c^2)\boldsymbol{e}_3 \cdot \boldsymbol{S} = (b/c^2)T_3$, and (4.19) operator becomes

$$H - a\boldsymbol{P} \cdot \boldsymbol{e}_1 - \frac{b}{c^2}T_3 = 0. \tag{4.20}$$

If we make the identification with the $H < 0$ solution of part (b) of Figure 4.2, the relation of the above observables is opposite to the previous one but now with the coefficients $-a$ and $-b$, respectively, *i.e.*, we get

$$H + a\boldsymbol{P} \cdot \boldsymbol{e}_1 + \frac{b}{c^2}T_3 = 0, \tag{4.21}$$

which clearly corresponds to the change $H \rightarrow -H$ in equation (4.20).

Multiplying (4.21) by (4.20) we obtain an expression which is satisfied by both particle and antiparticle

$$H^2 - \frac{a^2}{9}\boldsymbol{P}^2 \mathbb{I} - \frac{b^2 \hbar^2}{4c^4}\mathbb{I} = 0, \tag{4.22}$$

and which is an algebraic relation between H^2 and P^2. By identification of this expression with the Klein-Gordon operator (4.8), which also contains both $H > 0$ and $H < 0$ solutions, leads to $a = 3c$ and $b = 2mc^4/\hbar = c^3/R_0$ and by substitution in (4.20) we obtain Dirac's equation:

$$H - c\boldsymbol{P} \cdot \boldsymbol{\alpha} - \beta mc^2 = 0, \tag{4.23}$$

where Dirac's matrices $\boldsymbol{\alpha}$ and β are represented by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{4.24}$$

and thus Dirac's gamma matrices are

$$\gamma^0 \equiv \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \boldsymbol{\gamma} \equiv \gamma^0 \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{4.25}$$

i.e., Pauli-Dirac representation, where $3\boldsymbol{e}_1$ plays the role of a unit vector in the direction of the velocity. Substitution into (4.21) corresponds to the equivalent representation with the change $\gamma^\mu \rightarrow -\gamma^\mu$.

This representation is compatible with the acceleration $d\boldsymbol{u}/dt$ lying along the vector $\boldsymbol{e}_2$. In fact, in the center of mass frame and in the Heisenberg representation, Dirac's Hamiltonian reduces to $H = \beta mc^2$, and the time derivative of any observable A is obtained as

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A] + \frac{\partial A}{\partial t}, \tag{4.26}$$

such that for the velocity operator $\boldsymbol{u} = c\boldsymbol{\alpha}$,

$$\frac{d\boldsymbol{u}}{dt} = \frac{i}{\hbar}[mc^2\beta, c\boldsymbol{\alpha}] = \frac{2mc^3}{\hbar}\begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix} = \frac{c^2}{R_0}3\boldsymbol{e}_2, \tag{4.27}$$

c^2/R_0 being the constant modulus of the acceleration in this frame, and where $3\boldsymbol{e}_2$ plays the role of a unit vector along that direction.

The time derivative of this Cartesian system is

$$\frac{d\boldsymbol{e}_1}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{e}_1] = \frac{c}{R_0}\boldsymbol{e}_2, \tag{4.28}$$

$$\frac{d\boldsymbol{e}_2}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{e}_2] = -\frac{c}{R_0}\boldsymbol{e}_1, \tag{4.29}$$

$$\frac{d\boldsymbol{e}_3}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{e}_3] = 0, \tag{4.30}$$

since $\boldsymbol{e}_3$ is orthogonal to the trajectory plane and does not change, and where $c/R_0 = \omega$ is the angular velocity of the internal orbital motion. This time evolution of the observables $\boldsymbol{e}_i$ is the correct one if assumed to be a rotating left-handed system of vectors as shown in Figure

4.2-(a). It is for this reason that we considered at the beginning of this chapter that the body frame rotates with the same angular velocity as the orbital motion of the charge.

To be consistent with the above consideration as $3e_i$ as unit vectors, this means that the spin in the center of mass frame should be along $3e_3$. This is the case for the upper components while for the lower components (which in this representation correspond to $H < 0$ states) the orientation is the opposite. This means that for particles the corresponding set of axis forms a left handed system while for antiparticles they behave as a right handed system, showing a clear chirality difference between particles and antiparticles.

In general

$$\frac{d\boldsymbol{S}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{S}] = \frac{i}{\hbar}[c\boldsymbol{P}\cdot\boldsymbol{\alpha} + \beta mc^2, \boldsymbol{S}] = c\boldsymbol{P}\times\boldsymbol{\alpha} \equiv \boldsymbol{P}\times\boldsymbol{u},$$

is not a constant of the motion, but for the center of mass observer, this spin operator $\boldsymbol{u}\times\boldsymbol{U}+\boldsymbol{W}$ reduces to the equivalent of the classical spin of the particle $\boldsymbol{S}$ and is constant in this frame:

$$\frac{d\boldsymbol{S}}{dt} = \frac{i}{\hbar}[\beta mc^2, \boldsymbol{S}] = 0. \tag{4.31}$$

Only the T_3 spin component on the body axis remains constant while the other two T_1 and T_2 change because of the rotation of the corresponding axis,

$$\frac{dT_1}{dt} = \frac{i}{\hbar}[\beta mc^2, T_1] = \frac{c}{R_0}T_2, \tag{4.32}$$

$$\frac{dT_2}{dt} = \frac{i}{\hbar}[\beta mc^2, T_2] = -\frac{c}{R_0}T_1, \tag{4.33}$$

$$\frac{dT_3}{dt} = \frac{i}{\hbar}[\beta mc^2, T_3] = 0. \tag{4.34}$$

When analyzed from the point of view of an arbitrary observer, the classical motion is a helix and the acceleration is not of constant modulus c^2/R_0, and the spin operator $\boldsymbol{S}$ is no longer a constant of the motion, because it is the total angular momentum $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{S}$ that is conserved.

Identification of the internal variables with different real linear combinations of the $\boldsymbol{e}_i$ matrices lead to different equivalent representations of Dirac's matrices, and thus to different expressions of Dirac's equation.

For instance if we make the identification suggested by Figure 4.3, $\boldsymbol{u} = -a\boldsymbol{e}_3$ and the observable $d\boldsymbol{u}/dt \times \boldsymbol{u} = b\boldsymbol{e}_1$ with positive constants a and b, we obtain by the same method

$$\beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{4.35}$$

and thus gamma matrices

$$\gamma^0 \equiv \beta = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} \equiv \gamma^0\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{4.36}$$

i.e., Weyl's representation.

When we compare both representations, we see that Weyl's representation is obtained from Pauli-Dirac representation if we rotate the body frame $\pi/2$ around $\boldsymbol{e}_2$ axis. Then the corresponding rotation operator

$$R(\pi/2, \boldsymbol{e}_2) = \exp(\frac{i}{\hbar}\frac{\pi}{2}\boldsymbol{e}_2\cdot\boldsymbol{S}) = \exp(\frac{i}{\hbar}\frac{\pi}{2}T_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & -\mathbb{I} \\ \mathbb{I} & \mathbb{I} \end{pmatrix}.$$
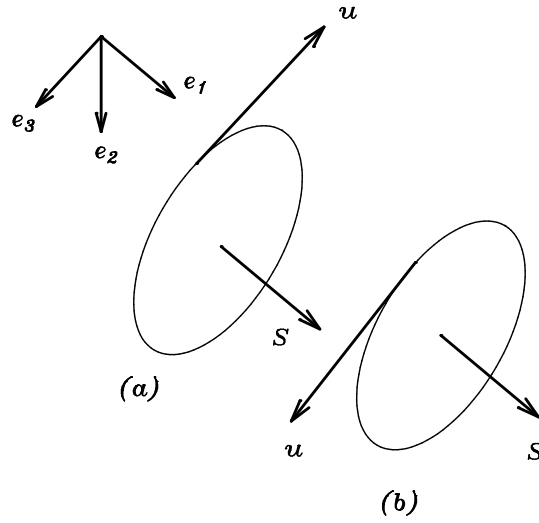
Figure 4.3: **Orientation in the Weyl representation.**

We can check that $R\,\gamma^\mu_{PD}\,R^\dagger = \gamma^\mu_W$, where γ^μ_{PD} and γ^μ_W are gamma matrices in the Pauli-Dirac and Weyl representation, respectively.

We can similarly obtain Dirac's equation in the case of zitterbewegung $z = 1$, by using the set of matrices (4.16) instead of (4.15), because they are multiples of each other and only some intermediate constant factor will change.
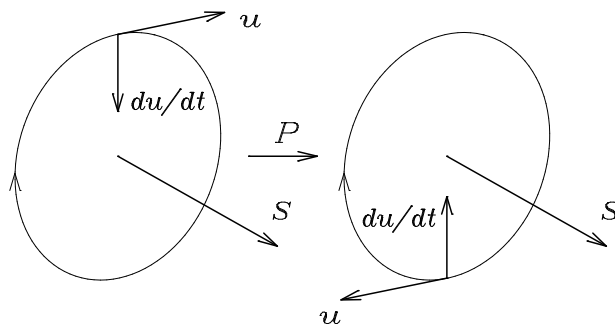
4.2.1 PCT Invariance

Figure 4.4: **Space reversal of the electron in the center of mass frame is equivalent to a rotation of value π along $\boldsymbol{S}$.**

In Figure 4.4 we represent the parity reversal of the description of the electron as given by this model of luxon which is circling around the center of mass at the velocity c and that under P and in the center of mass frame it changes according to

$$P : \{\boldsymbol{r} \to -\boldsymbol{r}, \boldsymbol{u} \to -\boldsymbol{u}, d\boldsymbol{u}/dt \to -d\boldsymbol{u}/dt, \boldsymbol{S} \to \boldsymbol{S}, H \to H\}.$$

In the Pauli-Dirac representation as we see in Figure 4.2, this amounts to a rotation of value π around axis $\boldsymbol{e}_3$ and thus

$$P \equiv R(\pi, \boldsymbol{e}_3) = \exp(i\pi\boldsymbol{e}_3 \cdot \boldsymbol{S}/\hbar) = \exp(i\pi T_3/\hbar) = i\gamma_0,$$

which is one of the possible representations of the parity operator $\pm\gamma_0$ or $\pm i\gamma_0$. In Weyl's representation this is a rotation of value π around $\boldsymbol{e}_1$ which gives again $P \equiv i\gamma_0$.
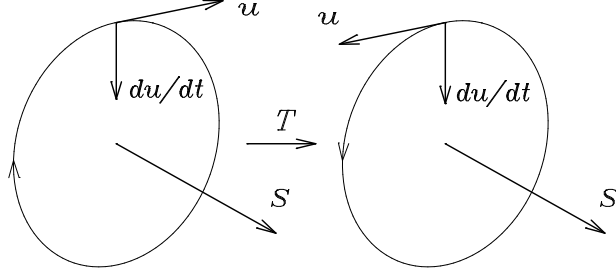
Figure 4.5: **Time reversal of the electron produces a particle of negative energy.**

In Figure 4.5 we represent its time reversal also in the center of mass frame

$$T : \{\boldsymbol{r} \to \boldsymbol{r}, \boldsymbol{u} \to -\boldsymbol{u}, d\boldsymbol{u}/dt \to d\boldsymbol{u}/dt, \boldsymbol{S} \to \boldsymbol{S}, H \to -H\},$$

but this corresponds to a particle of of $H < 0$ such that the relative orientation of spin, velocity and position, given by equation (2.127) agrees with the motion depicted in this figure.

A Dirac particle is a mechanical system whose intrinsic attributes are mass $m > 0$ and spin $\hbar/2$. We also see that the sign of H is also Poincaré invariant and it is also an intrinsic property which establishes two different systems of the same value of m and S. The system with $H > 0$ is called the particle and the other with $H < 0$ the antiparticle. The value of the mass attribute is introduced by hand. To characterize its interaction with an external electromagnetic field, we also introduce by hand another intrinsic property the electric charge e, located at the point $\boldsymbol{r}$. This implies that in addition to the mechanical properties m and S the system has as electromagnetic properties the electric charge e and because of the charge location separated from its center of mass and its motion at the speed of light, an electric dipole moment $\boldsymbol{d}$ and a magnetic moment $\boldsymbol{\mu}$, respectively. The electric charge can also have either a positive or negative sign.

$$\begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix} \quad P \Rightarrow \quad \begin{pmatrix} S \\ m \\ H \\ e \\ \boldsymbol{\mu} \\ -\boldsymbol{d} \end{pmatrix} \quad T \Rightarrow \quad \begin{pmatrix} S \\ m \\ -H \\ e \\ -\boldsymbol{\mu} \\ -\boldsymbol{d} \end{pmatrix} \quad C \Rightarrow \quad \begin{pmatrix} S \\ m \\ -H \\ -e \\ \boldsymbol{\mu} \\ \boldsymbol{d} \end{pmatrix}$$

The *PCT* transformation transforms particle into antiparticle and conversely, while keeping invariant the mechanical attributes m and S and the electromagnetic attributes $\boldsymbol{\mu}$ and $\boldsymbol{d}$. The *PCT* invariance of the system establishes a relationship between the sign of H and the sign

of e, although an indeterminacy exists in the election of the sign of the charge of the particle. The product eH is PCT invariant.

This implies that particle and antiparticle have a magnetic moment and an oscillating electric dipole in a plane orthogonal to the spin. Once the spin direction is fixed, the magnetic moment of both have the same relative orientation with the spin, either parallel or antiparallel, according to the election of the sign of the electric charge. The electric dipole moment oscillates leftwards for particles and rightwards for antiparticles which shows a difference between them which is called *chirality*. If as usual we call the electron to the system of negative electric charge *the particle*, the above PCT transformation transforms the system (a) of figure 4.6 into the system (b). If what we call the particle is of positive electric charge, then the spin and magnetic moment are opposite to each other for both particle and antiparticle.
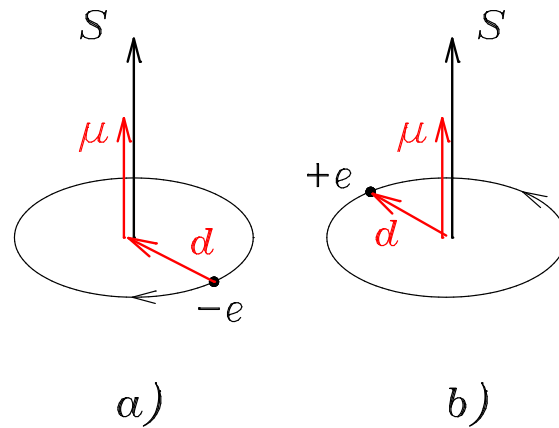
Figure 4.6: **Electromagnetic attributes $\boldsymbol{\mu}$ and $\boldsymbol{d}$ for (a) a negatively charged particle and its PCT transformed (b), and their relative orientation with the spin, in the center of mass frame. The electric dipole of the particle oscillates leftwards and rightwards for the antiparticle.**

However, to our knowledge no explicit direct measurement of the relative orientation between spin and magnetic moment of the free electron, can be found in the literature although very high precision experiments are performed to obtain the absolute value of g, the gyromagnetic ratio.

4.2.2   Two plausible experiments

A plausible indirect experiment [1] has been proposed to measure the relative orientation between spin and magnetic moment for one outer electron atoms like Rb or Cs.

Rb^{87} atoms have one electron at the level $5s$. Its nucleus has spin $3/2$ and the ground state of the atom has a total spin 1, and therefore the outer electron has its spin in the opposite

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[1] M. Rivas, *Are the electron spin and magnetic moment parallel or antiparallel vectors?*, LANL ArXiv:physics/0112057.

direction to the spin of the nucleus. The magnetic moment of the atom is basically the magnetic moment of this outer electron because the inner shells are full and the magnetic moment of the nucleus is relatively smaller.

Ultracold Rb^{87} atoms in an external magnetic field will be oriented with their magnetic moments pointing along the field direction. If in this direction we send a beam of circularly polarized photons of sufficient energy ~ 6.8GHz to produce the corresponding hyperfine transition to flip the electron spin in the opposite direction and thus leaving the atom in a spin 2 state, only those photons with the spin opposite to the spin of the outer electron will be absorbed. Measuring the spin orientation of the circularly polarized beam will give us the spin orientation of the electron thus showing its relationship with the magnetic moment orientation. Now the task is to check also the relative orientation for positrons.

This indefiniteness in the sign of the charge of matter is also present in Dirac's formalism. This prediction is consistent with the known structures formed by a particle and the corresponding antiparticle. As a matter of fact, the positronium (electron-positron bound sytem) has a ground state of spin 0 and magnetic moment 0. This means that the spins of both electron and positron are antiparallel to each other and the same thing happens to the corresponding magnetic moments. Therefore, for the electron and positron there would exist the same relative orientation between the spin and magnetic moment. The neutral pion π^0 is a linear combination of the quark-antiquark bound systems $u\bar{u}$, $d\bar{d}$ and sometimes the pair $s\bar{s}$ is also included. It is a system of 0 spin and 0 magnetic moment. Because each of the above quarks have different masses and charges, and thus different magnetic moments, the possibility is that each quark-antiquark pair is a system of 0 spin and 0 magnetic moment, and, therefore each quark and the corresponding antiquark must have the same relative orientation between the spin and magnetic moment.

Another experiment is the measurement of the preccession direction of the spin of e^+ and e^- and of μ^+ and μ^- in a storage ring. If e^+ and e^- and μ^+ and μ^- have the same relative orientation between spin and magnetic moment, then the torque and thus the preccession will be the same.

$$\boldsymbol{\mu} \times \boldsymbol{B} = \frac{d\boldsymbol{S}}{dt}$$

Nevertheless, if we inject into the accelerator particles and antiparticles with the spin up, and because the magnetic field of the ring has to be reversed for the antiparticle, then the preccession direction of both beams will be opposite to each other. If it is possible to detect the precession direction this will confirm the prediction and also the relative orientation between spin and magnetic moment.

4.2.3 Chirality

The classical model which satisfies Dirac's equation when quantized gives rise to two possible physical systems of $H > 0$ and $H < 0$. The $H > 0$ is usually called the particle. According to the previous analysis the internal motion of the charge takes place on a plane orthogonal to the spin direction and in a leftward sense when we fix as positive the spin direction. For the antiparticle the motion is rightwards. For particles, the local orientable frame of unit vectors $\boldsymbol{e}_i$ behaves as a left handed system rotating with an angular velocity in the opposite direction to the spin, while for antiparticles it can be considered as a right handed one.

If we should have started the analysis by considering first the antiparticle, then in order to get the same Pauli-Dirac representation as before we have to consider the body axis as the
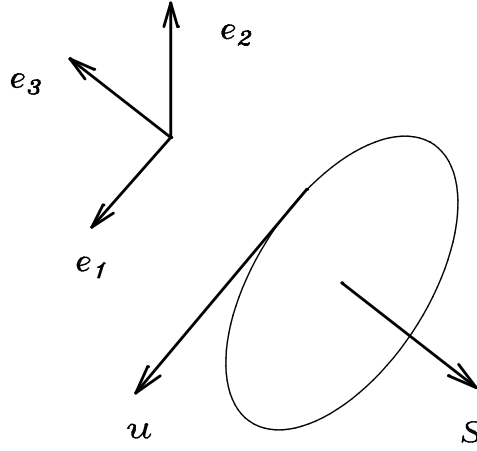
Figure 4.7: **Relative orientation of the body axis for the antiparticle that leads to Pauli-Dirac representation. It behaves as a rotating right handed Cartessian frame around the spin direction.**

ones depicted in figure 4.7, *i.e.*, in the opposite direction to the ones we chose before and this leads by the same arguments that the γ^μ matrices have to replaced by the $-\gamma^\mu$, so that the Hamiltonian in the center of mass frame is $-\beta mc^2$. In this way the motion of the body frame, instead of (4.28-4.30) is

$$\frac{d\boldsymbol{e}_1}{dt} = \frac{i}{\hbar}[-\beta mc^2, \boldsymbol{e}_1] = -\omega \boldsymbol{e}_2, \tag{4.37}$$

$$\frac{d\boldsymbol{e}_2}{dt} = \frac{i}{\hbar}[-\beta mc^2, \boldsymbol{e}_2] = \omega \boldsymbol{e}_1, \tag{4.38}$$

$$\frac{d\boldsymbol{e}_3}{dt} = \frac{i}{\hbar}[-\beta mc^2, \boldsymbol{e}_3] = 0, \tag{4.39}$$

with $\omega = c/R_0$, which clearly corresponds to a rotating right handed system with an angular velocity around the spin direction.

Matter is left and antimatter is right in this kind of models as far as the charge motion and the rotation of the local body frame are concerned, so that particles and antiparticles show a clear chirality.

