Falling elastic bars and springs

J. M. Aguirregabiria, A. Hernández, and M. Rivas
Theoretical Physics, The University of the Basque Country, P. O. Box 644, 48080 Bilbao, Spain

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We analyze the initial motion of an elastic bar that is suddenly released after being hung from one end. The analytical solutions uncover some unexpected properties, which can be checked with a digital camera or camcorder in an alternative setup in which a spring is substituted for the bar. The model and the experiments are useful for understanding the similarities and differences between the elastic properties of bars and springs. Students can use the simple experiments to improve their understanding of elastic waves.
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I. INTRODUCTION

Undergraduate physics students sometimes have difficulties realizing that the useful but idealized conceptual limit of rigid bodies can be misleading if applied to the analysis of some problems, for instance, the "pole and barn paradox"[1] and the "detonator paradox"[2] in special relativity. In both cases one has to realize that the parts of the object do not stop all at the same time.

To avoid the difficulty of the relativity of simultaneity, let us consider a simple problem in elasticity. A long thin elastic bar is vertically suspended from one end by a string. After equilibrium is attained, the string is suddenly cut. Will all points on the bar have the same acceleration just after the string is cut? Most students will quickly answer that all points will fall with acceleration g, and it takes some time to convince them that although the tension in the upper end disappears when the string is cut, it will take a finite time before the tension and the deformation change occurs at other points. The change will propagate along the bar in the form of a wave, so that initially the lower end does not move at all.

We will show that the theoretical analysis for a metallic bar is elementary, but the deformation would be too small and the change too fast to be seen, except with a sophisticated experimental setup. Instead we used a plastic spring,[3] which when stretched behaves much like the elastic bar, but has elastic properties that change completely when the loops are in contact with each other. Spring deformations are large and change slowly enough to be recorded with a digital camera. In our first try[4] we could easily see that the lower end did not start falling until the deformation change reached it.

We might think that if the center of mass moves with acceleration g while the points at the lower end are still at rest, the upper points must move with greater acceleration. We will see that this motion is not the case with the bar: what happens is simpler but (probably) less intuitive. We can go beyond the qualitative analysis and calculate the evolution from the string being cut to the deformation change reaching the lower end with minimal mathematics by using concepts known to students in introductory physics courses. The analysis of Sec. II shows that after the deformation change has reached a point, it will start moving with a velocity independent of position and time, that is, without acceleration. This result is a consequence of assuming that the force exerted by the string vanishes instantaneously; this assumption is a good approximation in other appropriate cases. To check the theoretical prediction we extracted some consecutive frames from our video[5] and obtained results such as in Fig. 1, where we can see that the upper end moves with a more or less constant velocity, while the lower end remains at rest until the elastic wave reaches this point.

Except for the first few frames in Fig. 1, there was not good agreement with the theoretical analysis because the upper coils quickly became completely compressed and touched one another. The reason for the disagreement can be understood by using the model we will discuss in Sec. III: matter quickly moves faster than the elastic wave and the dynamical problem changes completely (see Ref. 6).

We can obtain better agreement between the theory of elastic waves and our experiments with the spring by modifying the problem by attaching a mass on top of a bar and spring. As described in Sec. IV, some properties of the solution for the bar change qualitatively however small the mass, and we are able to apply the theoretical model of the bar for longer times.

Several loosely related problems with falling chains have been considered recently[7,8] using analytical mechanics, and some experimental results have been obtained by means of high-speed photography.[9] Our mechanical system is simpler and can be used by undergraduates to illustrate elasticity and waves by using only elementary results. Simple illustrative experiments can be performed with no laboratory equipment other than a digital camera.

II. THE FALLING BAR

We label each point P on the bar by the distance x measured from end A when the bar is not strained, as shown in Fig. 2(a). At time t the distance between the suspension point and P is $x+u(t,x)$, where $u(t,x)$ is the deformation field.

