A COMPOSITION LAW OF THE LORENTZ GROUP

M. Rivas, J.M. Aguirregabiria and A. Hernández
Department of Theoretical Physics, University of the Basque Country,
Apdo. 644-48080 Bilbao, Spain
E-mail: wtpripem@lg.ehu.es

By using the homomorfism between $SL(2, C)$ and $SO(3, 1)$ we obtain explicit analytic expressions for the composition law of the Lorentz group in terms of the physical parameters, relative velocity and orientation among inertial observers. This proves to be appropriate to analyze the group contractions.

1 Poincaré group

It is the group of linear transformations of Minkowski's space-time that leaves invariant the separation between any two close space-time events $ds^2 = dx^\mu dx_\mu$. We shall consider the contravariant components $x^\mu \equiv (ct, \boldsymbol{r})$, and $x' = gx$ is expressed as $x^{\mu'} = \Lambda^{\mu'}{}_\nu\, x^\nu + a^{\mu'}$, in terms of a constant matrix Λ and a constant translation four-vector $a^\mu \equiv (cb, \boldsymbol{a})$. We take for the covariant components of Minkowski's metric tensor $g_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$. Then $dx^{\mu'} = \Lambda^{\mu'}{}_\nu dx^\nu$ and $ds^2 = g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = g_{\sigma\rho} dx^\sigma dx^\rho$ implies for the matrix Λ

$$g_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = g_{\sigma\rho}. \tag{1}$$

Relations (1) represent 10 conditions among the 16 components of the matrix Λ, so that each matrix depends on 6 essential parameters. These parameters can be chosen in many ways. We shall take 3 of them as the components of the relative velocity $\boldsymbol{v}$ between inertial observers and the other 3 as the orientation $\boldsymbol{\alpha}$ of their Cartesian frames, expressed in a suitable parametrization of the rotation group.

Therefore, every element of the Poincaré group $\mathcal{P}$ will be represented by the 10 parameters $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\alpha})$ and the group action on a space-time point $x \equiv (t, \boldsymbol{r})$ will be interpreted in the form $x' = gx$:

$$x' = \exp(bH) \exp(\boldsymbol{a} \cdot \boldsymbol{P}) \exp(\boldsymbol{\beta} \cdot \boldsymbol{K}) \exp(\boldsymbol{\alpha} \cdot \boldsymbol{J})x,$$

as the action of a rotation followed by a boost or pure Lorentz transformation and finally a space and time translation. It is explicitely given on the space-time variables by

$$t' = \gamma t + \gamma(\boldsymbol{v} \cdot R(\boldsymbol{\alpha})\boldsymbol{r})/c^2 + b \tag{2}$$

$$\boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \gamma \boldsymbol{v} t + \gamma^2(\boldsymbol{v} \cdot R(\boldsymbol{\alpha})\boldsymbol{r})\boldsymbol{v}/(1 + \gamma)c^2 + \boldsymbol{a} \tag{3}$$

where $\boldsymbol{\beta}$ is the normal parameter for the pure Lorentz transformations subgroup, that in terms of the relative velocity among observers $\boldsymbol{v}$ is expressed as $\boldsymbol{\beta}/\beta \tanh\beta = \boldsymbol{v}/c$ as we shall see below and where the dimensions and domains of the parameters $b \in R$, $\boldsymbol{a} \in R^3$, parameter $\boldsymbol{v} \in R^3$, with the upper bound $v < c$, and $\boldsymbol{\alpha}$ depends on the parametrization we use for the rotation group. We shall use for $\boldsymbol{\alpha}$ the $\tan\alpha/2$ parametrization, *i.e.*, $\boldsymbol{\mu} = \boldsymbol{n}\tan\alpha/2$ where $\boldsymbol{n}$ is the unit vector along the rotation axis and α the rotated angle. The factor $\gamma(v) = (1 - v^2/c^2)^{-1/2}$.

The composition law of the group is obtained from $x'' = \Lambda' x' + a' = \Lambda'(\Lambda x + a) + a'$ or $x'' = \Lambda'' x + a''$, that reduces to $\Lambda'' = \Lambda'\Lambda$ and $a'' = \Lambda' a + a'$, *i.e.*, the composition law of the Lorentz transformations that we will find in next section and a Poincaré transformation (Λ', a') of the four vector a^μ. In this parametrization $g'' = g'g$, is[1]

$$b^{''} = \gamma' b + \gamma'(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})/c^2 + b', \quad (4)$$