Although the local motion of the charge, which takes place in a region of order of Compton's wavelength, is probably physically unobservable, this motion corresponds nevertheless to the oscillation of the instantaneous electric dipole moment, which oscillates at very high frequency, but its sense of motion, once the spin direction is fixed, reflects this difference between particle and antiparticle. This electric dipole motion is independent of whether the particle is positively or negatively charged.

Finally, when we compare the spin operator and the vector $\boldsymbol{e}_3$ we see

$$\boldsymbol{S} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}.$$

that the two upper components of the Dirac spinor correspond to positive energy solutions and therefore the upper components of these operators are related by $\boldsymbol{S} \sim \boldsymbol{e}_3$, while the lower components correspond to negative energy solutions and for this components these operators behave as $\boldsymbol{S} \sim -\boldsymbol{e}_3$, a vector relationship which is clearly depicted in the figures 4.2 and 4.7 respectively.

4.3 Dirac's algebra

The three spatial spin components S_i, the three spin projections on the body frame T_j and the nine components of the body frame $(\boldsymbol{e}_i)_j$, $i, j = 1, 2, 3$, whose matrix representations are given in the $z = 0$ case in (4.15) or in (4.16) in the $z = 1$ case, together with the 4×4 unit matrix $\mathbb{I}$, form a set of 16 linearly independent hermitian matrices. They are a linear basis of Dirac's algebra, and satisfy the following commutation relations:

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k, \qquad [T_i, T_j] = -i\hbar\epsilon_{ijk}T_k, \qquad [S_i, T_j] = 0, \tag{4.40}$$

$$[S_i, (\boldsymbol{e}_j)_k] = i\hbar\epsilon_{ikr}(\boldsymbol{e}_j)_r, \qquad [T_i, (\boldsymbol{e}_j)_k] = -i\hbar\epsilon_{ijr}(\boldsymbol{e}_r)_k, \tag{4.41}$$

and the scaled $3\boldsymbol{e}_i$ vectors in the $z = 0$ case

$$[(3\boldsymbol{e}_i)_k, (3\boldsymbol{e}_j)_l] = \frac{4i}{\hbar}\left(\delta_{ij}\epsilon_{klr}S_r - \delta_{kl}\epsilon_{ijr}T_r\right), \tag{4.42}$$

showing that the $\boldsymbol{e}_i$ operators transform like vectors under rotations but they are not commuting observables. In the case $z = 1$, the scaled $-9\boldsymbol{e}_i$, satisfy the same relations.

If we fix the pair of indexes i, and j, then the set of four operators S^2, S_i, T_j and $(\boldsymbol{e}_j)_i$ form a complete commuting set. In fact, the wave functions Φ_i, $i = 1, \ldots, 4$, given before (4.81)-(4.84), are simultaneous eigenfunctions of S^2, S_3, T_3 and $(\boldsymbol{e}_3)_3$ with eigenvalues $s = 1/2$ and for s_3, t_3, and e_{33} are the following ones:

$$\Phi_1 = |1/2, 1/2, 1/3>, \qquad \Phi_2 = |-1/2, 1/2, -1/3>, \tag{4.43}$$

$$\Phi_3 = |1/2, -1/2, -1/3>, \qquad \Phi_4 = |-1/2, -1/2, 1/3>, \tag{4.44}$$

and similarly for the Ψ_j spinors of (3.76)-(3.79)

$$\Psi_1 = |1/2, 1/2, -1/9>, \qquad \Psi_2 = |-1/2, 1/2, 1/9>, \tag{4.45}$$

$$\Psi_3 = |1/2, -1/2, 1/9>, \qquad \Psi_4 = |-1/2, -1/2, -1/9> . \tag{4.46}$$

The basic observables satisfy the following anticommutation relations:

$$\{S_i, S_j\} = \{T_i, T_j\} = \frac{\hbar^2}{2}\delta_{ij}\mathbb{I}, \tag{4.47}$$

$$\{S_i, T_j\} = \frac{\hbar^2}{2}(3\boldsymbol{e}_j)_i, \tag{4.48}$$

$$\{S_i, (3\boldsymbol{e}_j)_k\} = 2\,\delta_{ik}T_j, \qquad \{T_i, (3\boldsymbol{e}_j)_k\} = 2\,\delta_{ij}S_k, \tag{4.49}$$

$$\{(\boldsymbol{e}_i)_j, (\boldsymbol{e}_k)_l\} = \frac{2}{9}\,\delta_{ik}\delta_{jl}\mathbb{I} + \frac{2}{3}\epsilon_{ikr}\epsilon_{jls}(\boldsymbol{e}_r)_s. \tag{4.50}$$

If we define the dimensionless normalized matrices:

$$a_{ij} = 3(\boldsymbol{e}_i)_j, \ (\text{or } a_{ij} = -9(\boldsymbol{e}_i)_j), \qquad s_i = \frac{2}{\hbar}S_i, \qquad t_i = \frac{2}{\hbar}T_i, \tag{4.51}$$

together with the 4×4 unit matrix $\mathbb{I}$, they form a set of 16 matrices Γ_λ, $\lambda = 1, \ldots, 16$ that are hermitian, unitary, linearly independent and of unit determinant. They are the orthonormal basis of the corresponding Dirac's Clifford algebra.

The set of 64 unitary matrices of determinant $+1$, $\pm\Gamma_\lambda$, $\pm i\Gamma_\lambda$, $\lambda = 1, \ldots, 16$ form a finite subgroup of $SU(4)$. Its composition law can be obtained from:

$$
\begin{align}
a_{ij}\,a_{kl} &= \delta_{ik}\delta_{jl}\mathbb{I} + i\delta_{ik}\epsilon_{jlr}\,s_r - i\delta_{jl}\epsilon_{ikr}\,t_r + \epsilon_{ikr}\epsilon_{jls}\,a_{rs}, \tag{4.52} \\
a_{ij}\,s_k &= i\epsilon_{jkl}\,a_{il} + \delta_{jk}\,t_i, \tag{4.53} \\
a_{ij}\,t_k &= -i\epsilon_{ikl}\,a_{lj} + \delta_{ik}\,s_j, \tag{4.54} \\
s_i\,a_{jk} &= i\epsilon_{ikl}\,a_{jl} + \delta_{ik}t_j, \tag{4.55} \\
s_i\,s_j &= i\epsilon_{ijk}\,s_k + \delta_{ij}\mathbb{I}, \tag{4.56} \\
s_i\,t_j &= t_j\,s_i = a_{ji}, \tag{4.57} \\
t_i\,a_{jk} &= -i\epsilon_{ijl}\,a_{lk} + \delta_{ij}s_k, \tag{4.58} \\
t_i\,t_j &= -i\epsilon_{ijk}\,t_k + \delta_{ij}\mathbb{I}, \tag{4.59}
\end{align}
$$

and similarly we can use these expressions to derive the commutation and anticommutation relations (4.40-4.50).

Dirac's algebra is generated by the four Dirac gamma matrices γ^μ, $\mu = 0, 1, 2, 3$ that satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}, \tag{4.60}$$

$\eta^{\mu\nu}$ being Minkowski's metric tensor.

Similarly it can be generated by the following four observables, for instance: S_1, S_2, T_1 and T_2. In fact by (4.56) and (4.59) we obtain S_3 and T_3 respectively and by (4.57), the remaining elements.

Classically, the internal orientation of an electron is characterized by the knowledge of the components of the body frame $(\boldsymbol{e}_i)_j$, $i, j = 1, 2, 3$ that altogether constitute an orthogonal matrix. To completely characterize in a unique way this orthogonal matrix we need at least four of these components. In the quantum version, the knowledge of four $(\boldsymbol{e}_i)_j$ matrices and by making use of (4.52)-(4.59), allows us to recover the remaining elements of the complete Dirac algebra. It is in this sense that the *internal orientation* of the electron completely characterizes its internal structure. Dirac's algebra of translation invariant observables of the electron can be algebraically generated by the orientation operators.

4.4   Additional spacetime symmetries

The kinematical variables of this classical Dirac particle are reduced to time t, position $\boldsymbol{r}$, velocity $\boldsymbol{u}$ and orientation $\boldsymbol{\alpha}$, but the velocity is always $u = c$. It is always 1 in natural units. If the particle has mass $m \neq 0$ and spin $s \neq 0$, we can also define a natural unit of length s/mc and a natural unit of time s/mc^2. The unit of length is the radius of the zitterbewegung motion of figure 2.2, and the unit of time is the time employed by the charge, in the centre of mass frame, during a complete turn. This implies that the whole set of kinematical variables

and their time derivatives can be taken dimensionless, and the classical formalism is therefore invariant under spacetime dilations which do not modify the speed of light.

It turns out that although we started with the Poincaré group as the basic spacetime symmetry group, this kind of massive spinning Dirac particles, has a larger symmetry group. It also contains at least spacetime dilations with generator D. The action of this transformation on the kinematical variables is

$$t' = e^\lambda t, \quad \boldsymbol{r}' = e^\lambda \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}.$$

The new conserved Noether observable takes the form

$$D = tH - \boldsymbol{r} \cdot \boldsymbol{P}. \tag{4.61}$$

Let $R(\boldsymbol{\beta})$ be an arbitrary rotation which changes observer's axes. The action of this arbitrary rotation $R(\boldsymbol{\beta})$ on the kinematical variables is

$$t' = t, \quad \boldsymbol{r}' = R(\boldsymbol{\beta})\boldsymbol{r}, \quad \boldsymbol{u}' = R(\boldsymbol{\beta})\boldsymbol{u}, \quad R(\boldsymbol{\alpha}') = R(\boldsymbol{\beta})R(\boldsymbol{\alpha}),$$

and this is the reason why the generators $\boldsymbol{J}$ of rotations involve differential operators with respect to all these variables, the time excluded.

The orientation of the particle, represented by the variables $\boldsymbol{\alpha}$, or the equivalent orthogonal rotation matrix $R(\boldsymbol{\alpha})$, is interpreted as the orientation of an hypothetical Cartesian frame of unit axis $\boldsymbol{e}_i$, $i = 1, 2, 3$, located at point $\boldsymbol{r}$. It has no physical reality but can be interpreted as the corresponding Cartesian frame of some instantaneous inertial observer with origin at that point. But the selection of this frame is completely arbitrary so that the formalism is independent of its actual value. This means that, in addition to the above rotation group which modifies the laboratory axes, there will be another rotation group of elements $R(\boldsymbol{\gamma})$ which modifies only the orientation variables $\boldsymbol{\alpha}$, without modifying the variables $\boldsymbol{r}$ and $\boldsymbol{u}$, i.e., the rotation only of the body frame:

$$t' = t, \quad \boldsymbol{r}' = \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad R(\alpha') = R(\boldsymbol{\gamma})R(\boldsymbol{\alpha}), \tag{4.62}$$

The generators of this new rotation group, which affects only the orientation variables, will be the projection of the angular momentum generators $\boldsymbol{W}$ onto the body axes. It is clear that the operations of rotation of observer axis and rotation of the body frame commute with each other. This last rotation represents an active rotation of the body axis. From Noether's theorem the corresponding classical conserved observables are

$$I_i = \boldsymbol{W} \cdot \boldsymbol{e}_i, \tag{4.63}$$

where $\boldsymbol{e}_i$ are the three orthogonal unit vectors which define the body axis. These last operators were denoted by T_i in previous sections.

If $R(\boldsymbol{\alpha})$ is the orthogonal rotation matrix which describes the orientation of the particle, when considered by columns these columns describe the components of the three orthogonal unit vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$. Equations (4.62) correspond to the transformation $\boldsymbol{e}'_i = R(\boldsymbol{\gamma})\boldsymbol{e}_i$ of the body frame.

The W_i operators represent the components of the angular momentum operators associated to the change of orientation of the particle and projected in the laboratory frame. The corresponding conserved quantities (4.63) are the components of the angular momentum operators

projected onto the body frame $I_i = \boldsymbol{e}_i \cdot \boldsymbol{W}$. When quantizing the system they are given by the differential operators (4.78)-(4.80) of the appendix below and satisfy

$$I^2 = W^2, \quad [I_i, I_j] = i\epsilon_{ijk}I_k,$$

$$[I_i, K_j] = [I_i, J_j] = [I_i, H] = [I_i, D] = [I_i, P_j] = 0.$$

We can see that the self-adjoint operators I_i generate another $SU(2)$ group which is the representation of the rotation group which modifies only the orientation variables, commutes with the rotation group generated by the J_j, and with the whole enlarged Poincaré group, including spacetime dilations.

Since we expect that the formalism is independent of the orientation variables we have another $SO(3)$ group of spacetime symmetries of the particle.

4.4.1   Analysis of the enlarged symmetry group

Let H, $\boldsymbol{P}$, $\boldsymbol{K}$ and $\boldsymbol{J}$ be the generators of the Poincaré group $\mathcal{P}$. With the usual identification of $p^\mu \equiv (H, \boldsymbol{P})$ as the four-momentum operators and $w^\mu \equiv (\boldsymbol{P} \cdot \boldsymbol{J}, H\boldsymbol{J} - \boldsymbol{K} \times \boldsymbol{P})$ as the Pauli-Lubanski four-vector operator, the two Casimir operators of the Poincaré group are

$$C_1 = p_\mu p^\mu, \quad C_2 = -w_\mu w^\mu.$$

These two Casimir operators, if measured in the centre of mass frame where $\boldsymbol{P} = \boldsymbol{K} = 0$, reduce respectively in an irreducible representation to $C_1 = m^2 c^2$ and $C_2 = H^2 J^2 = m^2 c^2 s(s + 1)\hbar^2$. The two parameters m and s, which characterize every irreducible representation of the Poincaré group, represent the intrinsic properties of a Poincaré invariant elementary particle.

Let us consider the additional spacetime dilations of generator D. The action of this transformation on the kinematical variables is

$$t' = e^\lambda t, \quad \boldsymbol{r}' = e^\lambda \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}.$$

Let us denote this enlargement of the Poincaré group, sometimes called the Weyl group, by $\mathcal{W}$. In the quantum representation, this new generator when acting on the above wavefunctions, has the form:

$$D = it\partial/\partial t + i\boldsymbol{r} \cdot \nabla. \tag{4.64}$$

It satisfies

$$[D, p^\mu] = -ip^\mu, \quad [D, J^{\mu\nu}] = 0.$$

This enlarged group has only one Casimir operator [2] which, for massive systems where the operator $C_1 \neq 0$ is invertible, is reduced to

$$C = C_2 C_1^{-1} = C_1^{-1} C_2 \equiv C_2/C_1 = s(s + 1)\hbar^2.$$

In the centre of mass frame this operator is reduced to $C = S^2$, the square of the spin operator.

By assuming also the spacetime dilation invariance this implies that the mass is not an intrinsic property. It is the spin which is the only intrinsic property of this elementary particle. In fact, since the radius of the internal motion is $R = s/mc$, a change of length and time scale corresponds to a change of mass while keeping s and c constant. By this transformation the

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[2]Abellanas L and Martinez Alonso L 1975 *J. Math. Phys.* **16** 1580

elementary particle of spin 1/2 modifies its internal radius and therefore its mass and goes into another mass state.

The structure of the differential operator $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{Z} + \boldsymbol{W}$, where the spin part $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$ has only $s = 1/2$ eigenvalue for the above model, implies that the eigenvalue of the W^2 corresponds to $w = 1/2$ while for the $\boldsymbol{Z}^2$ part can be reduced to the two possibilities $z = 0$ or $z = 1$.

In addition to the group $\mathcal{W}$ we also consider the representation of the local rotation group generated by the I_i with eigenvalue $w = i = 1/2$. We have thus a larger spacetime symmetry group with an additional $SU(2)$ structure when quantized.

The generators I_i commute with all generators of the group $\mathcal{W}$, and this new symmetry group can be written as $\mathcal{W} \otimes SU(2)_I$.

This new group has only two Casimir operators S^2 and I^2 of eigenvalues 1/2. This justifies that our wavefunction will be written as a four-component wavefunction. When choosing the complete commuting set of operators to classify its states we take the operator $I^2 = S^2$, the S_3 and I_3 which can take the values $\pm 1/2$ and for instance the $p^\mu p_\mu$ and the p^μ. In this way we can separate in the wavefunction the orientation and velocity variables from the spacetime variables,

$$\psi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}) = \sum_{i=1}^{i=4} \phi_i(t, \boldsymbol{r}) \chi_i(\boldsymbol{u}, \boldsymbol{\alpha})$$

where the four $\chi_i(\boldsymbol{v}, \boldsymbol{\alpha})$ can be classified according to the eigenvalues $|s_3, t_3>$. The functions $\phi_i(t, \boldsymbol{r})$ can be chosen as eigenfunctions of the Klein-Gordon operator

$$p_\mu p^\mu \phi_i(t, \boldsymbol{r}) = m_i^2 \phi_i(t, \boldsymbol{r}).$$

Because this operator $p_\mu p^\mu$ does not commute with the D observable, the mass eigenvalue m_i is not an intrinsic property and the corresponding value depends on the particular state ϕ_i we consider.

For the classification of the $\chi_i(\boldsymbol{u}, \boldsymbol{\alpha})$ states we have also to consider the $\boldsymbol{Z}$ angular momentum operators. Because $[Z^2, S^2] = [Z^2, I^2] = [Z^2, p^\mu] = 0$, we can choose Z^2 as an additional commuting observable. It can only take integer eigenvalues when acting on functions of the velocity variables, because it has the structure of an orbital angular momentum. But because the total spin $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, and the S^2 has eigenvalue 1/2, the possible eigenvalues of Z^2 can be $z = 0$ or $z = 1$. See the appendix below for the possible clasification of the $\chi_i(\boldsymbol{u}, \boldsymbol{\alpha})$ part, according to $z = 0$ which gives rise to the (4.81-4.84) eigenfunctions, and the $z = 1$ eigenfunctions (4.85-4.88). In this last case the eigenfunctions cannot be simultaneously eigenfunctions of Z_3. Nevertheless the expectation value of Z_3 in the $z = 0$ basis vectors Φ_i is 0, while its expectation value in the $z = 1$ basis Ψ_i is $\pm 2/3$.

4.4.2 Enlargement of the kinematical space

Once the kinematical group has been enlarged by including spacetime dilations, we have a new dimensionless group parameter asociated to this one-parameter subgroup which can also be used as a new kinematical variable, to produce a larger homogeneous space of the group. In fact, if we take the time derivative of the constant of the motion (4.61) we get

$$H = \boldsymbol{P} \cdot \boldsymbol{u}.$$

If we compare this with the equation (4.19), one term is lacking. This implies that we need, from the classical point of view, an additional kinematical variable, a dimensionless scale β, such that under the action of this new transformation the enlarged kinematical variables transform

$$t' = e^\lambda t, \quad \boldsymbol{r}' = e^\lambda \boldsymbol{r}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}, \quad \beta' = \lambda + \beta.$$

From the group theoretical point of view this new dimensionless variable corresponds to the normal dimensionles group parameter of the transformation generated by D.

From the Lagrangian point of view, the new Lagrangian has also to depend on β and $\dot{\beta}$, with a general structure

$$L = \dot{t}T + \dot{\boldsymbol{r}} \cdot \boldsymbol{R} + \dot{\boldsymbol{u}} \cdot \boldsymbol{U} + \boldsymbol{\omega} \cdot \boldsymbol{W} + \dot{\beta}B,$$

with $B = \partial L/\partial \dot{\beta}$. The constant of the motion associated to the invariance of the dynamical equations under this new transformation implies that

$$D = tH - \boldsymbol{r} \cdot \boldsymbol{P} - B,$$

and the new generator in the quantum version takes the form

$$D = it\partial/\partial t + i\boldsymbol{r} \cdot \nabla + i\frac{\partial}{\partial \beta}.$$

In this way the last term of (4.19) is related to the time derivative of this last term

$$\frac{dB}{dt} = \frac{1}{c^2}\boldsymbol{S} \cdot \left(\frac{d\boldsymbol{u}}{dt} \times \boldsymbol{u}\right).$$

This new observable B, with dimensions of action, has a positive time derivative for particles and a negative time derivative for antiparticles. This sign is clearly related to the sign of H. In the center of mass frame $\boldsymbol{P} = 0$, $H = \pm mc^2 = dB/dt$, with solution $B(t) = B(0) \pm mc^2 t$. In units of $\hbar$ this observable represents half the phase of the internal motion

$$B(t) = B(0) \pm \frac{1}{2}\hbar\omega t.$$

Because the additional local rotations generated by the I_i commute with the $\mathcal{W}$ group, the above kinematical variables also span a homogeneous space of the whole $\mathcal{W} \otimes SU(2)_I$ group and, therefore, they represent the kinematical variables of an elementary system which has this new group as its kinematical group of spacetime symmetries.