For times $t<0$ the bar is at rest hanging from its end A as shown in Fig. 2(b), so that the distance AP is $x+u_0(x)$ in terms of the initial deformation field u_0 which we will now calculate. The tension τ_0 is readily found by writing the equilibrium condition for PB using the fact that the tension vanishes at the free end B: $\tau_0(L)=0$. We obtain

$$\tau_0(x) = \rho g(L-x), \qquad (1)$$

where ρ is the mass density. Hooke's law and the boundary condition $u_0(0)=0$ (which states that A is at the suspension point) allow us to calculate the deformation:

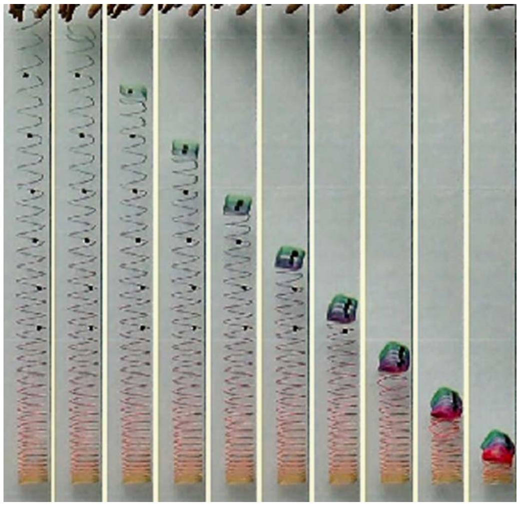

Fig. 1. Consecutive frames from a video sequence (Ref. 4) of an elastic spring hanging from one end and then released. The six black tags show that the coils remain at rest for a duration depending on their distance to the upper end.

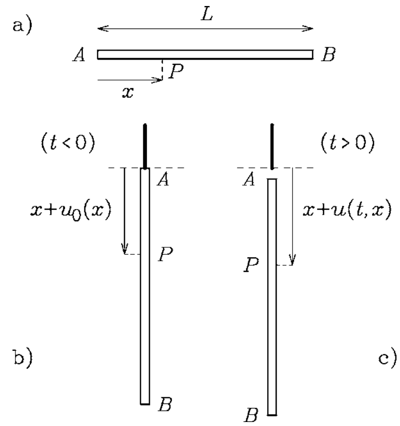

Fig. 2. Elastic bar (a) without strain, (b) hanging, and (c) released.

$$\tau_0 = E\frac{\partial u_0}{\partial x}, \tag{2}$$

which implies that

$$u_0(x) = \frac{g}{2c^2}(2Lx - x^2), \tag{3}$$

where E is Young's modulus and $c=\sqrt{E/\rho}$ is the speed of sound in the bar. We are assuming that the strain $\partial u_0/\partial x$ is sufficiently small to satisfy Hooke's law, which implies that $gL/c^2 \ll 1$.

At $t=0$ the string is cut so that the tension at A disappears instantaneously, and a stress discontinuity starts propagating along the bar with speed c. It is not difficult to use Newton's second law and Hooke's law to find the wave equation satisfied by the deformation field $u(t,x)$:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + g. \tag{4}$$

It will probably be easier for students to understand the calculation in a reference frame falling with the center of mass, where the deformation field is

$$u^* = u - \tfrac{1}{2}gt^2. \tag{5}$$

In such a frame matter appears weightless and the remaining forces are of elastic origin, so that the longitudinal equation is the homogeneous wave equation that is discussed in elementary physics courses,[10]

$$\frac{\partial^2 u^*}{\partial t^2} = c^2\frac{\partial^2 u^*}{\partial x^2}. \tag{6}$$

The solution of Eq. (6) is taught to students in d'Alembert's form, that is, as a superposition of two waves propagating in opposite directions:[10]

$$u^*(t,x) = f(x-ct) + h(x+ct). \tag{7}$$

[Using Eq. (5) after solving Eq. (6) is the standard mathematical technique for solving an inhomogeneous linear equation such as Eq. (4), but the reasoning here is more physical and can be presented to students before they study partial differential equations.]