$$\boldsymbol{a}^{''} = R(\boldsymbol{\mu}')\boldsymbol{a} + \gamma'\boldsymbol{v}' b + \frac{\gamma^{'2}}{(1+\gamma')c^2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})\boldsymbol{v}' + \boldsymbol{a}', \quad (5)$$

$$\boldsymbol{v}^{''} = \frac{R(\boldsymbol{\mu}')\boldsymbol{v} + \gamma'\boldsymbol{v}' + \frac{\gamma^{'2}}{(1+\gamma')c^2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{v})\boldsymbol{v}'}{\gamma'(1 + \boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\,\boldsymbol{v})/c^2)}, \quad (6)$$

$$\boldsymbol{\mu}^{''} = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu} + \boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu} + G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})}, \quad (7)$$

where $\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})$ and $G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu})$ are the real functions:

$$\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu}) = \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}[\boldsymbol{v} \times \boldsymbol{v}' + \boldsymbol{v}(\boldsymbol{v}' \cdot \boldsymbol{\mu}') + \boldsymbol{v}'(\boldsymbol{v} \cdot \boldsymbol{\mu}) + \boldsymbol{v} \times (\boldsymbol{v}' \times \boldsymbol{\mu}')$$

$$+(\boldsymbol{v} \times \boldsymbol{\mu}) \times \boldsymbol{v}' + (\boldsymbol{v} \cdot \boldsymbol{\mu})(\boldsymbol{v}' \times \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu})(\boldsymbol{v}' \cdot \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu}) \times (\boldsymbol{v}' \times \boldsymbol{\mu}')], \quad (8)$$

$$G(\boldsymbol{v}', \boldsymbol{\mu}', \boldsymbol{v}, \boldsymbol{\mu}) = \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}[\boldsymbol{v} \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot (\boldsymbol{v}' \times \boldsymbol{\mu}') + \boldsymbol{v}' \cdot (\boldsymbol{v} \times \boldsymbol{\mu})$$

$$-(\boldsymbol{v} \cdot \boldsymbol{\mu})(\boldsymbol{v}' \cdot \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu}) \cdot (\boldsymbol{v}' \times \boldsymbol{\mu}')]. \quad (9)$$

The unit element of the group is $(0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ and the inverse of $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ is

$$(-\gamma b + \gamma \boldsymbol{v} \cdot \boldsymbol{a}/c^2, -R(-\boldsymbol{\mu})(\boldsymbol{a} - \gamma \boldsymbol{v} b + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot \boldsymbol{a})\boldsymbol{v}), -R(-\boldsymbol{\mu})\boldsymbol{v}, -\boldsymbol{\mu}).$$

2 Lorentz group

The Lorentz group $\mathcal{L}$ is the subgroup of transformations of the form $(0, \mathbf{0}, \boldsymbol{v}, \boldsymbol{\mu})$, and every Lorentz transformation $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$ will be interpreted as the composition of a rotation followed by a boost in the way $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$ as mentioned before where $L(\boldsymbol{v})$ is a boost or pure Lorentz transformation and $R(\boldsymbol{\mu})$ a spatial rotation.

In the 4-dimensional representation of the Lorentz group on Minkowski space-time, a boost is expressed as $L(\boldsymbol{\beta}) = \exp(\boldsymbol{\beta} \cdot \boldsymbol{K})$ in terms of the dimensionless normal parameters β_i and of the 4×4 boost generators K_i given by

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

so that the final expression for $L(\boldsymbol{\beta})$ is

$$\begin{pmatrix} C & (\beta_1/\beta)S & (\beta_2/\beta)S & (\beta_3/\beta)S \\ (\beta_1/\beta)S & 1 + \beta_1\beta_1(C-1)/\beta^2 & \beta_1\beta_2(C-1)/\beta^2 & \beta_1\beta_3(C-1)/\beta^2 \\ (\beta_2/\beta)S & \beta_2\beta_1(C-1)/\beta^2 & 1 + \beta_2\beta_2(C-1)/\beta^2 & \beta_2\beta_3(C-1)/\beta^2 \\ (\beta_3/\beta)S & \beta_3\beta_1(C-1)/\beta^2 & \beta_3\beta_2(C-1)/\beta^2 & \beta_3\beta_3(C-1)/\beta^2 \end{pmatrix}$$

where $S = \sinh\beta$ and $C = \cosh\beta$ and $\beta = (\beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2}$. What is the physical interpretation of the β_i? Let us assume that observers O and O' relate their space-time measurements x and x' by $x^{\mu'} = L(\boldsymbol{\beta})^{\mu'}{}_{\nu} x^{\nu}$. Observer O sends at time t and at a later time $t + dt$ two light signals from a source placed at the origin of its Cartesian frame. These two signals when measured by O' take place at points $\boldsymbol{r}'$ and $\boldsymbol{r}' + d\boldsymbol{r}'$ and at instants t' and $t' + dt'$, respectively. Then they are related by