4.4.3   Relationship with the standard model

We have analyzed the spacetime symmetry group of a relativistic model of a Dirac particle. Matter described by this model ($H > 0$ states), is left handed while antimatter ($H < 0$), is right handed, as far as the relative orientation between the spin and the motion of the charge, is concerned. For matter, once the spin direction is fixed, the motion of the charge is counterclockwise when looking along the spin direction. It is contained in a plane orthogonal to the spin direction, with the usual sign convention for multivectors in the geometric algebra. The motion is clockwise for antimatter.

This particle has as symmetry group of the Lagrangian $\mathcal{W} \otimes SO(3)_I$ and $\mathcal{W} \otimes SU(2)_I$ in its quantum description, which is larger than the Poincaré group we started with as the initial kinematical group of the model. It contains in its quantum description, in addition to the Poincaré transformations, a $U(1)$ group which is a unitary representation of the spacetime dilations and also a $SU(2)_I$ group which is the unitary representation of the symmetry group of local rotations of the body frame. The whole group has two Casimir operators S^2, the Casimir of $\mathcal{W}$ and I^2 the Casimir of $SU(2)_I$, which take the eigenvalues $s = i = 1/2$ for the Dirac particle considered here.

Some of the features we get have a certain resemblance to the standard model of elementary particles, as far as kinematics is concerned. If we interpret the generators I_i of the unitary representation of the local rotations as describing isospin and the angular momentum operators $\boldsymbol{Z}$ related to the zitterbewegung as describing colour, an elementary particle described by this formalism is a massive system of spin $1/2$, isospin $1/2$, of undetermined mass and charge. It can be in a $s_3 = \pm 1/2$ spin state and also in a $i_3 = \pm 1/2$ isospin state. There are two nonequivalent irreducible representations according to the value of the zitterbewegung part of the spin z. It can only be a colourless particle $z = 0$ (lepton?) or a coloured one $z = 1$ in any of three possible colour states $z_3 = 1, 0, -1$, (quark?) but no greater z value is allowed. The basic states can thus also be taken as eigenvectors of $\boldsymbol{Z}^2$ but not of Z_3, so that the corresponding colour is unobservable. There is no possibility of transitions between the coloured and colourless particles because of the orthogonality of the corresponding irreducible representations.

Because the eigenvalues of Z_3 are unobservable we also have an additional unitary group of transformations $SU(3)$ which transforms the three Z_3 eigenvectors Y_i^j of (4.89) among themselves and which do not change the $z = 1$ value of the eigenstates Ψ_i. Nevertheless, the relationship between this new $SU(3)$ internal group, which is not a spacetime symmetry group, and $\mathcal{W} \otimes SU(2)_I$ is not as simple as a direct product and its analysis is left to a subsequent research. In another context, the $z = 0$ states corresponds to the motion of the charge pasing through the centre of mass and therefore no closed current loop and thus no magnetic moment.

This formalism is pure kinematical. We have made no mention to any electromagnetic, weak or strong interaction among the different models. So that, if we find this comparison with the standard model a little artificial, the mentioned model of Dirac particle just represents a massive system of spin $1/2$, spin projection on the body frame $1/2$, of undetermined mass and charge. It can be in a $s_3 = \pm 1/2$ spin state and also in a $i_3 = \pm 1/2$ when the spin is projected on the body axis. There are two different models of these Dirac particles according to the value of the orbital or zitterbewegung spin, $z = 0$ or $z = 1$, in any of the three possible orbital spin states $z_3 = 1, 0, -1$, which are unobservable, but no particle of greater z value is allowed. It is the spin, with its twofold structure orbital and rotational, the only intrinsic attribute of this Dirac elementary particle.

4.5   An interaction Lagrangian for two Dirac particles

An elementary particle can be annihilated by the interaction with the corresponding antiparticle, but if it is not destroyed, we made the assumption that the structure of an elementary particle is not modified by any interaction so that its intrinsic properties, the spin S and the spin projection on the body frame I cannot be altered by the interaction with an external field or by the presence in its neigbourhood of any other particle.

Let us consider a compound system formed by two spinning particles with the same kind

of kinematical variables. We shall use a subscript $a = 1, 2$ to distinguish the variables corresponding to each particle. Then the kinematical space of the compound system is spanned by the variables $(t_a, \boldsymbol{r}_a, \boldsymbol{u}_a, \boldsymbol{\alpha}_a, \beta_a), a = 1, 2$. The Lagrangian of the system will be written as

$$L = L_1 + L_2 + L_I$$

where the L_a, $a = 1, 2$, are the free Lagrangians of each particle and L_I is the interaction Lagrangian we are looking for. Both L_a are invariant under the enlarged group $\mathcal{S}$ and we are going to find an interaction Lagrangian L_I also invariant under $\mathcal{S}$. The general structure of the free Lagrangian L_a of each particle, which only depends on the corresponding kinematical variables of particle a, is

$$L_a = T_a \dot{t}_a + \boldsymbol{R}_a \cdot \dot{\boldsymbol{r}}_a + \boldsymbol{U}_a \cdot \dot{\boldsymbol{u}}_a + \boldsymbol{W}_a \cdot \boldsymbol{\omega}_a + B_a \dot{\beta}_a$$

where $T_a = \partial L_a / \partial \dot{t}_a$, $\boldsymbol{R}_a = \partial L_a / \partial \dot{\boldsymbol{r}}_a$, $\boldsymbol{U}_a = \partial L_a / \partial \dot{\boldsymbol{u}}_a$, $\boldsymbol{W}_a = \partial L_a / \partial \boldsymbol{\omega}_a$ and $B_a = \partial L_a / \partial \dot{\beta}_a$, because of the homogeneity of each L_a in terms of the $\tau-$derivatives of the corresponding kinematical variables. The spin and the spin projection on the body frame for each particle, are

$$\boldsymbol{S}_a = \boldsymbol{u}_a \times \boldsymbol{U}_a + \boldsymbol{W}_a, \quad I_{ai} = \boldsymbol{e}_{ai} \cdot \boldsymbol{W}_a$$

where $\boldsymbol{e}_{ai}$, $i = 1, 2, 3$ are three orthogonal unit vectors with origin at point $\boldsymbol{r}_a$.

The interaction Lagrangian between these two particles L_I will be in general a function of the kinematical variables of both particles and of their $\tau-$derivatives. If both intrinsic properties S_a and I_a of each particle are not modified by any interaction then the interaction Lagrangian cannot be a function of the derivatives of the kinematical variables $\dot{\boldsymbol{u}}_a$ and $\boldsymbol{\omega}_a$, $a = 1, 2$. Otherwise the functions $\boldsymbol{U}_a$ and $\boldsymbol{W}_a$ will be different than in the free case. In this case the functions $\boldsymbol{U}_a$ and $\boldsymbol{W}_a$, which give rise to the definition of the spin, are obtained only from the corresponding free Lagrangian L_a.

Then, as far as the $\tau-$derivatives of the kinematical variables are concerned, the interaction Lagrangian L_I will only depend on the variables $\dot{t}_a$, $\dot{\boldsymbol{r}}_a$ and $\dot{\beta}_a$, $a = 1, 2$. In addition to this, it will also be a function of the kinematical variables t_a, $\boldsymbol{r}_a$, $\boldsymbol{u}_a$ and β_a, but not of $\boldsymbol{\alpha}_a$ because of the invariance under the local rotation group $SO(3)_L$. Spacetime dilation invariance implies that the Lagrangian is a function of the phase difference $\beta_1 - \beta_2$, and of $\dot{\beta}_1 - \dot{\beta}_2$, but being both phases completely arbitrary and independent of each other it means that must be independent of these variables.

Because of the constraint $\boldsymbol{u}_a = \dot{\boldsymbol{r}}_a / \dot{t}_a$, the interaction Lagrangian will thus be finally a function

$$L_I = L_I(t_a, \boldsymbol{r}_a, \dot{t}_a, \dot{\boldsymbol{r}}_a),$$

and a homogeneous function of first degree of the derivatives $\dot{t}_a, \dot{\boldsymbol{r}}_a$, $a = 1, 2$.

If we call as usual the Minkowski four-vector $x_a^\mu \equiv (t_a, \boldsymbol{r}_a)$, translation invariance implies that the Lagrangian must be a function of $x_1^\mu - x_2^\mu$. The following two terms $\eta_{\mu\nu} \dot{x}_1^\mu \dot{x}_2^\nu$ and $\eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu)$, where $\eta_{\mu\nu}$ is Minkowski's metric tensor, are Poincaré invariant. If we consider that the evolution parameter τ is dimensionless, these terms have both dimensions of length squared. It therefore implies that its quotient is dimensionless and therefore invariant under spacetime dilations. The other requirement is that the Lagrangian is a homogeneous function of first degree in the $\tau-$derivatives of the kinematical variables. The squared root will do the job. A final discrete symmetry will be assumed because when the two particles are the same, and therefore indistinguishable, the interaction Lagrangian must be invariant under

the interchange $1 \leftrightarrow 2$ between the labels of the two particles. We thus arrive to the $\mathcal{S}$ group invariant Lagrangian

$$L_I = g\sqrt{\frac{\eta_{\mu\nu}\dot{x}_1^\mu \dot{x}_2^\nu}{\eta_{\mu\nu}(x_1^\mu - x_2^\mu)(x_2^\nu - x_1^\nu)}} = g\sqrt{\frac{\dot{t}_1 \dot{t}_2 - \dot{\boldsymbol{r}}_1 \cdot \dot{\boldsymbol{r}}_2}{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2 - (t_2 - t_1)^2}}$$

where g is a coupling constant.

Alternative Lagrangians which fulfill these requirements can be constructed. For instance, $L = g(\dot{x}_1 - \dot{x}_2)^\mu (x_1 - x_2)_\mu / (x_1 - x_2)^2$, but this one is a total $\tau-$derivative of the function $\log(x_1 - x_2)^2$. Another could be $L = g(\dot{x}_1 + \dot{x}_2)^\mu (x_1 - x_2)_\mu / (x_1 - x_2)^2$, also dimensionless and linear in the derivatives of the kinematical variables, but it reverses its sign under the interchange $1 \leftrightarrow 2$, and thus all interaction observables, like the interaction energy are reversed, which is physically meaningless for two alike particles.

The interaction between two Dirac particles is not unique. We know that among leptons and quarks there are short range interactions like the weak and strong interactions and a short and long range one like the electromagnetic interaction. The proposed Lagrangian has the advantage of describing an interaction which is scale invariant and thus it is valid as a long and short range interaction and which has a Coulomb-like behaviour when the spin is supressed, as we shall see in the next section. In this way it suplies a kind of generalization of an action at a distance electromagnetic interaction. The novelty is that this interaction Lagrangian has been obtained by assuming a spacetime symmetry group larger than the Poincaré group.

4.5.1 Synchronous description

Once an inertial observer is fixed we shall consider a synchronous time description, i.e. to use as evolution parameter the own observer's time t which is the same as the two time variables t_1 and t_2. In this case, $t = t_1 = t_2$, $\dot{t}_1 = \dot{t}_2 = 1$, and thus

$$L_I = g\sqrt{\frac{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{(\boldsymbol{r}_2 - \boldsymbol{r}_1)^2}} = g\frac{\sqrt{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}}{r} \tag{4.65}$$

where $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is the instantaneous separation between the corresponding charges in this frame and $\boldsymbol{u}_a = d\boldsymbol{r}_a/dt$ the velocity of the charge of particle a.

An average over the charge position and velocity in the centre of mass of particle 1 implies that $< \boldsymbol{r}_1 > = \boldsymbol{q}_1$ and $< \boldsymbol{u}_1 > = 0$, so that the interaction becomes the instantaneous Coulomb interaction, between the centre of mass of the first particle and the charge position of the other. The average over the other then corresponds to the instantaneous Coulomb interaction of two spinless point particles because when neglecting the zitterbewegung we are suppressing the spin structure.

It is suggesting that $g \sim \pm e^2$ in terms of the electric charge of each particle and where the $\pm$ sign depends on the kind of particles either of opposite or equal charge.

4.5.2 Analysis of a two-particle system

We have seen in section 2.6.2 that the dynamical equation of a free Dirac particle and for any inertial observer is a fourth-order differential equation for the position of the charge $\boldsymbol{r}$

which can be separated into a system of coupled second order differential equations for the centre of mass $\boldsymbol{q}$ and centre of charge $\boldsymbol{r}$ in the form (2.148):

$$\ddot{\boldsymbol{q}} = 0, \quad \ddot{\boldsymbol{r}} = \frac{1 - \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{r}}}{(\boldsymbol{q} - \boldsymbol{r})^2}(\boldsymbol{q} - \boldsymbol{r}),$$

where now the dot means time derivative. The first equation represents the free motion of the centre of mass and the second a kind of relativistic harmonic oscillation of point $\boldsymbol{r}$ around point $\boldsymbol{q}$ which preserves the constant absolute value c of the velocity $\dot{\boldsymbol{r}}$. In fact, if $\dot{q} \ll \dot{r} = 1$, $|\boldsymbol{q} - \boldsymbol{r}| \sim 1$ and the equation is just the harmonic motion $\ddot{\boldsymbol{r}} + \boldsymbol{r} \simeq \boldsymbol{q}$, of point $\boldsymbol{r}$ around $\boldsymbol{q}$. The factor $(1 - \dot{\boldsymbol{q}} \cdot \dot{\boldsymbol{r}})/(\boldsymbol{q} - \boldsymbol{r})^2$ prevents that when we take the boundary value $\dot{r}(0) = 1$, the solution does not modify this absolute value of the velocity of the charge.

In the case of interaction this second equation remains the same because it corresponds to the definition of the centre of mass position which is unchanged by the interaction, because it only involves the $\boldsymbol{U}$ and $\boldsymbol{W}$ functions. The first equation for particle a is going to be replaced by $d\boldsymbol{p}_a/dt = \boldsymbol{F}_a$ where $\boldsymbol{p}_a$ is the corresponding linear momentum of each particle expressed as usual in terms of the centre of mass velocity

$$\boldsymbol{p}_a = \gamma(\dot{\boldsymbol{q}}_a)m\dot{\boldsymbol{q}}_a, \qquad \gamma(\dot{\boldsymbol{q}}_a) = (1 - \dot{\boldsymbol{q}}_a^2)^{-1/2},$$

and the force $\boldsymbol{F}_a$ is computed from the interaction Lagrangian (4.65)

$$\boldsymbol{F}_a = \frac{\partial L_I}{\partial \boldsymbol{r}_a} - \frac{d}{dt}\left(\frac{\partial L_I}{\partial \boldsymbol{u}_a}\right)$$

For particle 1 it takes the form:

$$\boldsymbol{F}_1 = -g\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\sqrt{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2} + \frac{d}{dt}\left(\frac{g\boldsymbol{u}_2}{2|\boldsymbol{r}_1 - \boldsymbol{r}_2|\sqrt{1 - \boldsymbol{u}_1 \cdot \boldsymbol{u}_2}}\right) \qquad (4.66)$$

where it contains velocity terms which behave like $1/r^2$ and acceleration terms which go as $1/r$ in terms of the separation of the charges $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. In this new notation $\boldsymbol{u}_a = \dot{\boldsymbol{r}}_a$.

Then the system of second order differential equations to be solved are

$$\ddot{\boldsymbol{q}}_a = \frac{\alpha}{\gamma(\dot{\boldsymbol{q}}_a)}\left(\boldsymbol{F}_a - \dot{\boldsymbol{q}}_a(\boldsymbol{F}_a \cdot \dot{\boldsymbol{q}}_a)\right) \qquad (4.67)$$

$$\ddot{\boldsymbol{r}}_a = \frac{1 - \dot{\boldsymbol{q}}_a \cdot \dot{\boldsymbol{r}}_a}{(\boldsymbol{q}_a - \boldsymbol{r}_a)^2}(\boldsymbol{q}_a - \boldsymbol{r}_a), \quad a = 1, 2 \qquad (4.68)$$

where α is the fine structure constant once all the variables are taken dimensionless. For that, we take the space scale factor $R = \hbar/2mc$ and the time scale as $T = \hbar/2mc^2$. All terms of equation (4.67) which depend on the acceleration of the charges have to be replaced by the expressions of (4.68).

It would be desirable to find analytical solutions of the above equations (4.67-4.68). Nevertheless we have not succeded in finding such a goal. However we shall analyse different solutions obtained by numerical integration. We are going to use the computer program Dynamics Solver [?]. The quality of the numerical results is tested by using the different integration schemes this program allows, ranging from the very stable embedded Runge-Kutta code of eighth order, due to Dormand and Prince, to very fast extrapolation routines. All codes have adaptive step
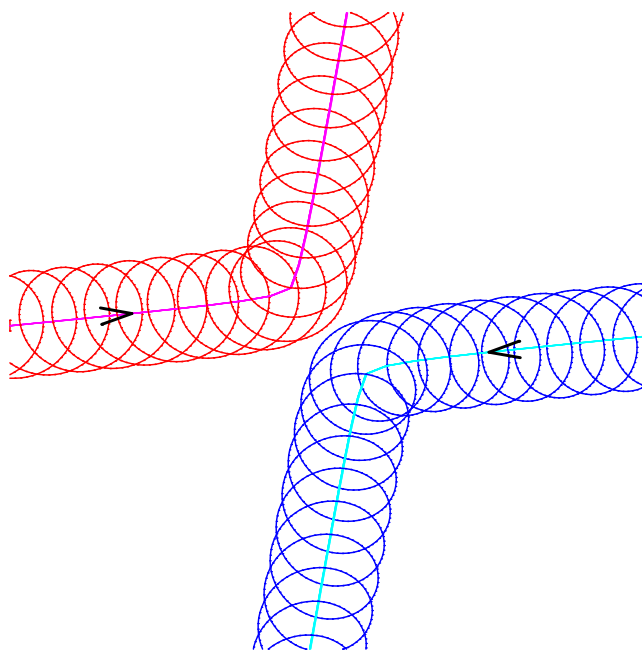
Figure 4.8: **The trajectories of the centres of mass and charge of two spinning particles with an initial centre of mass velocity $\dot{q}_a = 0.1$ and a small impact parameter.**

size control and we check that smaller tolerances do not change the results. Another advantage is that it can be prepared to analyse solutions corresponding to a wide range of boundary conditions, automatically.

See in figure 4.8 the scattering of two equal charged particles with parallel spins. The centre of mass motion of each particle is depicted with an arrow. If the two particles do not approach each other too much these trajectories correspond basically to the trajectories of two spinless point particles interacting through an instantaneous Coulomb force. By too much we mean that their relative separation between the corresponding centres of mass is always much greater than Compton's wavelength. This can be understood because of the above discussion about the Coulomb behaviour of the averaged interaction Lagrangian, if the average position of each charge is far from the other. For high energy interaction the two particles approach each other below that separation and therefore the average analysis no longer works because the charges approach each other to very small distances where the interaction term and the exact position of both charges, becomes important. In this case new phenomena appear. We can have, for instance, a forward scattering like the one depicted on figure 4.9, which is not described in the classical spinless case, or even the formation of bound pairs for particles of the same charge, which we shall analyse in what follows.

