Two Spin observables

We can find in the physics literature the use of two different spin observables for describing the angular momentum of matter. One is the Bargmann-Michel-Telegdi Spin and the other is Dirac's spin operator. They satisfy two different dynamical equations and the reason is that an angular momentum is a physical property defined with respect to some specific point. BMT spin is the angular momentum with respect to the center of mass and Dirac's spin is defined with respect to the charge position, which is a different point than the center of mass for a Dirac particle.

Bargmann-Michel-Telegdi Spin

It is based upon the idea that the evolution of the particle is described classically by a point $\boldsymbol{q}$, which represents the position of the center of mass of the system, and associated to it there is a spin four-vector S^μ, which in the rest frame of the particle takes the form $S_0^\mu \equiv (0, \boldsymbol{S})$, with the additional "magnetic condition", $S^\mu v_\mu = 0$. The four velocity $v^\mu \equiv (\gamma(v)c, \gamma(v)\boldsymbol{v})$, with $\boldsymbol{v} = d\boldsymbol{q}/dt$, the center of mass velocity.

The linear momentum of the particle is $\boldsymbol{p} = \gamma(v)m\boldsymbol{v}$, such that in the presence of an external electromagnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$, evaluated at the point $\boldsymbol{q}$, satisfies

$$\frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{1}$$

The spin dynamical equation is the (Thomas-BMT equation)

$$\frac{dS^\mu}{d\tau} = \frac{e}{m}\left(\frac{g}{2}F^{\mu\nu}S_\nu + \left(\frac{g}{2} - 1\right)(S_\alpha F^{\alpha\beta}v_\beta)v^\mu\right) \tag{2}$$

so that if the external electromagnetic field $F^{\mu\nu}$ vanishes, it is a constant of the motion.

Dirac Spin operator

At first, a Dirac particle has no classical analog. Dirac's spinor $\psi(t, \boldsymbol{r})$ is a four-component object defined on the space-time variables $(t, \boldsymbol{r})$, such that the total angular momentum of the Dirac particle with respect to the origin of the observer's frame is

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{S}_D,$$

where $\boldsymbol{p}$ is the linear momentum of the particle and $\boldsymbol{S}_D$ is Dirac's spin operator, written as usual in the Pauli-Dirac or in the Weyl representation as

$$\boldsymbol{S}_D = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

in terms of $\boldsymbol{\sigma}$ Pauli matrices. It thus represents the angular momentum of the particle with respect to the point $\boldsymbol{r}$.

The velocity of the point $\boldsymbol{r}$ is

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt} = \frac{i}{\hbar}[H, \boldsymbol{r}] = c\boldsymbol{\alpha},$$

in terms of Dirac's $\boldsymbol{\alpha}-$matrices. It thus suggests that point $\boldsymbol{r}$ is moving at the speed of light and therefore it does not represent the center of mass position of the particle. In fact, the linear momentum is not written in terms of this velocity but in terms of some average value $\boldsymbol{v} =< \boldsymbol{u} >$, as usual $\boldsymbol{p} = \gamma(v)m\boldsymbol{v}$. It is the average value of point $\boldsymbol{r}$ which can be interpreted as the center of mass of the particle $\boldsymbol{q} =< \boldsymbol{r} >$. In Dirac's theory, this average separation $|\boldsymbol{r} - \boldsymbol{q}| \sim \hbar/2mc$ is half Compton's wavelength.

The dynamical equation satisfied by the linear momentum is

$$\frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}). \tag{3}$$

thus justifying that point $\boldsymbol{r}$ represents the position of the charge of the particle because it is the velocity of point $\boldsymbol{r}$ which appears in the magnetic force term above, and the external fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are precisely evaluated at this point.

The dynamical equation satisfied by Dirac's spin operator is

$$\frac{d\boldsymbol{S}_D}{dt} = \boldsymbol{p} \times \boldsymbol{u}, \tag{4}$$

even in the presence of an external electromagnetic field, so that Dirac's spin operator is not a constant of the motion for the free particle. This dynamical equation has no relationship with Thomas-BMT equation (2). The reason is that Dirac's spin operator is not the angular momentum in the center of mass frame.

Dirac's instantaneous electric dipole moment of the particle is

$$\boldsymbol{d} = \frac{ie\hbar}{2mc}\boldsymbol{\alpha} = e(\boldsymbol{r} - \boldsymbol{q}),$$

so that in Dirac's theory the separation between the center of mass $\boldsymbol{q}$ and the center of charge $\boldsymbol{r}$ is in fact the operator

$$\boldsymbol{r} - \boldsymbol{q} = \frac{i\hbar}{2mc}\boldsymbol{\alpha}$$

so that the angular momentum of the Dirac particle with respect to its center of mass is

$$\boldsymbol{S} = \boldsymbol{S}_D + \frac{i\hbar}{2mc}\boldsymbol{\alpha} \times \boldsymbol{p}.$$

It is this angular momentum operator which has to be used for each one of the three quarks to obtain the angular momentum of the proton with respect to its center of mass. Otherwise, the addition of the three Dirac spin operators of the three quarks can never give rise to the spin of the proton because they are three angular momenta defined with respect to three different points and not with respect to their common center of mass.

All these features concerning the analysis of a Dirac particle can be better understoood by the classical analysis of a spinning elementary particle which satisfies Dirac's equation when quantized. See the publications concerning this classical description of elementary particles in my web page, below.

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