$$cdt' = L^0{}_0 cdt, \qquad dx^{i'} = L^{i'}{}_0 cdt$$

but the quotient $dx^{i'}/dt'$ is just the velocity of the light source, *i.e.* of the origin of O frame as measured by observer O', and then $v^i = cL^i{}_0/L^0{}_0$, such that the relation between the normal parameters and the relative velocity between observers is

$$\frac{\boldsymbol{v}}{c} = \frac{\boldsymbol{\beta}}{\beta} \tanh\beta$$

and therefore $\tanh\beta = v/c$. The function $\cosh\beta \equiv \gamma(v) = (1 - v^2/c^2)^{-1/2}$ and when expressed the transformation in terms of the relative velocity it takes the form of the symmetric matrix:

$$L(\boldsymbol{v}) = \begin{pmatrix} \gamma & \gamma v_x/c & \gamma v_y/c & \gamma v_z/c \\ \gamma v_x/c & 1+\frac{v_x^2}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_x v_y}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_x v_z}{c^2}\frac{\gamma^2}{\gamma+1} \\ \gamma v_y/c & \frac{v_y v_x}{c^2}\frac{\gamma^2}{\gamma+1} & 1+\frac{v_y^2}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_y v_z}{c^2}\frac{\gamma^2}{\gamma+1} \\ \gamma v_z/c & \frac{v_z v_x}{c^2}\frac{\gamma^2}{\gamma+1} & \frac{v_z v_y}{c^2}\frac{\gamma^2}{\gamma+1} & 1+\frac{v_z^2}{c^2}\frac{\gamma^2}{\gamma+1} \end{pmatrix} \quad (10)$$

The inverse transformation $L^{-1}(\boldsymbol{v}) = L(-\boldsymbol{v})$.

The composition law is obtained by the homomorphism between the Lorentz group $\mathcal{L}$ and the group $SL(2,C)$ of unimodular complex matrices. A rotation of angle α along a rotation axis given by the unit vector $\boldsymbol{n}$ is given by the 2×2 unitary matrix $\exp(\boldsymbol{\alpha}\cdot\boldsymbol{J})$,

$$R(\boldsymbol{\alpha}) = \cos(\alpha/2)\sigma_0 - i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin(\alpha/2), \quad (11)$$

that in terms of the vector $\boldsymbol{\mu} = \tan(\alpha/2)\boldsymbol{n}$ it looks:

$$R(\boldsymbol{\mu}) = \frac{1}{\sqrt{1+\mu^2}}(\sigma_0 - i\boldsymbol{\mu}\cdot\boldsymbol{\sigma}), \quad (12)$$

where σ_0 is the 2×2 unit matrix and σ_i are Pauli spin matrices. A pure Lorentz transformation of normal parameters β_i is represented by the hermitian matrix $\exp(\boldsymbol{\beta}\cdot\boldsymbol{K})$, where the $\boldsymbol{K}$ generators are represented by the 2×2 hermitian matrices $K_i = \sigma_i/2$. This matrix is:

$$L(\boldsymbol{\beta}) = \cosh(\beta/2)\sigma_0 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{\beta}}{\beta}\sinh(\beta/2), \quad (13)$$

such that in terms of the relative velocity parameters it gives

$$L(\boldsymbol{v}) = \sqrt{\frac{1+\gamma}{2}}\left(\sigma_0 + \frac{\gamma}{1+\gamma}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{v}}{c}\right). \quad (14)$$

Then, every element of $SL(2,C)$ is parametrized by the six real numbers $(\boldsymbol{v},\boldsymbol{\mu})$, and interpreted in the way

$$A(\boldsymbol{v},\boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu}), \quad (15)$$

We thus see that every 2×2 matrix $A \in SL(2,C)$ can be written in terms of a complex four-vector a^μ and the four Pauli matrices σ_μ, as $A = a^\mu\sigma_\mu$, and $\det A = 1$ leads to $(a^0)^2 - \boldsymbol{a}^2 = 1$. The general form of (15) is

$$A(\boldsymbol{v},\boldsymbol{\mu}) = \sqrt{\frac{1+\gamma}{2(1+\mu^2)}}\left[\sigma_0\left(1 - i\frac{\boldsymbol{\mu}\cdot\boldsymbol{u}}{1+\gamma}\right) + \boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{u}+\boldsymbol{u}\times\boldsymbol{\mu}}{1+\gamma} - i\boldsymbol{\mu}\right)\right], \quad (16)$$

where vector $\boldsymbol{u} = \gamma(v)\boldsymbol{v}/c$.