In figure 4.10 we represent an initial situation for two equal charged particles with parallel spins such that the corresponding centres of mass are separated by a distance below Compton's wavelength. Remember that the radius of the internal motion is half Compton's wavelength. We locate the charge labels e_a at the corresponding points $\boldsymbol{r}_a$ and the corresponding mass labels m_a to the respective centre of mass $\boldsymbol{q}_a$. We depict in part (a) the situation when the two particles have the same phase $\beta_1 = \beta_2$. The forces $\boldsymbol{F}_a$, on each particle $a = 1, 2$, are computed in terms of the positions, velocities and accelerations of both charges, according to (4.66), and are also depicted on the corresponding centres of mass as a consequence of the structure of the equations (4.67). We see that a repulsive force between the charges produces
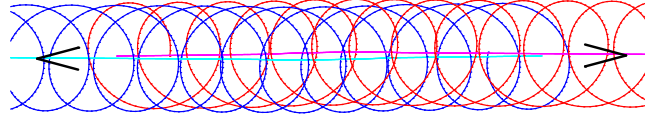
Figure 4.9: **Forward scattering of two spinning particles of the same charge with an initial separation $2q_a(0) = 10$, centre of mass velocity $|\dot{q}_a(0)| = 0.18$ and a very small impact parameter. The two centres of mass cross very close to each other, with a small deviation.**

also a repulsive force between the centres of mass in this situation. However, in part (b) both charges have opposite phases $\beta_1 = -\beta_2$, and now the repulsive force between the charges implies an atractive force between the corresponding centres of mass. If the initial situation is such that the centres of mass separation is greater than Compton's wavelength, the force is always repulsive irrespective of the internal phases of the particles.
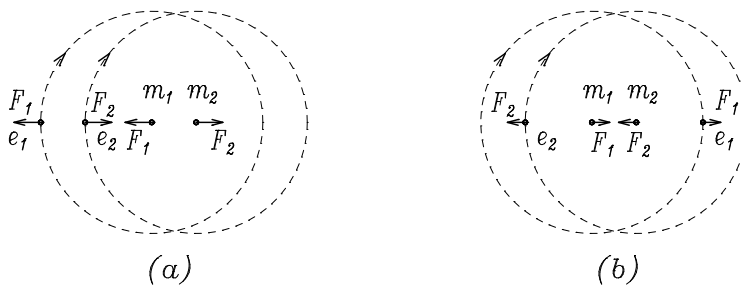
Figure 4.10: **Boundary values for two Dirac particles with parallel spins and with a separation between the centres of mass below Compton's wavelength. The dotted lines represent the previsible clockwise motion of each charge. In (a) both particles have the same phase and the repulsive force between charges produces a repulsive force between their centres of mass, while in (b), with opposite phases, the force between the centres of mass is atractive.**

In figure 4.11 we have another situation of opposite phases and where the initial separation

between the centres of mass is larger but still smaller than Compton's wavelength.
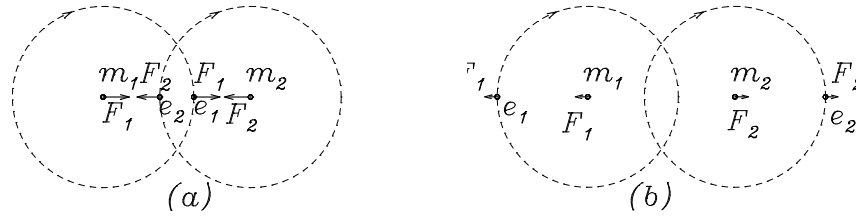
Figure 4.11: **(a) Another situation of two charges with opposite phases which produce an atractive force between the centres of mass provided they are separated below Compton's wavelength. In part (b), after half a cycle of the motion of the charges, the force becomes repulsive between the centres of mass, but its intensity is much smaller than the atractive force in (a) so that the resulting motion is also a bound motion.**

To analyse this situation, which is going to produce bound motions, we proceed as follows: We start the numerical integration by imposing the boundary condition that both centres of mass are at rest and located at the origin of the reference frame $\boldsymbol{q}_a(0) = \dot{\boldsymbol{q}}_a(0) = 0$. For particle 2 we take the initial phase $\beta_2(0) = 0$ and for β_1 we start with $\beta_1(0) = 0$ and, will be increased step by step in one degree in the automatic process, up to reach the whole range of 2π radians. The boundary values of the variables $\boldsymbol{r}_a(0)$ and $\dot{\boldsymbol{r}}_a(0)$, with the constraint $|\dot{\boldsymbol{r}}_a(0)| = 1$, are taken as the corresponding values compatible with these phases. The whole system is analysed in its centre of mass frame, so that for subsequent boundary values these variables are restricted to $\boldsymbol{q}_1(0) = -\boldsymbol{q}_2(0)$ and $\dot{\boldsymbol{q}}_1(0) = -\dot{\boldsymbol{q}}_2(0)$. The automatic integration is performed in such a way that when the two particles separate, i.e., when their centre of mass separation is above Compton's wavelength, the integration stops and starts again with a new boundary value of the phase $\beta_1(0)$ of one degree more, and the new values of the variables $\boldsymbol{r}_a(0)$ and $\dot{\boldsymbol{r}}_a(0)$. If the two particles do not separate we wait until the integration time corresponds to 10^6 turns of the charges around their corresponding centre of mass, stop the process, keep record of the phases and initial velocities, and start again with new boundary values. This corresponds, in the case of electrons, to a bound state leaving during a time greater than 10^{-15} seconds. For some particular boundary values, with opposite phases, we have left the program working during a whole week and the bound state prevails. This represents a time of life of the bound state greater than 10^{-9} seconds. Leaving the computation program running for a year will only increase this lower bound in two orders of magnitude. The general feeling is that the bound states are sufficiently stable, because even the possible numerical integration errors do not destroy the stability. This process is repeated again and again by changing slightly the initial values of the centre of mass variables $\boldsymbol{q}_a(0)$ and $\dot{\boldsymbol{q}}_a(0)$, in steps of 0.0001 in these dimensionless units and with $\beta_2(0) = 0$, and the same procedure with $\beta_1(0)$, as above. To test the acuracy of the integration method, we check every 10^3 integration steps that the velocities of the charges of both particles remain of absolute value 1, within a numerical error smaller than 10^{-20}.

The whole process is repeated by changing the initial $\beta_2(0)$ phase to any other arbitrary value. We are interested to see whether different results are produced depending on the values of the phase difference $\beta_2(0) - \beta_1(0)$ and of the centre of mass variables $\boldsymbol{q}_a(0)$ and $\dot{\boldsymbol{q}}_a(0)$. We collect all data which produce bound motions, and find the following results:

1. The initial velocity of their centres of mass must be $|\dot{\boldsymbol{q}}_a(0)| < 0.01c$. Otherwise the bound motion is not stable and the two particles, after a few turns, go off.

2. For each velocity $|\dot{q}_a(0)| < 0.01c$ there is a range Δ of the pase $\beta_1(0) = \beta_2(0) + \pi \pm \Delta$ for which the bound motion is stable. The greater the centre of mass velocity of each particle the narrower this range, so that the bound motion is more likely when the phases are opposite to each other.

3. We have found bound motions for an initial separation between the centres of mass up to 0.8 times Compton's wavelength, like the one depicted in figure 4.11, provided the above phases and velocities are kept within the mentioned ranges.

In figure 4.12 we show the bound motion of both particles when their centres of mass are initially separated $q_{1x} = -q_{2x} = 0.2\times$Compton's wavelength, $\dot{q}_{1x} = -\dot{q}_{2x} = 0.008$ and $\dot{q}_{1y} = -\dot{q}_{2y} = 0.001$, $\beta_2 = 0$ and $\beta_1 = \pi$. Now the force between the charges is repulsive but nevertheless, if the internal phases β_1 and β_2 are opposite to each other, it becomes an atractive force between their centres of mass in accordance to the mechanism shown in figure 4.10 (b).

This possibility of formation of low energy metastable bound pairs of particles of the same charge is not peculiar of this interaction Lagrangian. By using the electromagnetic interaction or even the instantaneous Coulomb interaction between the charges of two spinning Dirac particles we found in [3] also this behaviour. This bound motion is not destroyed by external electric fields and also by an external magnetic field along the spin direction. Nevertheless, a transversal magnetic field destroys this bound pair system.
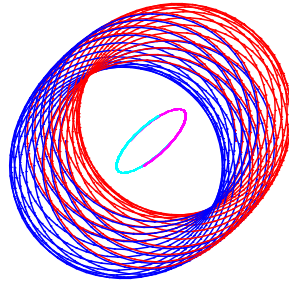
Figure 4.12: **Bound motion of the centres of mass and charge of two spinning particles with parallel spins and with a centre of mass velocity** $v \simeq 0.0082$, **for an initial separation between the centres of mass of** $0.2\times$**Compton's wavelength.**

When we make the average of the position $\boldsymbol{r}_a$ it becomes the centre of mass $\boldsymbol{q}_a$ and the repulsive force between the charges is also a repulsive force between the corresponding centres of mass and therefore when we suppress the zitterbewegung spin content of the particles there is no possibility of formation of bound pairs.

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[3]M. Rivas *J. Phys. A: Math. Gen.* **36** 4703 (2003), (*Preprint* physics/0112005)

Although this result produces a classical mechanism for the formation of a spin 1 bound system from two equal charged fermions we must be careful about its conclusions. First, it is a classical description and although the range of energies which produce this phenomenon is a wide one it does not mean that two electrons can reach that binding energy. This Dirac particle is a system of 7 degrees of freedom: 3 represent the position $\boldsymbol{r}$, another 3 the orientation $\boldsymbol{\alpha}$ and finally the phase β. If we accept the equipartition theorem for the energy, then for the maximum kinetic energy which produces a bound motion $mv^2/2 = 7\kappa T/2$, where κ is Boltzmann's constant and $v = 0.01c$ the maximum velocity of the center of mass of each particle, then it means that a gas of polarized electrons (like the conducting electrons in a quantum Hall effect) could form bound states up to a temperature below $T = 8.47 \times 10^5 \mathrm{K}$, which is a very high temperature. In a second place, matter at this level behaves according to quantum mechanical rules and therefore we must solve the corresponding quantum mechanical bound state to establish the proper energies and angular momenta at which these bound states would be stationary. This problem has not been solved yet, but the existence of this classical possibility of formation of bound pairs justifies an effort in this direction. If the phases of the two particles are the same (or almost the same) there is no possibility of formation of a bound state. The two fermions of the bound state have the same spin and energy. They differ that their phases and linear momenta are opposite to each other. Is this difference in the phase a way to overcome at the classical level, the Pauli exclusion principle?

4.6 Appendix: The group $\mathcal{W} \otimes SU(2)_I$

Under infinitesimal time and space translations of parameters $\delta\tau$ and $\delta\boldsymbol{b}$, respectively, the kinematical variables transform as

$$t' = t + \delta\tau, \quad \boldsymbol{r}' = \boldsymbol{r} + \delta\boldsymbol{b}, \quad \boldsymbol{u}' = \boldsymbol{u}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}, \quad \beta' = \beta$$

so that the self-adjoint generators of translations are

$$H = i\frac{\partial}{\partial t}, \quad \boldsymbol{P} = -i\nabla, \qquad [H, \boldsymbol{P}] = 0.$$

Under an infinitesimal spacetime dilation of normal parameter $\delta\lambda$, they transform in the way:

$$t' = t + t\delta\lambda, \quad \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{r}\delta\lambda, \quad \boldsymbol{v}' = \boldsymbol{v}, \quad \boldsymbol{\alpha}' = \boldsymbol{\alpha}, \quad \beta' = \beta + \delta\lambda$$

so that the generator takes the form ($\hbar = 1$)

$$D = it\frac{\partial}{\partial t} + i\boldsymbol{r} \cdot \nabla + i\frac{\partial}{\partial \beta} = tH - \boldsymbol{r} \cdot \boldsymbol{P} - B, \qquad [D, H] = -iH, \quad [D, P_j] = -iP_j.$$

To describe orientation we can represent every element of the rotation group by the three-vector $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, where α is the rotated angle and $\boldsymbol{n}$ is a unit vector along the rotation axis. This is the normal or canonical parameterization. Alternatively we can represent every rotation by the three-vector $\boldsymbol{\rho} = \tan(\alpha/2)\boldsymbol{n}$. In this case, every rotation matrix takes the form

$$R(\boldsymbol{\rho})_{ij} = \frac{1}{1 + \rho^2}\left((1 - \rho^2)\delta_{ij} + 2\rho_i\rho_j + 2\epsilon_{ikj}\rho_k\right).$$

The advantage of this parameterization is that the composition of rotations $R(\boldsymbol{\rho}') = R(\boldsymbol{\mu})R(\boldsymbol{\rho})$ takes the simple form

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\mu} + \boldsymbol{\rho} + \boldsymbol{\mu} \times \boldsymbol{\rho}}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}}.$$

Under an infinitesimal rotation of parameter $\delta\boldsymbol{\mu} = \delta\boldsymbol{\alpha}/2$, in terms of the normal parameter, the kinematical variables transform as

$$
\begin{aligned}
\delta t &= 0, \quad \delta\beta = 0 \\
\delta r_i &= -2\epsilon_{ijk}r_j\delta\mu_k \\
\delta u_i &= -2\epsilon_{ijk}u_j\delta\mu_k \\
\delta\rho_i &= \left(\delta_{ik} + \rho_i\rho_k + \epsilon_{ikl}\rho_l\right)\delta\mu_k,
\end{aligned}
$$

so that the variation of the kinematical variables per unit of normal rotation parameter $\delta\alpha_k$ is

$$
\begin{aligned}
\delta t_k &= 0, \quad \delta\beta_k = 0 \\
\delta r_{ik} &= -\epsilon_{ijk}r_j \\
\delta u_{ik} &= -\epsilon_{ijk}u_j \\
\delta\rho_{ik} &= \frac{1}{2}\left[\delta_{ik} + \rho_i\rho_k + \epsilon_{ikl}\rho_l\right],
\end{aligned}
$$

and the self-adjoint generators J_k, are

$$J_k = i\epsilon_{ijk}r_j\frac{\partial}{\partial r_i} + i\epsilon_{ijk}u_j\frac{\partial}{\partial u_i} + \frac{1}{2i}\left(\frac{\partial}{\partial\rho_k} + \rho_k\rho_i\frac{\partial}{\partial\rho_i} + \epsilon_{ikl}\rho_l\frac{\partial}{\partial\rho_i}\right).$$

They can be separated into three parts, according to the differential operators involved, with respect to the three kinds of kinematical variables $\boldsymbol{r}$, $\boldsymbol{u}$ and $\boldsymbol{\rho}$, respectively:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{Z} + \boldsymbol{W},$$

$$L_k = i\epsilon_{ijk}r_j\frac{\partial}{\partial r_i},$$

$$Z_k = i\epsilon_{ijk}u_j\frac{\partial}{\partial u_i}, \quad W_k = \frac{1}{2i}\left(\frac{\partial}{\partial\rho_k} + \rho_k\rho_i\frac{\partial}{\partial\rho_i} + \epsilon_{ikl}\rho_l\frac{\partial}{\partial\rho_i}\right). \tag{4.69}$$

They satisfy the angular momentum commutation rules and commute among themselves:

$$[L_j, L_k] = i\epsilon_{jkl}L_l, \quad [Z_j, Z_k] = i\epsilon_{jkl}Z_l, \quad [W_j, W_k] = i\epsilon_{jkl}W_l,$$

$$[\boldsymbol{L}, \boldsymbol{Z}] = [\boldsymbol{L}, \boldsymbol{W}] = [\boldsymbol{Z}, \boldsymbol{W}] = 0.$$

and thus

$$[J_j, J_k] = i\epsilon_{jkl}J_l, \quad [\boldsymbol{J}, H] = [\boldsymbol{J}, D] = 0, \quad [J_j, P_k] = i\epsilon_{jkl}P_l.$$

The above orientation variable $\boldsymbol{\rho}$, under a general boost of velocity $\boldsymbol{v}$, transforms as

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\rho} + \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho})}{1 + G(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho})},$$

where

$$\boldsymbol{F}(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho}) = \frac{\gamma(v)}{1 + \gamma(v)}\left(\boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{\rho}) + (\boldsymbol{u} \times \boldsymbol{\rho}) \times \boldsymbol{v}\right),$$

$$G(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{\rho}) = \frac{\gamma(v)}{1 + \gamma(v)} \left(\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{\rho}) \right), \quad \gamma(v) = (1 - v^2)^{-1/2}.$$

Finally, under an infinitesimal boost of value $\delta\boldsymbol{v}$, $\gamma(v) \approx 1$, the kinematical variables transform as

$$
\begin{aligned}
\delta t &= \boldsymbol{r} \cdot \delta\boldsymbol{v} \\
\delta\boldsymbol{r} &= t\delta\boldsymbol{v} \\
\delta\boldsymbol{u} &= \delta\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{u} \cdot \delta\boldsymbol{v}) \\
\delta\boldsymbol{\rho} &= -\left[\boldsymbol{\rho}(\boldsymbol{u} \cdot \delta\boldsymbol{v}) + \boldsymbol{\rho}\left((\boldsymbol{u} \times \boldsymbol{\rho}) \cdot \delta\boldsymbol{v}\right) - \boldsymbol{u} \times \delta\boldsymbol{v} - \delta\boldsymbol{v}(\boldsymbol{u} \cdot \boldsymbol{\rho}) - \right. \\
&\qquad \left. (\boldsymbol{u} \times \boldsymbol{\rho}) \times \delta\boldsymbol{v} \right]/2, \\
\delta\beta &= 0,
\end{aligned}
$$

and the variation of these variables per unit of infinitesimal velocity parameter δv_j is

$$
\begin{aligned}
\delta t_j &= r_j \\
\delta r_{ij} &= t\delta_{ij} \\
\delta v_{ij} &= \delta_{ij} - u_i u_j \\
\delta\rho_{ij} &= -\frac{1}{2} \left[\rho_j u_i + \rho_i \epsilon_{jkl} u_k \rho_l - \epsilon_{ikj} u_k - \delta_{ij} u_k \rho_k \right], \\
\delta\beta_j &= 0,
\end{aligned}
$$

so that the boost generators K_j have the form

$$
\begin{aligned}
K_j &= ir_j \frac{\partial}{\partial t} + it \frac{\partial}{\partial r_j} + i \left(\frac{\partial}{\partial u_j} - u_j u_i \frac{\partial}{\partial u_i} \right) + \\
&\quad \frac{1}{2i} \left(\rho_j u_i \frac{\partial}{\partial \rho_i} + \rho_i \epsilon_{jkl} u_k \rho_l \frac{\partial}{\partial \rho_i} - \epsilon_{ikj} u_k \frac{\partial}{\partial \rho_i} - u_k \rho_k \frac{\partial}{\partial \rho_j} \right)
\end{aligned}
$$

Similarly, the generators K_j can be decomposed into three parts, according to the differential operators involved and we represent them with the same capital letters as in the case of the $\boldsymbol{J}$ operators but with a tilde:

$$\boldsymbol{K} = \widetilde{\boldsymbol{L}} + \widetilde{\boldsymbol{Z}} + \widetilde{\boldsymbol{W}}, \quad \widetilde{L}_j = ir_j \frac{\partial}{\partial t} + it \frac{\partial}{\partial r_j}, \quad \widetilde{Z}_j = i \left(\frac{\partial}{\partial u_j} - u_j u_i \frac{\partial}{\partial u_i} \right),$$

$$\widetilde{W}_j = \frac{1}{2i} \left(\rho_j u_i \frac{\partial}{\partial \rho_i} + \rho_i \epsilon_{jkl} u_k \rho_l \frac{\partial}{\partial \rho_i} + \epsilon_{jki} u_k \frac{\partial}{\partial \rho_i} - u_k \rho_k \frac{\partial}{\partial \rho_j} \right)$$

They satisfy the commutation rules:

$$[\widetilde{L}_j, \widetilde{L}_k] = -i\epsilon_{jkl} L_l, \quad [\widetilde{Z}_j, \widetilde{Z}_k] = -i\epsilon_{jkl} Z_l, \quad [\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{Z}}] = [\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{W}}] = 0,$$

and also

$$[K_j, K_k] = -i\epsilon_{jkl} J_l.$$

We can check that

$$\widetilde{\boldsymbol{Z}} = \boldsymbol{u} \times \boldsymbol{Z}, \quad \widetilde{\boldsymbol{W}} = \boldsymbol{u} \times \boldsymbol{W}.$$

If we define the spin operator $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, and the part of the kinematical momentum $\widetilde{\boldsymbol{S}} = \widetilde{\boldsymbol{Z}} + \widetilde{\boldsymbol{W}} = \boldsymbol{u} \times \boldsymbol{S}$, they satisfy

$$[S_j, S_k] = i\epsilon_{jkl}S_l, \quad [S_j, \widetilde{S}_k] = i\epsilon_{jkl}\widetilde{S}_l, \quad [\widetilde{S}_j, \widetilde{S}_k] = -i\epsilon_{jkl}S_l,$$

where in the last expression we have used the constraint $u^2 = 1$. They generate the Lie algebra of a Lorentz group which commutes with spacetime translations $[\boldsymbol{S}, p^\mu] = [\widetilde{\boldsymbol{S}}, p^\mu] = 0$.

In the $\boldsymbol{\rho}$ parameterization of the rotation group, the unit vectors of the body frame $\boldsymbol{e}_i$, $i = 1, 2, 3$ have the following components:

$$(\boldsymbol{e}_i)_j = R(\boldsymbol{\rho})_{ji},$$

so that the $T_k = \boldsymbol{e}_k \cdot \boldsymbol{W}$ operators of projecting the rotational angular momentum $\boldsymbol{W}$ onto the body frame are given by

$$T_k = \frac{1}{2i}\left(\frac{\partial}{\partial \rho_k} + \rho_k \rho_i \frac{\partial}{\partial \rho_i} - \epsilon_{ikl}\rho_l \frac{\partial}{\partial \rho_i}\right). \tag{4.70}$$

They differ from the W_k in (4.69) by the change of $\boldsymbol{\rho}$ by $-\boldsymbol{\rho}$, followed by a global change of sign. They satisfy the commutation relations

$$[T_j, T_k] = -i\epsilon_{jkl}T_l. \tag{4.71}$$

The minus sign on the right hand side of (4.71) corresponds to the difference between the active and passive point of view of transformations. The rotation of the laboratory axis (passive rotation) has as generators the $\boldsymbol{J}$, which satisfy $[J_j, J_k] = i\epsilon_{jkl}J_l$. The T_i correspond to the generators of rotations of the particle axis (active rotation), so that, the generators $-T_i$ will also be passive generators of rotations and satisfy $[-T_j, -T_k] = i\epsilon_{jkl}(-T_l)$.

In the normal parameterization of rotations $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, if we describe the unit vector $\boldsymbol{n}$ along the rotation axis by the usual polar and azimuthal angles θ and ϕ, respectively, so that $\boldsymbol{n} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the above W_i generators take the form

$$\begin{aligned}
W_1 &= \frac{1}{2i}\left[2\sin\theta\cos\phi\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\cos\phi}{\tan(\alpha/2)} - \sin\phi\right)\frac{\partial}{\partial\theta} - \right.\\
&\quad \left.\left(\frac{\sin\phi}{\tan(\alpha/2)\sin\theta} + \frac{\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi}\right],
\end{aligned} \tag{4.72}$$

$$\begin{aligned}
W_2 &= \frac{1}{2i}\left[2\sin\theta\sin\phi\frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta\sin\phi}{\tan(\alpha/2)} + \cos\phi\right)\frac{\partial}{\partial\theta} - \right.\\
&\quad \left.\left(\frac{\cos\theta\sin\phi}{\sin\theta} - \frac{\cos\phi}{\tan(\alpha/2)\sin\theta}\right)\frac{\partial}{\partial\phi}\right],
\end{aligned} \tag{4.73}$$

$$W_3 = \frac{1}{2i}\left[2\cos\theta\frac{\partial}{\partial\alpha} - \frac{\sin\theta}{\tan(\alpha/2)}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\right], \tag{4.74}$$

$$\begin{aligned}
W^2 &= -\left[\frac{\partial^2}{\partial\alpha^2} + \frac{1}{\tan(\alpha/2)}\frac{\partial}{\partial\alpha} + \right.\\
&\quad \left.\frac{1}{4\sin^2(\alpha/2)}\left\{\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\right],
\end{aligned} \tag{4.75}$$

$$W_+ = W_1 + iW_2 = \frac{e^{i\phi}}{2i} \left[2\sin\theta \frac{\partial}{\partial\alpha} + \frac{\cos\theta}{\tan(\alpha/2)} \frac{\partial}{\partial\theta} + i \frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\phi} + \right.$$
$$\left. \frac{i}{\tan((\alpha/2))\sin\theta} \frac{\partial}{\partial\phi} \right], \tag{4.76}$$

$$W_- = W_1 - iW_2 = \frac{e^{-i\phi}}{2i} \left[2\sin\theta \frac{\partial}{\partial\alpha} + \frac{\cos\theta}{\tan(\alpha/2)} \frac{\partial}{\partial\theta} - i \frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta} \frac{\partial}{\partial\phi} - \right.$$
$$\left. \frac{i}{\tan(\alpha/2)\sin\theta} \frac{\partial}{\partial\phi} \right], \tag{4.77}$$

and the passive T_i generators take the form

$$T_1 = \frac{-i}{2} \left[2\sin\theta \cos\phi \frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta \cos\phi}{\tan(\alpha/2)} + \sin\phi \right) \frac{\partial}{\partial\theta} - \right.$$
$$\left. \left(\frac{\sin\phi}{\tan(\alpha/2)\sin\theta} - \frac{\cos\theta \cos\phi}{\sin\theta} \right) \frac{\partial}{\partial\phi} \right], \tag{4.78}$$

$$T_2 = \frac{-i}{2} \left[2\sin\theta \sin\phi \frac{\partial}{\partial\alpha} + \left(\frac{\cos\theta \sin\phi}{\tan(\alpha/2)} - \cos\phi \right) \frac{\partial}{\partial\theta} - \right.$$
$$\left. \left(-\frac{\cos\theta \sin\phi}{\sin\theta} - \frac{\cos\phi}{\tan(\alpha/2)\sin\theta} \right) \frac{\partial}{\partial\phi} \right], \tag{4.79}$$

$$T_3 = \frac{-i}{2} \left[2\cos\theta \frac{\partial}{\partial\alpha} - \frac{\sin\theta}{\tan(\alpha/2)} \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\phi} \right]. \tag{4.80}$$

T_i are related to W_i by changing α into $-\alpha$.

The normalized eigenvectors of $W^2 = T^2$ and W_3 and T_3 for $w = i = 1/2$, written in the form $|w_3, t_3 >$, (which are also eigenvectors of Z^2 with $z = 0$) are written as $|0; s_3, t_3 >$

$$\Phi_1 = |1/2, -1/2> = i\sqrt{2}\sin(\alpha/2)\sin\theta e^{i\phi}, \tag{4.81}$$
$$\Phi_2 = |-1/2, -1/2> = \sqrt{2}\left(\cos(\alpha/2) - i\cos\theta \sin(\alpha/2)\right) \tag{4.82}$$
$$\Phi_3 = |1/2, 1/2> = -\sqrt{2}\left(\cos(\alpha/2) + i\cos\theta \sin(\alpha/2)\right), \tag{4.83}$$
$$\Phi_4 = |-1/2, 1/2> = -i\sqrt{2}\sin(\alpha/2)\sin\theta e^{-i\phi}. \tag{4.84}$$

The rising and lowering operators $W_\pm$ and the corresponding $T_\pm$ transform them among each other. $\{\Phi_1, \Phi_2\}$ are related by $W_\pm$, and similarly the $\{\Phi_3, \Phi_4\}$ while the sets $\{\Phi_1, \Phi_3\}$ and $\{\Phi_2, \Phi_4\}$ are separately related by $T_\pm$. For instance

$$W_-\Phi_1 = \Phi_2, \quad W_-\Phi_2 = 0, \quad W_-\Phi_3 = \Phi_4,$$

$$T_-\Phi_1 = \Phi_3, \quad T_-\Phi_3 = 0, \quad T_-\Phi_2 = \Phi_4.$$

They form an orthonormal set with respect to the normalised invariant measure defined on $SU(2)$

$$dg(\alpha, \theta, \phi) = \frac{1}{4\pi^2} \sin^2(\alpha/2)\sin\theta \, d\alpha \, d\theta \, d\phi,$$

$$\alpha \in [0, 2\pi], \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi].$$

$$\int_{SU(2)} dg(\alpha, \theta, \phi) = 1.$$

The wavefunction ψ can be separated in the form

$$\psi(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\alpha}) = \sum_{i=1}^{i=4} \phi_i(t, \boldsymbol{r}) \chi_i(\boldsymbol{u}, \boldsymbol{\alpha})$$

where the four χ_i can be classified according to the eigenvalues $|s_3, t_3 >$. The functions $\phi_i(t, \boldsymbol{r})$ can be chosen as eigenfunctions of the Klein-Gordon operator

$$p_\mu p^\mu \phi_i(t, \boldsymbol{r}) = m_i^2 \phi_i(t, \boldsymbol{r}).$$

The functions $\chi(\boldsymbol{u}, \boldsymbol{\alpha})$ can also be separated because the total spin $\boldsymbol{S}$ with $s = 1/2$, is the sum of the two parts $\boldsymbol{S} = \boldsymbol{Z} + \boldsymbol{W}$, with $[\boldsymbol{Z}, \boldsymbol{W}] = 0$, so that since the $\boldsymbol{W}$ part contributes with $w = 1/2$ then the $\boldsymbol{Z}$ part contributes with $z = 0$ or $z = 1$. The $z = 0$ contribution corresponds to the functions $\chi_i(\boldsymbol{\alpha})$ independent of the velocity variables and the orthonormal set are the above Φ_i, $i = 1, 2, 3, 4$, which can also be written in the form $|z; s_3, t_3 >$, with $z = 0$.

Because $\boldsymbol{Z} = -i\boldsymbol{u} \times \nabla_u$, for the $z = 1$ part the eigenvectors of Z^2 and Z_3 are the spherical harmonics $Y_1^i(\widetilde{\theta}, \widetilde{\phi})$, $i = -1, 0, 1$. The variables $\widetilde{\theta}$ and $\widetilde{\phi}$ represent the direction of the velocity vector $\boldsymbol{u}$. Because $[Z_i, W_j] = 0$, we can again separate the variables in the functions $\chi(\boldsymbol{u}, \boldsymbol{\alpha})$. In this case the $\chi(\boldsymbol{u}, \boldsymbol{\alpha}) = \sum \phi_i(\widetilde{\theta}, \widetilde{\phi}) \lambda_i(\alpha, \theta, \phi)$. The four orthonormal vectors, eigenvectors of S_3, Z^2 with $z = 1$ and T_3, $|1; s_3, t_3 >$, are now

$$\Psi_1 \;=\; |1; 1/2, 1/2 >= \frac{1}{\sqrt{3}} \left(Y_1^0(\widetilde{\theta}, \widetilde{\phi}) \Phi_1 - \sqrt{2} Y_1^1(\widetilde{\theta}, \widetilde{\phi}) \Phi_2 \right), \tag{4.85}$$

$$\Psi_2 \;=\; |1; -1/2, 1/2 >= \frac{1}{\sqrt{3}} \left(-Y_1^0(\widetilde{\theta}, \widetilde{\phi}) \Phi_2 + \sqrt{2} Y_1^{-1}(\widetilde{\theta}, \widetilde{\phi}) \Phi_1 \right), \tag{4.86}$$

$$\Psi_3 \;=\; |1; 1/2, -1/2 >= \frac{1}{\sqrt{3}} \left(Y_1^0(\widetilde{\theta}, \widetilde{\phi}) \Phi_3 - \sqrt{2} Y_1^1(\widetilde{\theta}, \widetilde{\phi}) \Phi_4 \right), \tag{4.87}$$

$$\Psi_4 \;=\; |1; -1/2, -1/2 >= \frac{1}{\sqrt{3}} \left(-Y_1^0(\widetilde{\theta}, \widetilde{\phi}) \Phi_4 + \sqrt{2} Y_1^{-1}(\widetilde{\theta}, \widetilde{\phi}) \Phi_3 \right). \tag{4.88}$$

where Φ_i are the same as those in (4.81-4.84) and the spherical harmonics $Y_1^i(\widetilde{\theta}, \widetilde{\phi})$ are

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin(\widetilde{\theta}) e^{i\widetilde{\phi}}, \;\; Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos(\widetilde{\theta}), \;\; Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin(\widetilde{\theta}) e^{-i\widetilde{\phi}}. \tag{4.89}$$

The Z_i operators are given by

$$Z_1 = i\sin\widetilde{\phi} \frac{\partial}{\partial\widetilde{\theta}} + i\frac{\cos\widetilde{\theta}}{\sin\widetilde{\theta}} \cos\widetilde{\phi} \frac{\partial}{\partial\widetilde{\phi}}, \quad Z_2 = -i\cos\widetilde{\phi} \frac{\partial}{\partial\widetilde{\theta}} + i\frac{\cos\widetilde{\theta}}{\sin\widetilde{\theta}} \sin\widetilde{\phi} \frac{\partial}{\partial\widetilde{\phi}},$$

$$Z_3 = -i\frac{\partial}{\partial\widetilde{\phi}}.$$

The rising and lowering operators $Z_\pm$ are

$$Z_\pm = e^{\pm i\widetilde{\phi}} \left(\pm\frac{\partial}{\partial\widetilde{\theta}} + i\frac{\cos\widetilde{\theta}}{\sin\widetilde{\theta}} \frac{\partial}{\partial\widetilde{\phi}} \right),$$

so that

$$Z_- Y_1^1 = \sqrt{2}\, Y_1^0, \quad Z_- Y_1^0 = \sqrt{2}\, Y_1^{-1}.$$

The four spinors Ψ_i are orthonormal with respect to the invariant measure

$$dg(\alpha, \theta, \phi\,; \widetilde{\theta}, \widetilde{\phi}) = \frac{1}{4\pi^2} \sin^2(\alpha/2) \sin\theta \sin\widetilde{\theta}\, d\alpha\, d\theta\, d\phi\, d\widetilde{\theta} d\widetilde{\phi}$$

$$\alpha \in [0, 2\pi], \quad \widetilde{\theta}, \theta \in [0, \pi], \quad \widetilde{\phi}, \phi \in [0, 2\pi].$$

Similarly as before, the rising and lowering operators $S_\pm = Z_\pm + W_\pm$ and the corresponding $T_\pm$ transform the Ψ_i among each other. In particular $\{\Psi_1, \Psi_2\}$ are related by $S_\pm$, and similarly $\{\Psi_3, \Psi_4\}$ while the sets $\{\Psi_1, \Psi_3\}$ and $\{\Psi_2, \Psi_4\}$ are separately related by $T_\pm$. This is the reason why the general spinor in this representation is a four-component object.

In the $z = 0$ basis Φ_i (4.81-4.84), the spin operators and the basis vectors of the body frame take the form

$$\boldsymbol{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} = \boldsymbol{W},$$

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i\mathbb{I} \\ i\mathbb{I} & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix},$$

$$\boldsymbol{e}_1 = \frac{-1}{3}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \frac{-1}{3}\begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{-1}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix},$$

in terms of the Pauli $\boldsymbol{\sigma}$ matrices and the 2×2 unit matrix $\mathbb{I}$.

In the $z = 1$ basis Ψ_i (4.85-4.88), the operators S_i and T_i take the same matrix form as above, while the $\boldsymbol{e}_i$ are

$$\boldsymbol{e}_1 = \frac{1}{9}\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \frac{1}{9}\begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \frac{1}{9}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}.$$

In all cases, the 6 Hermitian traceless matrices S_i, T_j, the nine Hermitian traceless matrices e_{ij} and the 4×4 unit matrix are linearly independent and they completely define a Hermitian basis for Dirac's algebra, so that any other translation invariant observable of the particle will be expressed as a real linear combination of the above 16 Hermitian matrices. We have used this fact previously to explicitly obtain Dirac's equation for this model.

Both representations are orthogonal to each other, $< \Phi_i | \Psi_j >= 0$, and they produce two different irreducible representations of the group, so that they describe two different kinds of particles of the same spin $1/2$.

The matrix representation of the Z_i and W_i operators in the basis Ψ_i are given by

$$\boldsymbol{Z} = \frac{2}{3}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{W} = \frac{-1}{6}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

although Ψ_i are not eigenvectors of Z_3 and W_3.

Chapter 5

Electromagnetic structure of the electron

5.1 Electromagnetic structure of the electron

Let us consider that the classical electron is described by the model whose charge is moving in circles at the speed of light in the center of mass frame.

One of the immediate questions concerning the classical structure of the electron is, what is the associated electromagnetic field of the particle? We see that the charge is accelerated and according to the classical electromagnetic theory, the particle must necessarily radiate continuously. However, from the mechanical point of view we have produced a classical free system, such that properties like mechanical energy and mechanical linear and angular momentum are conserved in time. The Lagrangian that describes the system is Poincaré invariant, and if we think about a free system, the corresponding field structure cannot produce loss of energy and linear momentum. The free particle has to have associated an electromagnetic field without radiation. Radiation has to be produced whenever the center of mass of the particle is accelerated, *i.e.*, when the particle is no longer free.

There must exist radiationless solutions of Maxwell's equations, associated to point charges moving in circles at the speed of light. One possibility is to consider solutions derived from the Liénard-Wiechert potentials $(A^\mu_{ret} + A^\mu_{adv})/2$, where A^μ_{ret} and A^μ_{adv} are the corresponding retarded and advanced potentials. But, even if we take as the probable electric field $(\boldsymbol{E}_{ret} + \boldsymbol{E}_{adv})/2$, it is neither static nor Coulomb-like, and therefore it does not look like the estimated electric field of a point electron. We shall consider next a particular static solution: the time average field during a complete turn of the charge.

5.1.1 The time average electric and magnetic field

Let us assume that we have a test charge in the neighborhood of the electron. The frequency of the zitterbewegung is very high, of order $\sim 10^{21}$ s^{-1}. If our test particle is moving slowly, then presumably the detected electric field will be some time average field during a complete turn of the charge.

The retarded (or advanced) electric field of a point charge at the observation point $\boldsymbol{x}$ at time t is given by [1]

$$\boldsymbol{E} = \boldsymbol{E}_\beta + \boldsymbol{E}_a,$$

---

[1] A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, Dover, N.Y. (1980).

where

$$\boldsymbol{E}_\beta(t, \boldsymbol{x}) = \frac{e(1 - \beta^2)}{R^2(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3}\,(\boldsymbol{n} - \boldsymbol{\beta})\,, \tag{5.1}$$

$$\boldsymbol{E}_a(t, \boldsymbol{x}) = \frac{e}{Rc^2(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3}\,\boldsymbol{n} \times ((\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a})\,, \tag{5.2}$$

are the velocity and acceleration fields, respectively. Observables $\boldsymbol{r}$, $\boldsymbol{u} = d\boldsymbol{r}/dt$ and $\boldsymbol{a} = d\boldsymbol{u}/dt$, are the position, velocity and acceleration of the charge, evaluated at the retarded (or advanced) time $\tilde{t} = t - R/c$, (or $\tilde{t} = t + R/c$). Vector $\boldsymbol{\beta} = \boldsymbol{u}/c$, and

$$\boldsymbol{n} = \frac{\boldsymbol{x} - \boldsymbol{r}}{|\boldsymbol{x} - \boldsymbol{r}|}, \quad R = |\boldsymbol{x} - \boldsymbol{r}|.$$

The corresponding magnetic field is $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}/c$. Because for the electron the charge is moving at the speed of light $\beta = 1$, the velocity field $\boldsymbol{E}_\beta$ vanishes, and it seems that the only field contribution behaves as $1/R$.

The complete analytical expression of a time average field at any arbitrary point has not yet been obtained. However, to obtain an estimate, let us compute the average field on some particular point. Let us consider that the electron is at rest, with the center of mass at the origin of a reference frame. The constant spin is pointing along the OZ axis. We shall try to calculate this average field at a point P of coordinate z in this OZ axis. In Figure 5.1, we represent the different magnitudes at the retarded time $\tilde{t}$, needed to apply equation (5.2).
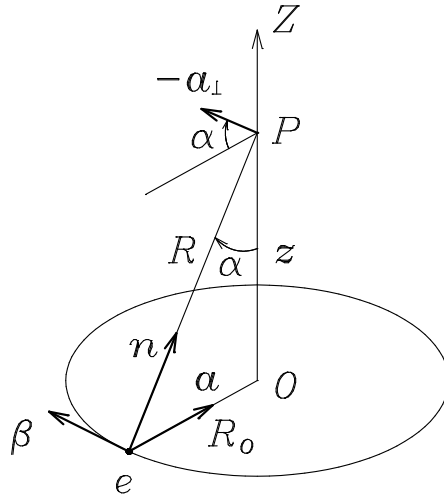
Figure 5.1: **Instantaneous electric field of the electron at point P has a component along $-a_\perp$ and $-\boldsymbol{\beta}$.**

In that particular point shown in the figure, $\boldsymbol{n} \cdot \boldsymbol{\beta} = 0$, and thus

$$\boldsymbol{E} = \frac{e}{c^2 R}\,(\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{a}) - \boldsymbol{n} \times (\boldsymbol{\beta} \times \boldsymbol{a})) = \frac{e}{c^2 R}\,(-\boldsymbol{a}_\perp - \boldsymbol{\beta}(\boldsymbol{n} \cdot \boldsymbol{a}))\,,$$

where vector $\boldsymbol{a}_\perp = \boldsymbol{a} - \boldsymbol{n}(\boldsymbol{a} \cdot \boldsymbol{n})$, is the component of the acceleration orthogonal to the unit vector $\boldsymbol{n}$. For the observation point P, the expression $\boldsymbol{n} \cdot \boldsymbol{a}$ is constant at any retarded point,

and the time average of $\boldsymbol{\beta}$ during a complete turn is zero, and for the vector $\boldsymbol{a}_\perp$ it reduces to its z-component $a_\perp \sin \alpha$. Since the acceleration in this frame is $a = c^2/R_0$, $a_\perp = a \cos \alpha$ and $\sin \alpha = R_0/R$ and $\cos \alpha = z/R$, the time average electric field at point P is

$$\boldsymbol{E}(z) = \frac{ez}{(R_0^2 + z^2)^{3/2}} \hat{\boldsymbol{z}}, \qquad (5.3)$$

where $\hat{\boldsymbol{z}}$ is a unit vector along the OZ axis. The advanced field has exactly the same expression. This is a radial field from the origin of the reference frame with a Coulomb-like behaviour $1/z^2$, but it does not diverge at the origin. We depict this field in Figure 5.2, for comparison with the Coulomb field of a point charge at the origin, where we take as a unit of length the radius R_0 of the internal motion.