Conversely, since $(1/2)\mathrm{Tr}\ (\sigma_\mu \sigma_\nu) = \delta_{\mu\nu}$, we obtain $a^\mu = (1/2)\mathrm{Tr}\ (A\sigma_\mu)$ and if we express (16) in the form $A(\boldsymbol{v}, \boldsymbol{\mu}) = a^\mu \sigma_\mu$ we can determine $\boldsymbol{\mu}$ and $\boldsymbol{u}$ from the components of the complex four-vector a^μ, and it leads to:

$$\boldsymbol{\mu} = -\frac{\mathrm{Im}\ \boldsymbol{a}}{\mathrm{Re}\ a^0} \tag{17}$$

$$\boldsymbol{u} = 2(\mathrm{Re}\ a^0 \mathrm{Re}\ \boldsymbol{a} + \mathrm{Im}\ a^0 \mathrm{Im}\ \boldsymbol{a} + \mathrm{Re}\ \boldsymbol{a} \times \mathrm{Im}\ \boldsymbol{a}), \tag{18}$$

where $\mathrm{Re}\ a^\mu$ and $\mathrm{Im}\ a^\mu$ are the real and imaginary parts of the corresponding components of the four-vector a^μ. When $\mathrm{Re}\ a^0 = 0$ expression (17) is defined and represents a rotation of value π along the axis in the direction of vector $\mathrm{Im}\ \boldsymbol{a}$.

If we represent every Lorentz transformation in terms of a rotation and a boost in the reverse order, $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v})$, then the general expression of A is the same as (16) with a change of sign in the cross product term $\boldsymbol{u} \times \boldsymbol{\mu}$. Therefore, the decomposition is also unique, the rotation $R(\boldsymbol{\mu})$ is the same as before but the new Lorentz boost is given by

$$\boldsymbol{u} = 2(\mathrm{Re}\ a^0 \mathrm{Re}\ \boldsymbol{a} + \mathrm{Im}\ a^0 \mathrm{Im}\ \boldsymbol{a} + \mathrm{Im}\ \boldsymbol{a} \times \mathrm{Re}\ \boldsymbol{a}),$$

where the only difference is that the third term is reversed.

The orthogonal 4×4 rotation matrix takes the form

$$\begin{pmatrix} 1 & 0 \\ 0 & R(\boldsymbol{\mu}) \end{pmatrix}, \tag{19}$$

where $R(\boldsymbol{\mu})$ is a 3×3 orthogonal matrix. When expressed a Lorentz transformation in the form $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})R(\boldsymbol{\mu})$, then by construction the first column of $\Lambda(\boldsymbol{v}, \boldsymbol{\mu})$ is just the first column of (10) where the velocity parameters $\boldsymbol{v}$ are defined and therefore $L(\boldsymbol{v})$ is known and we can multiply on the left such that $L(-\boldsymbol{v})\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})$ is in fact a rotation matrix of the form (19). If expressed in the reverse order $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v})$, then it is the first row of Λ that coincides with the first row of (10) and it turns out that for a fixed arbitrary general Lorentz transformation we have $L(\boldsymbol{v})R(\boldsymbol{\mu}) = R(\boldsymbol{\mu})L(\boldsymbol{v}')$ with the same rotation on both sides as derived in (17) and $L(\boldsymbol{v}') = R(-\boldsymbol{\mu})L(\boldsymbol{v})R(\boldsymbol{\mu}) = L(R(-\boldsymbol{\mu})\boldsymbol{v})$, *i.e*, the velocity $\boldsymbol{v}' = R(-\boldsymbol{\mu})\boldsymbol{v}$. In any case, the decomposition of a Lorentz transformation as a product of a rotation and a boost is a unique one, in terms of the same rotation $R(\boldsymbol{\mu})$ and a boost to be determined, depending on the order we take these two operations.