We can clearly see the fitting of the average field and the Coulomb field for large z. The maximum of the average field takes place at $z = R_0/\sqrt{2}$. If we consider that the static field of a pointlike electron is this time average field, then the electrostatic energy does not diverge. But to obtain the expression for the energy is necessary to find the field at any point, a goal that we have not attained.
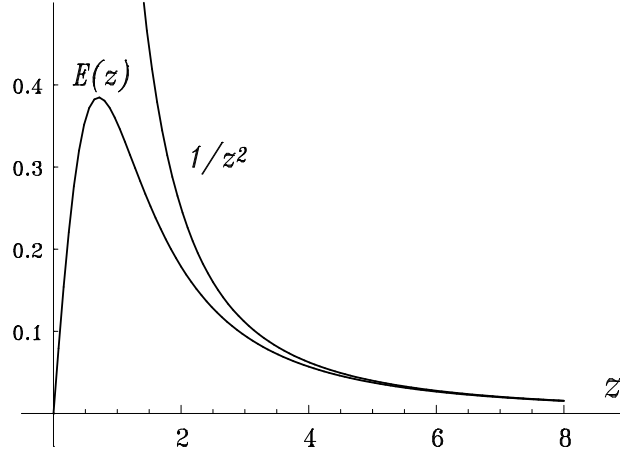
Figure 5.2: **Average retarded (or advanced) electric field (5.3) and Coulomb field along the OZ axis.**

However, if we are involved in high energy processes, our test particle is moving sufficiently fast relative to the electron, then the field it senses is the instantaneous $1/R$ field, which is greater than the average field, and becomes important for points closer to the electron. This means that the average energy density of the local instantaneous field is greater than the average Coulomb-like energy density, and we can naively interpret this difference, from the classical point of view, as the energy associated to the cloud of virtual photons in the surroundings of the particle. Is this the corresponding infinite energy which is usually cancelled out in the renormalization of quantum electrodynamics?

To compute numerically the average field at an arbitrary position, let us consider the different magnitudes depicted in Figure 5.3.

If at time $t = 0$ the charge is located at point A on the OX axis, then at time t the different observables shown in the figure are described in Cartesian coordinates and in the laboratory frame by

$$\boldsymbol{k} = R_0[\cos \omega t, \sin \omega t, 0] \equiv R_0 \widetilde{\boldsymbol{k}}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{u}}{c} = [-\sin \omega t, \cos \omega t, 0],$$
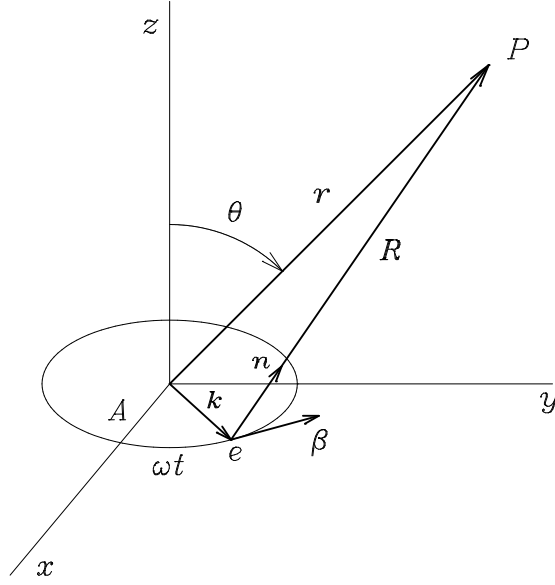
Figure 5.3: **Charge motion and observation point P.**

$$\boldsymbol{r} = [x, y, z], \quad \boldsymbol{a} = \frac{d\boldsymbol{u}}{dt} = \frac{c^2}{R_0}[-\cos\omega t, -\sin\omega t, 0] = \frac{c^2}{R_0}\widehat{\boldsymbol{a}},$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{k} = R_0(\widetilde{\boldsymbol{r}} - \widetilde{\boldsymbol{k}}), \quad \boldsymbol{n} = \frac{\boldsymbol{R}}{R}, \quad R = |\boldsymbol{R}| = R_0\widetilde{R}.$$

With these definitions, field (5.2) can be written as

$$\boldsymbol{E}(t, \boldsymbol{r}) = \left(\frac{e}{R_0^2}\right) \frac{\boldsymbol{n} \times ((\boldsymbol{n} - \boldsymbol{\beta}) \times \widehat{\boldsymbol{a}})}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3 \widetilde{R}}.$$

We want to compare the time average value of this field with the static Coulomb field of a point charge e at the center of mass

$$\boldsymbol{E}_0(\boldsymbol{r}) = \left(\frac{e}{R_0^2}\right) \frac{\widehat{\boldsymbol{r}}}{r^2},$$

where $\widehat{\boldsymbol{r}}$ is a unit vector in the radial direction. The constant factor in brackets in front of these formulae will be dropped out from now on. In this way the unit of length is the zitterbewegung radius R_0.

When the charge is at the point indicated in Figure 5.3, the retarded field it produces at point P is evaluated at the observation time $t_o = t + R/c$. Thus $dt_o = dt + d\widetilde{R}/\omega$, because $R_0/c = 1/\omega$. If we express $d\widetilde{R}$ in terms of dt, we get $dt_o = (N(t)/\widetilde{R}(t))dt$, where N and $\widetilde{R}$ are explicitly given by

$$\widetilde{R}(t) = \sqrt{(\widetilde{x} - \cos\omega t)^2 + (\widetilde{y} - \sin\omega t)^2 + \widetilde{z}^2},$$

$$N(t) = \widetilde{R}(t) + \widetilde{x}\sin\omega t - \widetilde{y}\cos\omega t.$$

We are going to average the field at P with respect to the observation time at that point during a complete period of the motion of the charge T. If we define a dimensionless evolution time

$\tau = \omega t$, then $\omega T = 2\pi$ and thus

$$\frac{1}{T}\int_0^T E(t_o)\,dt_o = \frac{1}{T}\int_0^T E(t)\frac{N(t)}{\widetilde{R}(t)}dt = \frac{1}{2\pi}\int_0^{2\pi} E(\tau)\frac{N(\tau)}{\widetilde{R}(\tau)}d\tau. \tag{5.4}$$

In terms of the τ evolution the different expressions are

$$\boldsymbol{n}\times(\boldsymbol{n}\times\widehat{\boldsymbol{a}}) = \boldsymbol{n}(\boldsymbol{n}\cdot\widehat{\boldsymbol{a}}) - \widehat{\boldsymbol{a}},$$

and

$$\boldsymbol{n}(\boldsymbol{n}\cdot\widehat{\boldsymbol{a}}) = \frac{1 - \widetilde{x}\cos\tau - \widetilde{y}\sin\tau}{\widetilde{R}^2}[\widetilde{x} - \cos\tau, \widetilde{y} - \sin\tau, \widetilde{z}],$$

$$\widehat{\boldsymbol{a}} = [-\cos\tau, -\sin\tau, 0],$$

while

$$\boldsymbol{n}\times(\boldsymbol{\beta}\times\widehat{\boldsymbol{a}}) = \frac{1}{\widetilde{R}}[\widetilde{y} - \sin\tau, -\widetilde{x} + \cos\tau, 0],$$

and

$$1 - \boldsymbol{n}\cdot\boldsymbol{\beta} = \frac{1}{\widetilde{R}}\left(\widetilde{R} + \widetilde{x}\sin\tau - \widetilde{y}\cos\tau\right).$$

We are interested in the radial and transversal part of the field $E_r = \boldsymbol{E}\cdot\widehat{\boldsymbol{r}}$, $E_\theta = \boldsymbol{E}\cdot\widehat{\boldsymbol{\theta}}$, and $E_\phi = \boldsymbol{E}\cdot\widehat{\boldsymbol{\phi}}$, respectively. Here $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\phi}}$ are respectively the usual unit vectors in polar spherical coordinates. If we consider that the observation point P is on the plane XOZ, then we have to take $\widetilde{x} = r\sin\theta$, $\widetilde{y} = 0$ and $\widetilde{z} = r\cos\theta$, where r is the radial separation from the origin in units of R_0.

The final expressions for the field components are

$$E_r(r,\theta,\tau) = \frac{(\widetilde{R}^2 - r^2 - 1)\sin\theta\cos\tau + \widetilde{R}\sin\theta\sin\tau + r(1 + \sin^2\theta\cos^2\tau)}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$E_\theta(r,\theta,\tau) = \frac{\left[(\widetilde{R}^2 - 1)\cos\tau + \widetilde{R}\sin\tau + r\sin\theta\cos^2\tau\right]\cos\theta}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$E_\phi(r,\theta,\tau) = \frac{(\widetilde{R}^2 - 1)\sin\tau + \widetilde{R}(r\sin\theta - \cos\tau) + r\sin\theta\sin\tau\cos\tau}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

with

$$\widetilde{R} = \sqrt{r^2 - 2r\sin\theta\cos\tau + 1}.$$

To take the time average value of the above fields we have to perform the integration (5.4) so that the above expressions of E_r, E_θ and E_ϕ have to be multiplied by $N(\tau)/\widetilde{R}(\tau)$, where now

$$N(\tau) = \widetilde{R} + r\sin\theta\sin\tau.$$

The average retarded radial electric field for $\theta = 0$ is already depicted in Figure 5.2 but we also include it in the next Figure 5.4. We see the Coulomb behavior of the radial component for the directions $\theta = 0, \pi/3, \pi/4, \pi/6$. Similarly, in Figure 5.5 is displayed the transversal component of the average retarded electric field $< E_\theta(r,\theta) >$ for the same directions, that goes
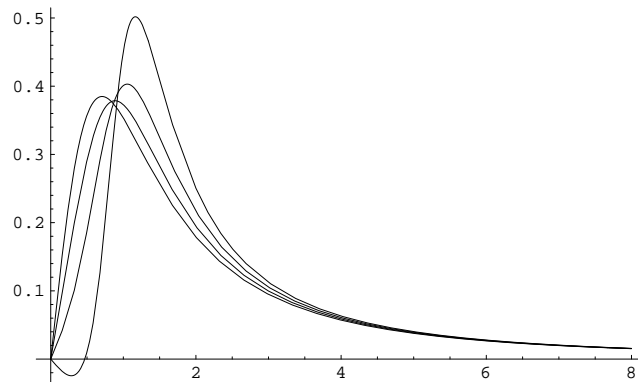
Figure 5.4: **Time average $< E_r(r) >$ of the radial component of the retarded electric field in the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$.**
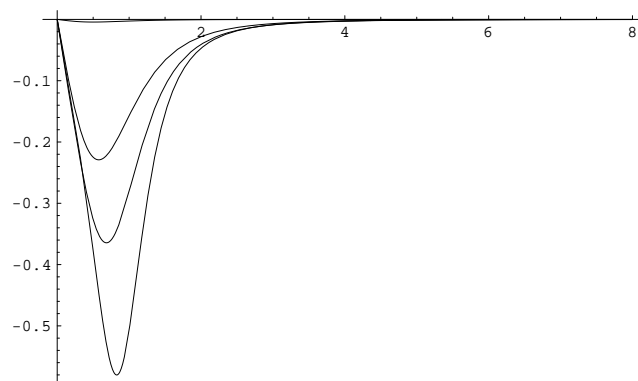

Figure 5.5: **Time average of the component $< E_\theta(r) >$ of the retarded electric field in the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$. It goes to zero very quickly. For $\theta = \pi/2$ it vanishes everywhere.**

to zero very quickly. For $\theta = \pi/2$, we see that $< E_\theta(r, \pi/2) >= 0$. The average $< E_\phi(r, \theta) >$ vanishes everywhere for any $\theta \neq \pi/2$. On the plane $\theta = \pi/2$ the numerical routine fails.

The average magnetic field can be computed in the same way. Here we shall consider only the retarded solution and we will compare it with the magnetic field produced by an intrinsic magnetic moment $\boldsymbol{\mu}$ placed at the center of mass. This magnetic field is [2]

$$\boldsymbol{B}_0(\boldsymbol{r}) = \frac{3\widehat{\boldsymbol{r}}(\widehat{\boldsymbol{r}} \cdot \boldsymbol{\mu}) - \boldsymbol{\mu}}{c^2 r^3}.$$

For our system the magnetic moment produced by the moving charge is of value $ecR_0/2$ in the direction of OZ, so that in units of R_0 it can be written as

$$\boldsymbol{B}_0(\boldsymbol{r}) = \left(\frac{e}{2cR_0^2}\right)\frac{3\widehat{\boldsymbol{r}}(\widehat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{z}}) - \widehat{\boldsymbol{z}}}{\widetilde{r}^3}.$$

The nonvanishing components are

$$B_{0r}(r, \theta) = \left(\frac{e}{cR_0^2}\right)\frac{\cos\theta}{\widetilde{r}^3}, \quad B_{0\theta}(r, \theta) = \left(\frac{e}{cR_0^2}\right)\frac{\sin\theta}{2\widetilde{r}^3}. \tag{5.5}$$

In our model, the instantaneous magnetic field is $\boldsymbol{B} = \boldsymbol{n} \times \boldsymbol{E}/c$. Their components can be written, after deleting a constant factor e/cR_0^2, as:

$$B_r(r, \theta, \tau) = \frac{(1 - r\sin\theta\cos\tau)\cos\theta}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$B_\theta(r, \theta, \tau) = \frac{r\cos\tau(1 + \sin^2\theta) - (1 + r^2)\sin\theta - \widetilde{R}r\sin\tau}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3},$$

$$B_\phi(r, \theta, \tau) = \frac{(\widetilde{R}\cos\tau + \sin\tau)r\cos\theta}{\left(\widetilde{R} + r\sin\theta\sin\tau\right)^3}.$$

To proceed with the retarded time average integral we have to multiply the above fields by $N(t)/\widetilde{R}(t)$, as before. The numerical integration is compared with the analytical expression of the magnetic field of a dipole (5.5) for different directions.

The magnetic dipole field (5.5) goes to infinity when $r \to 0$. In Figures 5.6-5.8 we show the matching of the $B_{0r}(r)$ components of the dipole and the computed time average value $< B_r(r, \theta) >$, for $r > R_0$ and in the directions given by $\theta = \pi/6, \pi/4$ and $\pi/3$. Similarly, in Figures 5.9-5.11, for the corresponding $B_{0\theta}(r, \theta)$ and $< B_\theta(r, \theta) >$ components.

The computed time averages $< B_r(r) >$ and $< B_\theta(r) >$ do not diverge at the origin but have the behavior depicted in 5.12 and 5.13, respectively, when represented along the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$, and they take the values $\cos\theta$ and $-\sin\theta$ respectively, at point $r = 0$.

The time average value of the transversal component $< B_\phi(r, \theta) >$ vanishes everywhere for all directions.

To end this section we can think about the possibility of computing the average fields using the advanced solutions in spite of the retarded ones.

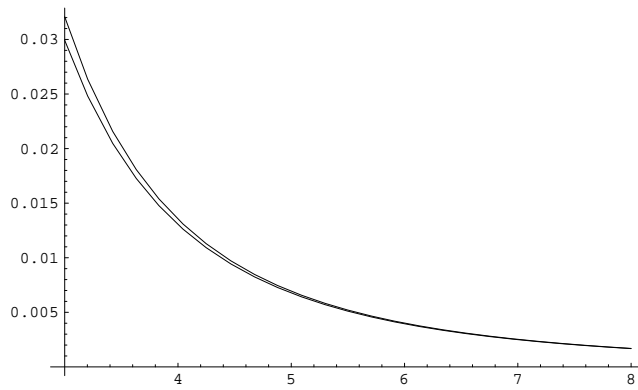[2] J.D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, NY 3rd. ed. (1998).

Figure 5.6: **Radial components of the dipole field $B_{0r}(r)$ and the time average retarded magnetic field $< B_r(r) >$, along the direction $\theta = \pi/6$.**
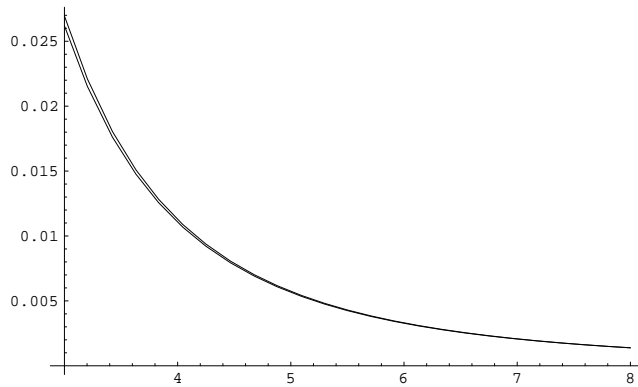

Figure 5.7: **Radial components of the dipole field $B_{0r}(r)$ and the time average retarded magnetic field $< B_r(r) >$, along the direction $\theta = \pi/4$.**
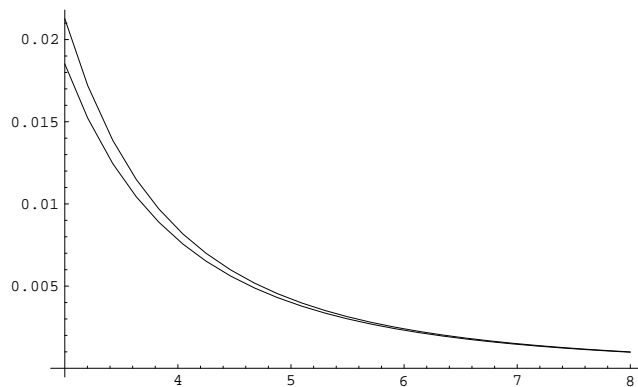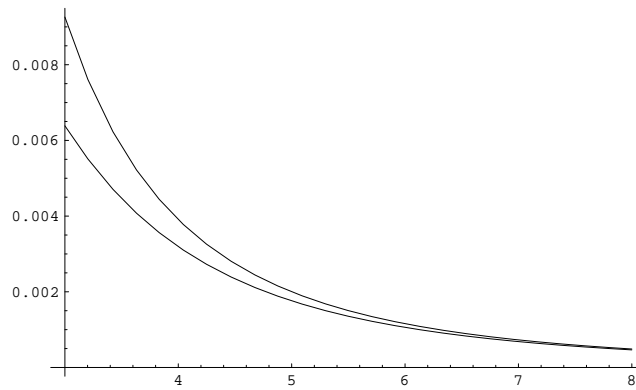

Figure 5.8: **Radial components of the dipole field $B_{0r}(r)$ and the time average retarded magnetic field $< B_r(r) >$, along the direction $\theta = \pi/3$.**

Figure 5.9: **Time average retarded magnetic field $< B_\theta(r) >$ and the dipole field $B_{0\theta}(r)$, along the direction $\theta = \pi/6$.**
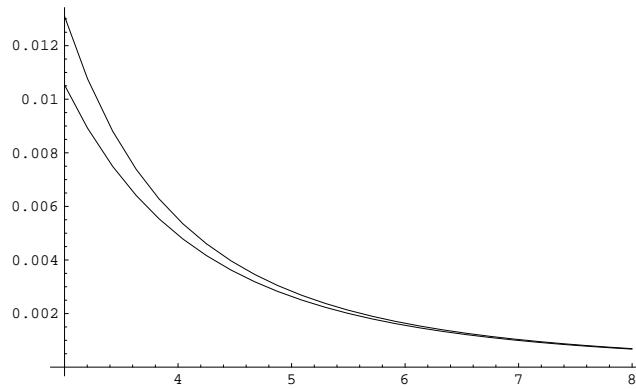
Figure 5.10: **Time average retarded magnetic field $< B_\theta(r) >$ and the dipole field $B_{0\theta}(r)$, along the direction $\theta = \pi/4$.**
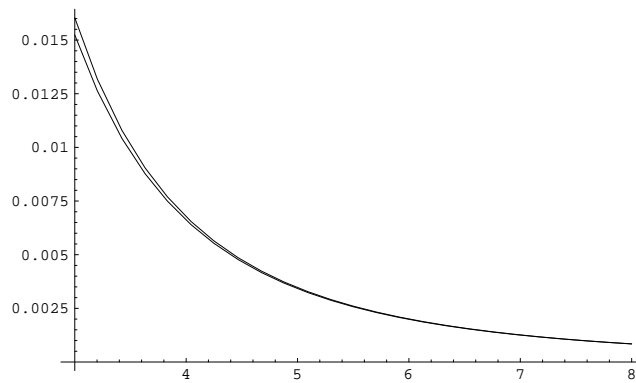
Figure 5.11: **Time average retarded magnetic field $< B_\theta(r) >$ and the dipole field $B_{0\theta}(r)$, along the direction $\theta = \pi/3$.**
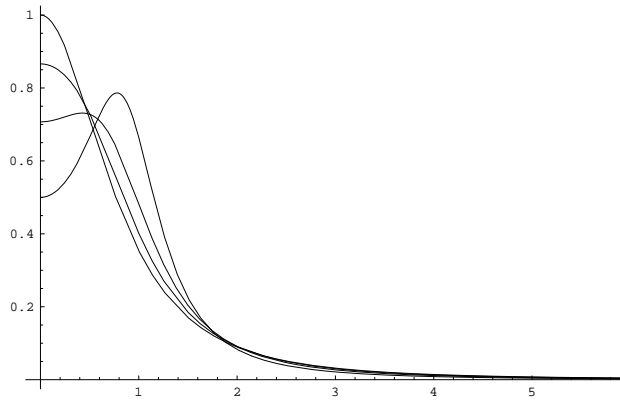
Figure 5.12: **Time average retarded magnetic field $< B_r(r) >$ along the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$ and its behavior at $r = 0$. For $\theta = \pi/2$ it vanishes everywhere.**
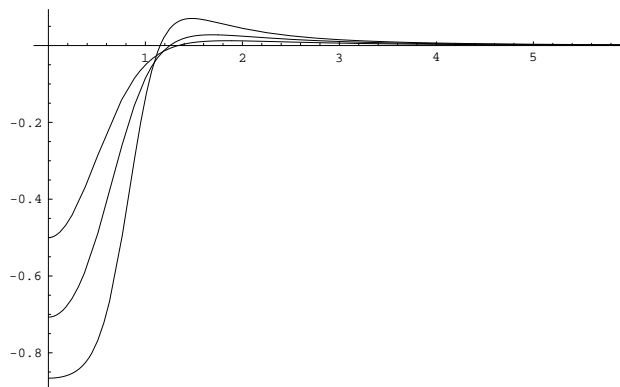
Figure 5.13: **Time average retarded magnetic field $< B_\theta(r) >$ along the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$ and its behavior at $r = 0$.**
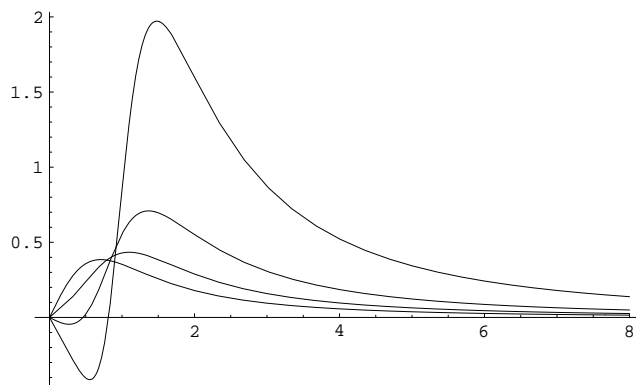
Figure 5.14: **Time average radial component $< E_r(r) >$ of the advanced electric field in the directions $\theta = 0, \pi/3, \pi/4$ and $\pi/6$.**

In that case the observation time will be related with the laboratory time by $t_o = t - R/c$, and therefore $dt_o = (M(t)/\widetilde{R}(t))dt$, where $\widetilde{R}(t)$ is the same as before, but

$$M(t) = \widetilde{R}(t) - \widetilde{x}\sin\omega t + \widetilde{y}\cos\omega t.$$

Then, if we depict, for instance, the advanced average radial electric field in Figure 5.14, for the same directions as in Figure 5.4, we see the different behavior in these radial directions and, although the field decreases for large distances, it nevertheless does not fit with a Coulomb field.

The numerical routine fails to compute the corresponding integrals for $\theta = \pi/2$ where we have some indefiniteness of the integrands for observation points lying on the XOY plane. There are no singularities for points inside the circle of radius R_0. We have a divergence of order $1/r$ for points on this circle, but this divergence can be removed by taking a principal value of the time integral. Finally, the quotient term $1 - \boldsymbol{n}\cdot\boldsymbol{\beta}$ can vanish for some observation points on the XOY plane outside the circle of radius R_0, whenever the retarded $\boldsymbol{n}$ and $\boldsymbol{\beta}$ become parallel vectors. But this can happen only for a single point of the retarded charge position in the average integral and perhaps some kind of principal value should be taken to properly obtain a finite average value. The difficulties of obtaining an analytical estimate for these integrals make this analysis incomplete. Nevertheless, the nice fitting of the average electric field with a Coulomb field and the average magnetic field with the field of a magnetic dipole, for distances of a few Compton wave lengths away, except on the $\theta = \pi/2$ plane where we have not been able to obtain an estimate, suggests that we devote some effort to renormalize and improve the model at a classical level.

Chapter 6

Some spin features

6.1 Gyromagnetic ratio

The $g = 2$ gyromagnetic ratio of the electron was considered for years a success of Dirac's electron theory. [1] Later, Levy-Leblond [2] obtained similarly $g = 2$ but from a $s = 1/2$ non-relativistic wave equation. Proca [3] found $g = 1$ for spin 1 particles and this led Belinfante [4] to conjecture that the gyromagnetic ratio for elementary systems is $g = 1/s$, irrespective of the value s of its spin. He showed this to be true for quantum systems of spin $3/2$, and a few years later the conjecture was analyzed and checked by Moldauer and Case [5] to be right for any half-integer spin, and by Tumanov [6] for the value $s = 2$. In all these cases a minimal electromagnetic coupling was assumed.

Weinberg [7] made the prediction $g = 2$ for the intermediate bosons of the weak interactions when analyzing the interaction of W bosons with the electromagnetic field by requiring a good high-energy behavior of the scattering amplitude. The discovery of the charged $W^{\pm}$ spin 1 bosons with $g = 2$, contradictory to Belinfante's conjecture, corroborated Weinberg's prediction and raised the question as to whether $g = 2$ for any elementary particle of arbitrary spin.

Jackiw [8] has given another dynamical argument confirming that the gyromagnetic ratio of spin-1 fields is $g = 2$, provided a nonelectromagnetic gauge invariance is accepted. He also gives some *ad hoc* argument for $s = 2$ fields, consistent with the $g = 2$ prescription.

Ferrara *et al.* [9] in a Lagrangian approach for massive bosonic and fermionic strings, by the requirement of a smooth fixed-charge $M \to 0$ limit, get $g = 2$ as the most natural value for particles of arbitrary spin. However the only known particles which fulfill this condition are leptons and charged $W^{\pm}$ bosons, *i.e.*, charged fermions and bosons of the lowest admissible values of spin. No other higher spin charged elementary particles have been found.

The aim of this section, instead of using dynamical arguments as in the previous attempts,

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[1] P.A.M. Dirac, *Proc. Roy. Soc. London* **A117**, 610 (1928).

[2] J.M. Levy-Leblond, *Comm. Math. Phys.* **6**, 286 (1967).

[3] A. Proca, *Compt. Rend.* **202**, 1420 (1936); *Journ. Phys. Radium*, **49**, 245 (1988).

[4] F.J. Belinfante, *Phys. Rev.* **92**, 997 (1953).

[5] P.A. Moldauer and K.M. Case, *Phys. Rev.* **102**, 279 (1956).

[6] V.S. Tumanov, *Sov. Phys. JETP*, **19**, 1182 (1964).

[7] S. Weinberg, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru and H. Pendleton, MIT press, Cambridge, MA (1970), p. 283.

[8] R. Jackiw, *Phys. Rev.* D **57**, 2635 (1998).

[9] S. Ferrara, M. Porrati and V.L. Telegdi, *Phys. Rev.* D **46**, 3529 (1992).

is to give a kinematical description of the gyromagnetic ratio of elementary particles [10] which is based upon the double content of their spin operator structure.

The general structure of the quantum mechanical angular momentum operator in either relativistic or nonrelativistic approach is

$$\boldsymbol{J} = \boldsymbol{r} \times \frac{\hbar}{i}\nabla + \boldsymbol{S} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{S}, \tag{6.1}$$

where the spin operator is

$$\boldsymbol{S} = \boldsymbol{u} \times \frac{\hbar}{i}\nabla_u + \boldsymbol{W}, \tag{6.2}$$

and ∇_u is the gradient operator with respect to the velocity variables and $\boldsymbol{W}$ is a linear differential operator that operates only on the orientation variables $\boldsymbol{\alpha}$ and therefore commutes with the other. For instance, in the $\boldsymbol{\rho} = \boldsymbol{n}\tan(\alpha/2)$ parametrization $\boldsymbol{W}$ is written as

$$\boldsymbol{W} = \frac{\hbar}{2i}\left[\nabla_\rho + \boldsymbol{\rho} \times \nabla_\rho + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \nabla_\rho)\right]. \tag{6.3}$$

The first part in (6.2), related to the zitterbewegung spin, has integer eigenvalues because it has the form of an orbital angular momentum in terms of the $\boldsymbol{u}$ variables. Half-integer eigenvalues come only from the operator (6.3). This operator $\boldsymbol{W}$ takes into account the change of orientation, *i.e.*, the rotation of the particle.

We have seen in either relativistic or non-relativistic examples that if the only spin content of the particle $\boldsymbol{S}$ is related to the zitterbewegung part $\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U}$, then the relationship between the magnetic moment and zitterbewegung spin is given by

$$\boldsymbol{\mu} = \frac{e}{2}\,\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{Z}, \tag{6.4}$$

i.e., with a normal up to a sign gyromagnetic ratio $g = 1$. If the electron has a gyromagnetic ratio $g = 2$, this implies necessarily that another part of the spin is coming from the angular velocity of the body, but producing no contribution to the magnetic moment.

Therefore for the electron, both parts $\boldsymbol{W}$ and $\boldsymbol{Z}$ contribute to the total spin. But the $\boldsymbol{W}$ part is related to the angular variables that describe orientation and does not contribute to the separation $\boldsymbol{k}$ between the center of charge and the center of mass. It turns out that the magnetic moment of a general particle is still related to the motion of the charge by the expression (6.4), *i.e.*, in terms of the $\boldsymbol{Z}$ part but not to the total spin $\boldsymbol{S}$. It is precisely when we try to express the magnetic moment in terms of the total spin that the concept of gyromagnetic ratio arises.

Now, let us assume that both $\boldsymbol{Z}$ and $\boldsymbol{W}$ terms contribute to the total spin $\boldsymbol{S}$ with their lowest admissible values.

For Dirac's particles, the classical zitterbewegung is a circular motion at the speed of light of radius $R = S/mc$ and angular frequency $\omega = mc^2/S$, in a plane orthogonal to the total spin. The total spin $\boldsymbol{S}$ and the $\boldsymbol{Z}$ part, are both orthogonal to this plane and can be either parallel or antiparallel. Let us define the gyromagnetic ratio by $Z = gS$. For the lowest admissible values of the quantized spins $z = 1$ and $w = 1/2$ in the opposite direction this gives rise to a total $s = 1/2$ perpendicular to the zitterbewegung plane and then $g = 2$.

For $s = 1$ particles the lowest possible values compatible with the above relative orientations are $z = 2$ and $w = 1$ in the opposite direction, thus obtaining again $g = 2$. The possibility $z = 1$ and $w = 0$ is forbidden in the relativistic case because necessarily $w \neq 0$ to describe vector bosons with a multicomponent wave-function.

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[10] M. Rivas, J.M.Aguirregabiria and A. Hernández, *Phys. Lett.* **A 257**, 21 (1999).

6.2   Instantaneous electric dipole

The internal motion of the charge of the electron in the center of mass frame is a circle at the speed of light. The position of the charge in this frame is related to the total spin by eq. (2.126), *i.e.*,

$$k = \frac{1}{mc^2} S \times u, \tag{6.5}$$

where S is the total constant spin and $u = dk/dt$, with $u = c$ is the velocity of the charge. In addition to this motion there is a rotation of a local frame linked to the particle that gives rise to some angular velocity, but this rotation has no effect on the electric dipole structure. (See Fig. 6.1 where the angular velocity and the local frame are not depicted).
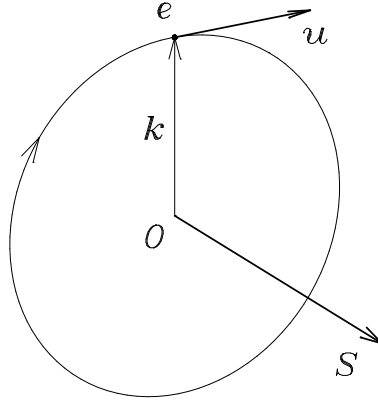
Figure 6.1: **Electron charge motion in the C.M. frame.**

Now, from the point of view of the center of mass observer, the particle behaves as though it has a magnetic moment related to the particle current by the usual classical expression

$$\mu = \frac{1}{2} \int k \times j \, d^3r = \frac{e}{2} \, k \times \frac{dk}{dt},$$

where e is the charge and $j(r - k) = e \, dk/dt \, \delta^3(r - k)$ is the particle current density. The orbital term $k \times dk/dt$ is related to the zitterbewegung part of spin that quantizes with integer values and which for spin 1/2 and spin 1 charged particles is twice the total spin S, giving rise to a pure kinematical interpretation of the gyromagnetic ratio $g = 2$ for this model as seen in the previous section.

But also in the center of mass frame the particle has an oscillating instantaneous electric dipole moment $d = ek$, that is thus related to the total spin by

$$d = \frac{e}{mc^2} S \times u. \tag{6.6}$$

This instantaneous electric dipole, which fulfills the usual definition of the momentum of the point charge e with respect to the origin of the reference frame, is translation invariant

because it is expressed in terms of a relative position vector $\boldsymbol{k}$. It can never be interpreted as some kind of fluctuation of a spherical symmetry of a charge distribution. Even in this kind of model, it is not necessary to talk about charge distributions, because all particle attributes are defined at single points.

In his original 1928 article, [11] Dirac obtains that the Hamiltonian for the electron has, in addition to the Hamiltonian of a free point particle of mass m, two new terms that in the presence of an external electromagnetic field are

$$\frac{e\hbar}{2m}\boldsymbol{\Sigma}\cdot\boldsymbol{B} + \frac{ie\hbar}{2mc}\boldsymbol{\alpha}\cdot\boldsymbol{E} = -\boldsymbol{\mu}\cdot\boldsymbol{B} - \boldsymbol{d}\cdot\boldsymbol{E}, \qquad (6.7)$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\alpha} = \gamma_0\boldsymbol{\gamma},$$

i.e., $\boldsymbol{\Sigma}$ is expressed in terms of $\boldsymbol{\sigma}$ Pauli-matrices and $\boldsymbol{\alpha}$ is Dirac's velocity operator when written in terms of Dirac's gamma matrices.

We shall show that the quantum counterpart of expression (6.6) is in fact the electric dipole term of Dirac's Hamiltonian (6.7). The remaining part of this section is to consider the representation of the 'cross' product in (6.6) in terms of the matrix (or geometric) product of the elements of Dirac's algebra that represent the quantum version of the above observables, so that a short explanation to properly interpret these observables as elements of a Clifford algebra is given in what follows.

Both, velocity operator $\boldsymbol{u} = c\boldsymbol{\alpha}$ and spin operator $\boldsymbol{S}$ are bivectors in Dirac's algebra, considered as elements of the Geometric or Clifford algebra of space-time in the sense of Hestenes. [12]

In fact, Dirac's alpha matrices are written as a product of two gamma matrices $\alpha_i = \gamma_0\gamma_i$ and also the spin components $S_j = (i\hbar/2)\gamma_k\gamma_l$, j, k, l cyclic $1, 2, 3$, and where the four gamma matrices, γ_μ, $\mu = 0, 1, 2, 3$ are interpreted as the four basic vectors of Minkowski's space-time that generate Dirac's Clifford algebra. They satisfy $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}$, i.e., $\gamma_0^2 = 1$ and $\gamma_i^2 = -1$, where the dot means the inner product in Dirac's Clifford algebra. We thus see that velocity and spin belong to the even subalgebra of Dirac's algebra and therefore they also belong to Pauli algebra or geometric algebra of three-dimensional space. Under spatial inversions $\gamma_0 \to \gamma_0$ and $\gamma_i \to -\gamma_i$, the velocity operator changes its sign and it is thus a spatial vector, while the spin is invariant under this transformation as it corresponds to a spatial bivector or pseudovector.

The relationship between the cross product and the outer and inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$ in Pauli algebra is,

$$\boldsymbol{a} \times \boldsymbol{b} = -i\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{b} \cdot (i\boldsymbol{a}), \qquad (6.8)$$

where $\wedge$ represents the symbol for the outer product in geometric algebra, the imaginary unit i represents the unit three-vector or pseudoscalar and $i\boldsymbol{a}$ is the dual bivector of vector $\boldsymbol{a}$.

The inner product of a vector $\boldsymbol{b}$ and a bivector A is expressed in terms of the geometric product in the form

$$\boldsymbol{b} \cdot A = \frac{1}{2}(\boldsymbol{b}A - A\boldsymbol{b}) \qquad (6.9)$$

where in Dirac's or Pauli algebra the geometric product $\boldsymbol{b}A$ is just the ordinary multiplication of matrices.

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[11] P.A.M. Dirac, *Proc. Roy. Soc. London*, **A117**, 610 (1928).

[12] D. Hestenes, *Space-Time algebra*, Gordon and Breach, NY (1966); D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, D. Reidel Pub. Co. Dordrecht, (1984).
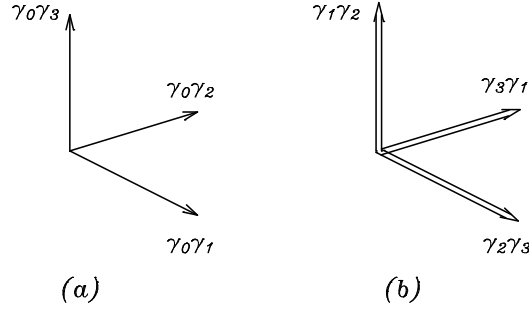
Figure 6.2: **A basis for vectors** (a) **and bivectors (pseudovectors)** (b) **of Pauli algebra.**

If we choose a basis of vectors and pseudovectors as in Fig. 6.2, where the double-lined objects of part (b) represent the dual vectors of the corresponding spatial bivectors, and express in these bases the observables of Fig. 6.1, then the spatial velocity vector $\boldsymbol{u} = c\gamma_0\gamma_2$ and the pseudovector $\boldsymbol{S} = (\hbar/2)\gamma_2\gamma_3$ and therefore, using (6.8) and (6.9) we get

$$\boldsymbol{S} \times \boldsymbol{u} = \boldsymbol{u} \cdot (i\boldsymbol{S}) = \frac{ic\hbar}{2}\left(\frac{1}{2}\left(\gamma_0\gamma_2\gamma_2\gamma_3 - \gamma_2\gamma_3\gamma_0\gamma_2\right)\right) = \frac{-ic\hbar}{2}\gamma_0\gamma_3.$$

Now vector $\boldsymbol{k} = R\gamma_0\gamma_3$ with $R = \hbar/2mc$, and substituting in (6.6) we get the desired result.

6.3 Classical Tunneling

As a consequence of the zitterbewegung and therefore of the separation between the center of mass and center of charge, we shall see that spinning particles can have a non-vanishing crossing of potential barriers.

Let us consider a spinning particle with spin of (anti)orbital type, as described in Section 2.2, under the influence of a potential barrier. The Langrangian of this system is given by:

$$L = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} - \frac{m}{2\omega^2}\frac{\dot{\boldsymbol{u}}^2}{\dot{t}} - eV(\boldsymbol{r})\dot{t}. \tag{6.10}$$

Sharp walls correspond classically to infinite forces so that we shall consider potentials that give rise to finite forces like those of the shape depicted in Fig. 6.3, where V_0 represents the top of the potential.

Then the external force $F(x)$, is constant and directed leftwards in the region $x \in (-a, 0)$ and rightwards for $x \in (0, b)$, vanishing outside these regions.

Potentials of this kind can be found for instance in the simple experiment depicted in Figure 6.4 in which an electron beam, accelerated with some acceleration potential V_a, is sent into the uniform field region of potential V_0 contained between the grids or plates A, C and B.