3 Contractions

A group contraction is a change of parametrization in terms of some arbitrary parameter ϵ, followed by a limit $\epsilon \to 0$. The necessary condition that this limit gives rise to a new group composition law [2] is that the manifold spanned by the unchanged parameters defines a subgroup of the original group. The Poincaré group has two important rotational invariant contractions. The set of elements of the form $(b, \mathbf{0}, \mathbf{0}, \boldsymbol{\mu})$ is a subgroup of $\mathcal{P}$ and therefore if we define the new parameters $b_0 = b$, $\epsilon \boldsymbol{a}_0 = \boldsymbol{a}$, $\epsilon \boldsymbol{v}_0 = \boldsymbol{v}$, $\boldsymbol{\mu}_0 = \boldsymbol{\mu}$, in the limit $\epsilon \to 0$, eqs. (4)-(7) transform into

$$b_0'' = b_0' + b_0, \tag{20}$$
$$\boldsymbol{a}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{a} + \boldsymbol{v}_0' b_0 + \boldsymbol{a}_0', \tag{21}$$
$$\boldsymbol{v}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{v}_0 + \boldsymbol{v}_0', \tag{22}$$
$$\boldsymbol{\mu}_0'' = \frac{\boldsymbol{\mu}'_0 + \boldsymbol{\mu}_0 + \boldsymbol{\mu}'_0 \times \boldsymbol{\mu}_0}{1 - \boldsymbol{\mu}'_0 \cdot \boldsymbol{\mu}_0}. \tag{23}$$

This corresponds to the composition law of the Galilei group that can also be obtained by the limit $c \to \infty$ that leads from the Poincaré group $\mathcal{P}$ to the Galilei group, so that the group action (2)-(3) is transformed into

$$t' = t + b, \tag{24}$$
$$\boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}. \tag{25}$$

Simmilarly, the set of elements of the form $(0, \boldsymbol{a}, \mathbf{0}, \boldsymbol{\mu})$ is another subgroup of $\mathcal{P}$ so that the change of parameters $\epsilon b_0 = b$, $\boldsymbol{a}_0 = \boldsymbol{a}$, $\epsilon \boldsymbol{v}_0 = \boldsymbol{v}$, $\boldsymbol{\mu}_0 = \boldsymbol{\mu}$, and by the substitution $\boldsymbol{v}/c^2 = \boldsymbol{w}$, in the limit $\epsilon \to 0$ leads to

$$b_0'' = b_0' + b_0 + \boldsymbol{w}_0' \cdot R(\boldsymbol{\mu}_0')\boldsymbol{a}_0, \tag{26}$$
$$\boldsymbol{a}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{a} + \boldsymbol{a}_0', \tag{27}$$
$$\boldsymbol{w}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{w}_0 + \boldsymbol{w}_0', \tag{28}$$
$$\boldsymbol{\mu}_0'' = \frac{\boldsymbol{\mu}'_0 + \boldsymbol{\mu}_0 + \boldsymbol{\mu}'_0 \times \boldsymbol{\mu}_0}{1 - \boldsymbol{\mu}'_0 \cdot \boldsymbol{\mu}_0}. \tag{29}$$

This is the composition law of the Carroll group [3] that can also be obtained as the limit of the Poincaré group when $c \to 0$, $\boldsymbol{v} \to 0$ faster than c, but $\boldsymbol{v}/c^2 \to \boldsymbol{w}$, where the real parameters $\boldsymbol{w}$ with dimensions of inverse of velocity are defined. In this case the action of the group on the space-time is transformed into

$$t' = t + \boldsymbol{w} \cdot R(\boldsymbol{\mu})\boldsymbol{r} + b, \tag{30}$$
$$\boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \boldsymbol{a}. \tag{31}$$

A physical difference between these three groups we can find is the following. In the Poincaré group all inertial observers measure different discrepancies $\delta t' \neq \delta t$ and $|\delta \boldsymbol{r}| \neq |\delta \boldsymbol{r}'|$ about the temporal and spatial separation between arbitrary space-time events. In the Galilei limit $\delta t' = \delta t$ while $|\delta \boldsymbol{r}| \neq |\delta \boldsymbol{r}'|$ in general, so that time intervals are absolute measurements while space intervals are relative to each observer. In the Carroll limit we have the converse $\delta t' \neq \delta t$ in general, while $|\delta \boldsymbol{r}| = |\delta \boldsymbol{r}'|$ where space intervals are absolute since the inertial observers have their relative motions frozen in the limit $\boldsymbol{v} \to 0$, but they still produce different time measurements characterized by the parameter $\boldsymbol{w}$, so that we can consider the Carroll group as the other non-relativistic limit of the Poincaré group that describes only the physics of tachyons. In fact, it has been obtained as the $c \to 0$ limit, together $\boldsymbol{v} \to 0$. Therefore, all physical phenomena travelling at velocities below c, including the relative motion of inertial observers, have been reduced to static systems, remaining for description only those phenomena with velocities above c.

Acknowledgments

This work has been partially supported by the Universidad del País Vasco/Euskal Herriko Unibertsitatea under contract UPV/EHU 172.310 EB150/98 and by DGICYT project PB96-0250.

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