In Figure 6.4 from a strict classical viewpoint a spinless electron stops at the dotted line and is rejected backwards. But a classical spinning electron can cross the barrier provided its kinetic energy is above some minimum value, although below the top of the potential. This minimum value depends on the separation between plates.
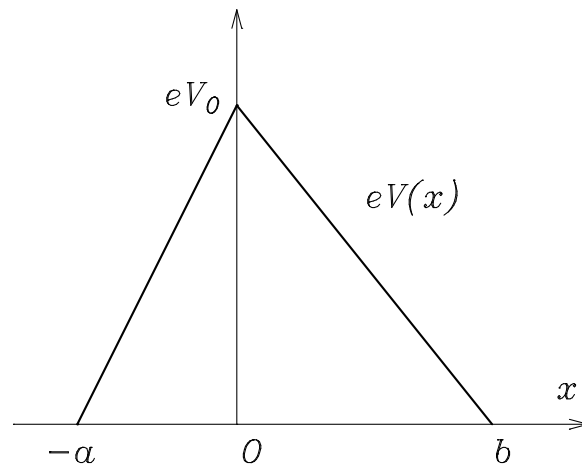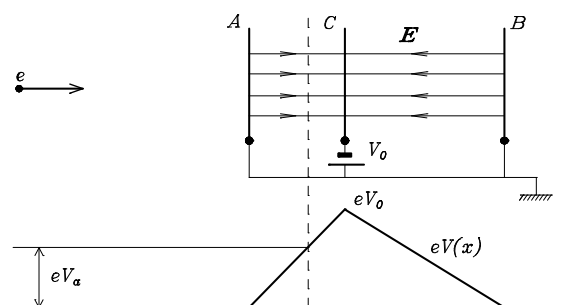
Figure 6.3: **Triangular potential barrier.**

Figure 6.4: **Electron beam into a potential barrier. A classical spinless electron never crosses the dotted line. A spinning particle of the same energy might cross the barrier.**

Let us assume for simplicity that the spin is pointing up or down in the z direction such that the point charge motion takes place in the XOY plane. Let q_x, q_y and $q_z = 0$, be the coordinates of the center of mass and x, y and $z = 0$, the position of the charge.

The dynamical equations are

$$\frac{d^2 q_x}{dt^2} = \frac{1}{m} F(x), \quad \frac{d^2 q_y}{dt^2} = 0, \tag{6.11}$$

$$\frac{d^2 x}{dt^2} + \omega^2 (x - q_x) = 0, \quad \frac{d^2 y}{dt^2} + \omega^2 (y - q_y) = 0, \tag{6.12}$$

where

$$F(x) = \begin{cases} -eV_0/a, & \text{for } x \in (-a, 0), \\ eV_0/b, & \text{for } x \in (0, b), \\ 0, & \text{otherwise.} \end{cases}$$

Equations (6.11) are nonlinear and we have not been able to obtain an analytical solution in closed form. We shall try to find a numerical solution. To make the corresponding numerical analysis we shall define different dimensionless variables. Let R be the average separation between the center of charge and center of mass. In the case of circular internal motion, it is just the radius R_0 of the zitterbewegung. Then we define the new dimensionless position variables:

$$\hat{q}_x = q_x/R, \quad \hat{q}_y = q_y/R, \quad \hat{x} = x/R, \quad \hat{y} = y/R, \quad \hat{a} = a/R, \quad \hat{b} = b/R.$$

The new dimensionless time variable $\alpha = \omega t$ is just the phase of the internal motion, such that the dynamical equations become

$$\frac{d^2 \hat{q}_x}{d\alpha^2} = A(\hat{x}), \quad \frac{d^2 \hat{q}_y}{d\alpha^2} = 0,$$

$$\frac{d^2 \hat{x}}{d\alpha^2} + \hat{x} - \hat{q}_x = 0, \quad \frac{d^2 \hat{y}}{d\alpha^2} + \hat{y} - \hat{q}_y = 0,$$

where $A(\hat{x})$ is given by

$$A(\hat{x}) = \begin{cases} -eV_0/\hat{a} m \omega^2 R^2, & \text{for } \hat{x} \in (-\hat{a}, 0), \\ eV_0/\hat{b} m \omega^2 R^2, & \text{for } \hat{x} \in (0, \hat{b}), \\ 0, & \text{otherwise.} \end{cases}$$

In the case of the relativistic electron, the internal velocity of the charge is $\omega R = c$, so that the parameter $e/mc^2 = 1.9569 \times 10^{-6} \text{V}^{-1}$, and for potentials of order of 1 volt we can take the dimensionless parameter $eV_0/m\omega^2 R^2 = 1.9569 \times 10^{-6}$.

If we choose as initial conditions for the center of mass motion

$$\hat{q}_y(0) = 0, \quad d\hat{q}_y(0)/d\alpha = 0,$$

then the center of mass is moving along the OX axis. The above system reduces to the analysis of the one-dimensional motion where the only variables are $\hat{q}_x$ and $\hat{x}$. Let us call from now on these variables q and x respectively and remove all hats from the dimensionless variables. Then the dynamical equations to be solved numerically are just

$$\frac{d^2 q}{d\alpha^2} = A(x), \quad \frac{d^2 x}{d\alpha^2} + x - q = 0, \tag{6.13}$$

where $A(x)$ is given by

$$A(x) = \begin{cases} -1.9569 \times 10^{-6}\, a^{-1}V_0, & \text{for } x \in (-a, 0), \\ 1.9569 \times 10^{-6}\, b^{-1}V_0, & \text{for } x \in (0, b), \\ 0, & \text{otherwise.} \end{cases} \qquad (6.14)$$

Numerical integration has been performed by means of the computer package *Dynamics Solver.* [13] The quality of the numerical results is tested by using the different integration schemes this program allows, ranging from the very stable embedded Runge-Kutta code of eight order due to Dormand and Prince to very fast extrapolation routines. All codes have adaptive step size control and we check that smaller tolerances do not change the results.
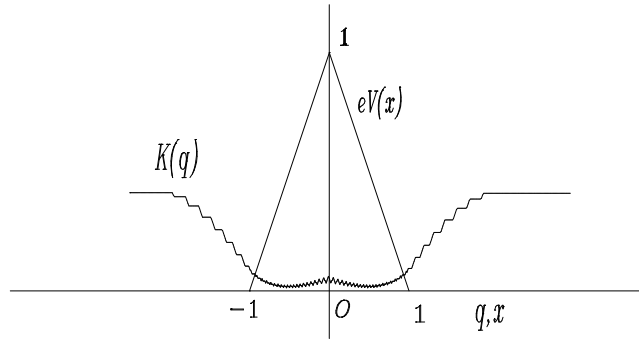

Figure 6.5: **Kinetic Energy during the crossing for the values** $a = b = 1$.

With $a = b = 1$, and in energy units such that the top of the barrier is 1, if we take an initial kinetic energy K below this threshold, $K = m\dot{q}(0)^2/2eV_0 = 0.41$ we obtain for the center of mass motion the graphic depicted in Fig. 6.5, where is shown the variation of the kinetic energy of the particle $K(q)$, with the center of mass position during the crossing of the barrier. There is always crossing with a kinetic energy above this value. In Fig. 6.6, the same graphical evolution with $a = 1$ and $b = 10$ and $K = 0.9055$ for a potential of 10^3 Volts in which the different stages in the evolution are evident. Below the initial values for the kinetic energy of 0.4 and 0.9 respectively, the particle does not cross these potential barriers and it is rejected backwards.

If in both examples the parameter a is ranged from 1 to 0.05, thus making the left slope sharper, there is no appreciable change in the crossing energy, so that with $a = 1$ held fixed we can compute the minimum crossing kinetic energies for different b values, $K_c(b)$.

To compare this model with the quantum tunnel effect, let us quantize the system. In the quantization of generalized Lagrangians developed in the Chapter 3, the wave function for this system is a squared-integrable function $\psi(t, \boldsymbol{r}, \boldsymbol{u})$, of the seven kinematical variables and the generators of the Galilei group have the form:

$$H = i\hbar\frac{\partial}{\partial t},\ \boldsymbol{P} = -i\hbar\nabla,\ \boldsymbol{K} = m\boldsymbol{r} - t\boldsymbol{P} + i\hbar\nabla_u,\ \boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{Z}, \qquad (6.15)$$

where ∇_u is the gradient operator with respect to the $\boldsymbol{u}$ variables. These generators satisfy

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[13] J.M. Aguirregabiria, *Dynamics Solver*, computer program for solving different kinds of dynamical systems, which is available from his author through the web site <http://tp.lc.ehu.es/jma.html> at the server of the Theoretical Physics dept. of The University of the Basque Country, Bilbao (Spain).
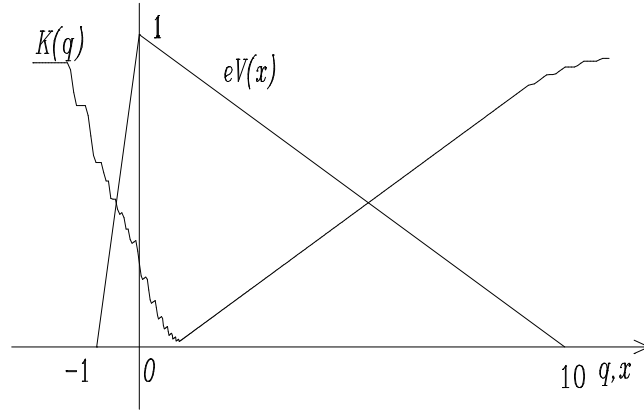
Figure 6.6: **Kinetic Energy during the crossing for the values** $a = 1$, $b = 10$.

the commutation relations of the extended Galilei group, [14] and the spin operator is given by $\boldsymbol{Z} = -i\hbar\boldsymbol{u} \times \nabla_u$.

One Casimir operator of this extended Galilei group is the Galilei invariant internal energy of the system $\mathcal{E}$, which in the presence of an external electromagnetic field and with the minimal coupling prescription is written as,

$$\mathcal{E} = H - eV - \frac{1}{2m}(\boldsymbol{P} - e\boldsymbol{A})^2, \qquad (6.16)$$

where V and $\boldsymbol{A}$ are the external scalar and vector potentials, respectively.

In our system $\boldsymbol{A} = 0$, and V is only a function of the x variable. It turns out that because of the structure of the above operators we can find simultaneous eigenfunctions of the following observables: the Casimir operator (6.16), H, P_y, P_z, Z^2 and Z_z. The particle moves along the OX axis, with the spin pointing in the OZ direction, and we look for solutions which are eigenfunctions of the above operators in the form:

$$\left(H - eV(x) - \frac{1}{2m}\boldsymbol{P}^2\right)\psi = \mathcal{E}\psi, \ H\psi = E\psi, \ P_y\psi = 0, \ P_z\psi = 0, \qquad (6.17)$$

$$Z^2\psi = s(s+1)\hbar^2\psi, \quad Z_z\psi = \pm s\hbar\psi, \qquad (6.18)$$

so that ψ is independent of y and z, and its time dependence is of the form $\exp(-iEt/\hbar)$. Since the spin operators produce derivatives only with respect to the velocity variables, we can look for solutions with the variables separated in the form:

$$\psi(t, x, \boldsymbol{u}) = e^{-iEt/\hbar}\phi(x)\chi(\boldsymbol{u}),$$

and thus

$$\left(\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + E - eV(x) - \mathcal{E}\right)\phi(x) = 0, \qquad (6.19)$$

$$Z^2\chi(\boldsymbol{u}) = s(s+1)\hbar^2\chi(\boldsymbol{u}), \quad Z_z\chi(\boldsymbol{u}) = \pm s\hbar\chi(\boldsymbol{u}), \qquad (6.20)$$

where the spatial part $\phi(x)$, is uncoupled with the spin part $\chi(\boldsymbol{u})$, and $E - eV(x) - \mathcal{E}$ represents the kinetic energy of the system. The spatial part satisfies the one-dimensional Schroedinger

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[14] J.M. Levy-Leblond, *Galilei Group and Galilean Invariance*, in E.M. Loebl, *Group Theory and its applications*, Acad. Press, NY (1971), vol. 2, p. 221.

equation, and the spin part is independent of the interaction, so that the probability of quantum tunneling is contained in the spatial part and does not depend on the particular value of the spin. If the particle is initially on the left-hand side of the barrier, with an initial kinetic energy $E_0 = E - \mathcal{E}$, then we can determine the quantum probability for crossing for $a = 1$ and different values of the potential width b.

The one-dimensional quantum mechanical problem of the spatial part for the same one-dimensional potential depicted in Fig. 6.3 is: [15]

$$\phi(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x \leq -a, \\ C_1\text{Ai}(D(1 - G + \frac{x}{a}) + C_2\text{Bi}(D(1 - G + \frac{x}{a}), & -a \leq x \leq 0, \\ C_3\text{Ai}(L(1 - G - \frac{x}{b})) + C_4\text{Bi}(L(1 - G - \frac{x}{b})), & 0 \leq x \leq b, \\ Te^{ikx}, & x \geq b, \end{cases} \tag{6.21}$$

where x is the same dimensionless position variable as before, and the constants

$$k = \sqrt{\frac{E}{2mc^2}}, \; D = \sqrt[3]{\frac{eV_0a^2}{2mc^2}}, \; L = \sqrt[3]{\frac{eV_0b^2}{2mc^2}}, \; G = \frac{E}{eV_0}. \tag{6.22}$$

Functions $\text{Ai}(x)$ and $\text{Bi}(x)$ are the Airy functions of x. The six integration constants R, T, and $C_i, i = 1, 2, 3, 4$, can be obtained by assuming continuity of the functions and their first order derivatives at the separation points of the different regions. The coefficient $|R|^2$ represents the probability of the particle to be reflected by the potential and $|T|^2$ its probability of crossing.
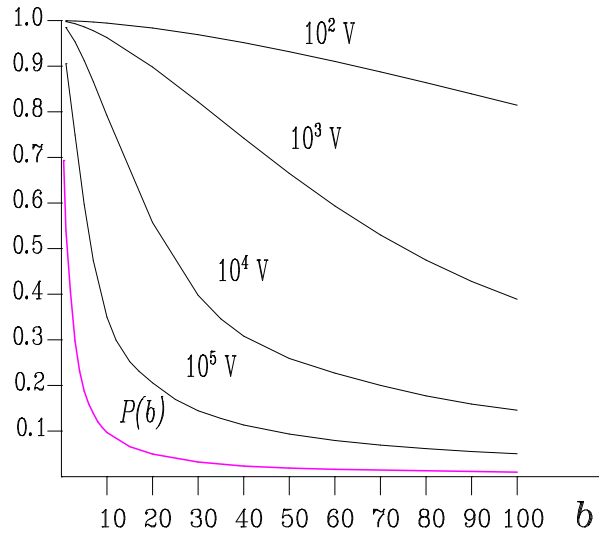

Figure 6.7: **Classical and Quantum Probability of crossing for different potentials.**

Computing the T amplitude for $a = 1$ and different values of the potential width b, and for energies below the top of the barrier eV_0, we show in Fig. 6.7, the average probability for quantum tunneling for four different potentials V_0 of 10^2, 10^3, 10^4 and 10^5 Volts. This average probability has been computed by assuming that on the left of the barrier there is a uniform distribution of particles of energies below eV_0.

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[15] L. Landau and E. Lifchitz, *Mécanique quantique*, Mir Moscow (1988), 3rd. edition.

If we consider for the classical spinning particle the same uniform distribution of particles, then, the function $P(b) = 1 - K_c(b)$, where $K_c(b)$ is the minimum dimensionless kinetic energy for crossing computed before, represents the ratio of the particles that with kinetic energy below the top of the potential cross the barrier because of the spin contribution.

This function $P(b)$, is also depicted in Fig. 6.7. We see that for the different potentials shown in that figure the classical average probability of crossing is smaller than the quantum one, but for stronger potentials this classical probability, coming from the spin contribution, becomes relatively important.

Because the tunnel effect is a function of $\hbar$ and the spin of elementary particles is also of order of $\hbar$ it is very difficult to separate from the outcome of a real experiment involving elementary particles, which part is due to a pure quantum effect and which is the contribution to crossing coming from the spin structure. From (6.19) and (6.20) it is clear that the quantum probability of tunneling is independent of the spin.

To test experimentally this contribution, it will be necessary to perform separate experiments with particles of the same mass and charge but with different values of the spin. Thus, the difference in the outcome will be related to the spin contribution. This can be accomplished for instance, by using ions of the type A^{++} that could be either in a singlet, $(s = 0)$ state or in a triplet $(s = 1)$ state.

But if there exists a contribution to crossing not included in the usual quantum mechanical analysis we have to modify the quantum mechanical equations. To be consistent with the above analysis the Schroedinger-Pauli equation should be modified to include the additional electric dipole term. A term of the form $-eER\cos\omega t$, where E is the external electric field and R the radius of the zitterbewegung, should be considered to solve the corresponding quantum wave function. This term is of the order of the separation R between the center of mass and center of charge, which is responsible for the classical crossing. This additional electric dipole term is already included in Dirac's equation but is suppressed when taking the low velocity limit, as it corresponds to this low energy example. Nevertheless, although this is a low energy process and the time average value of the electric dipole vanishes, there are very high field gradients.

We see that the separation between the center of mass and center of charge that gives rise to the spin structure of this particle model justifies that this system can cross a potential barrier even if its kinetic energy is below the top of the potential.

6.3.1 Spin polarized tunneling

I like to point out the following ideas to discuss whether they can be useful in connection with the interpretation of the giant magnetoresistance of polycrystaline films[16]. This is known in the literature as the **spin polarized tunneling**. [17]

The main feature of the "classical" spin polarized tunneling we have seen in the previous section is not a matter of whether tunneling is classical or not, because this is a nonsense question. Matter at this scale is interpreted under quantum mechanical rules. But if we use a model of a classical spinning particle that, when polarized orthogonal to the direction of motion, produces a crossing that is not predicted by the Schroedinger-Pauli equation, it means that

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[16]2007 Nobel Prize of Physics to Albert Fert and Peter Grünberg for the discovery in 1988 of Giant Magnetoresistence.

[17] V.N. Dobrovolsky, D.I. Sheka and B.V. Chernyachuk, Surface Science **397**, 333 (1998); P. Raychaudhuri, T.K. Nath, A.K. Nigam and R. Pinto, cond-mat/9805258, preprint.

this quantum mechanical equation is lacking some term. The coupling term $-\boldsymbol{\mu} \cdot \boldsymbol{B}$, between the magnetic moment and magnetic field that gives rise to the Pauli equation, is inherited from Dirac's electron theory. But Dirac's equation also predicts another term $-\boldsymbol{d} \cdot \boldsymbol{E}$, of the coupling of an instantaneous electric dipole with the electric field. It is this oscillating electric dipole term that we believe is lacking in quantum mechanical wave equations. In general, the average value of this term in an electric field of smooth variation is zero. But in high intensity fields or in intergranular areas in which the effective potentials are low, but their gradients could be very high, this average value should not be negligible.

The conduction of electrons in synterized materials is completely different than the conduction on normal conductors. The material is not a continuous crystal. It is formed by small grains that are bound together by the action of some external pressure. If we can depict roughly the electric current flow, this is done by the jumping of electrons from grain to grain, through a tunneling process in which there is some estimated effective potential barrier confined in the gap between grains. Therefore these materials show in general a huge resistivity when compared with true conductors.

The form of this potential is unknown. The simplest one is to assume a wall of thickness d, the average separation between grains, and height h. But it can also be estimated as one of the potentials of the former example. What we have shown previously is that for every potential barrier, there is always a minimum energy, below the top of the potential, that electrons above that energy cross with probability 1 when polarized orthogonal to the motion, even within a classical interpretation. But this effect is not predicted by "normal" quantum mechanics because tunneling is spin independent.

Now, let us assume that we are able to estimate some average effective potential barrier in the intergranular zone of this polycristaline material. If the corresponding minimum crossing energy of this barrier for polarized electrons is below the Fermi level, then, when we introduce a magnetic field in the direction of the film and the magnetic domains in the grains become polarized, all electrons above that minimum energy of crossing will flow from grain to grain as in a good conductor, with a classical probability 1. That's all. Here the difficulty is to estimate properly this potential barrier and therefore the corresponding classical crossing energy.

It can be argued that the presence of the magnetic field to polarize electrons produces a change in the energy of particles. Nevertheless, even for a magnetic field of the order of 1 Tesla and in a potential barrier of 1 Volt, the magnetic term $-\boldsymbol{\mu} \cdot \boldsymbol{B}$ contributes with an energy of order of $\pm 5.7 \times 10^{-5}$eV, which does not modify the quantum probability of crossing.

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