At time t the perturbation, which propagates with velocity c, has not yet reached points $x>ct$, so that for these points u^* is still given by Eq. (5) after replacing u by the initial deformation field u_0 of Eq. (2). Because all points with $x<ct$ are already moving, the equation of motion there will be Eq. (6), and we have to use one of the solutions given by Eq. (7). Hence for times $t<L/c$, that is, before the wave reaches the lower end, we seek a piecewise solution in the form

$$u^*(t,x) = \begin{cases} u_0(x) - \frac{1}{2}gt^2, & x > ct, \\ f(x-ct) + h(x+ct), & x < ct < L. \end{cases} \tag{8}$$

We now have to calculate the functions f and h, which is easily done by using the following two physical conditions:

(1) The bar does not break, so that u is continuous at the wavefront $x=ct$ where the two pieces of Eq. (8) must match. If we let $x=ct$ in Eq. (8) and remember Eq. (2), this condition may be written in the form:

$$-\frac{g}{4c^2}2x(2x-2L) = f(0) + h(2x), \tag{9}$$

which implies that

$$h(x) = \frac{g}{4c^2}x(2L-x) - f(0). \tag{10}$$

As a consequence of Eq. (10), any value for $f(0)$ will be canceled in the sum $f(x-ct)+h(x+ct)$, because only the difference $f(x-ct)-f(0)$ appears. Thus, there is no restriction on taking $f(0)=0$.

(2) In the free-falling reference frame the center of mass is at rest by definition, so that its velocity, given by the mean value

$$\frac{1}{L}\int_0^L \frac{\partial u^*(t,x)}{\partial t}\,dx = \frac{1}{L}\int_{ct}^L (-gt)dx + \frac{c}{L}\int_0^{ct}[h'(x+ct) - f'(x-ct)]\,dx \tag{11}$$

$$= \frac{c}{L}f(-ct) - \frac{g}{4L}t(2L-ct), \tag{12}$$

must vanish, which gives the functional form for f by replacing $-ct$ in Eq. (12) by the generic variable x:

$$f(x) = -\frac{g}{4c^2}x(2L+x). \tag{13}$$

We substitute Eqs. (9) and (13) into the solution (8), use the inverse of the transformation (5), and obtain our main result in the laboratory frame:

$$u(t,x) = \frac{g}{2c^2}\begin{cases} 2Lx - x^2, & x > ct, \\ 2Lct - x^2, & x < ct < L. \end{cases} \tag{14}$$

The stress is then calculated by applying Hooke's law using Eq. (14):

$$\tau(t,x) = E\frac{\partial u}{\partial x} = \rho g\begin{cases} L - x, & x > ct, \\ -x, & x < ct < L. \end{cases} \tag{15}$$

As expected, at the two free ends A and B we have $\tau(t,0) = \tau(t,L) = 0$ for all $0 < t < L/c$. The tension is discontinuous and becomes a compression at the wavefront $x=ct$, so that its value in Eq. (15) at the lower end $x=L$ goes to $-\rho g L$ as $t \to L/c$, which shows that a reflected wave must appear to make sure that $\tau(t,L)$ always vanishes. We are not interested here in this reflected wave, because it cannot be seen in our experiment.

The surprise arises when we calculate the velocity of each point by again using Eq. (14):

$$\frac{\partial}{\partial t}[x + u(t,x)] = \frac{\partial u}{\partial t} = \frac{gL}{c}\begin{cases} 0, & x > ct, \\ 1, & x < ct < L. \end{cases} \tag{16}$$

All points outside the wavefront $x=ct$ move without acceleration, but as time increases more and more points start moving with velocity gL/c, which, as stressed after Eq. (2), is smaller than the sound velocity c, so that the center of mass moves with increasing velocity gt. That is, at the wavefront $x=ct$ the velocity is discontinuous and thus the acceleration diverges. In terms of Dirac's delta function,[11] we can write the acceleration as

$$\frac{\partial^2 u}{\partial t^2} = gL\delta(x-ct), \quad t < L/c. \tag{17}$$

We now see that the answer to the question proposed at the beginning of Sec. I is that, in the limit in which the string is cut instantaneously, all points initially move without acceleration, except for those points lying at the wavefront, which have infinite acceleration. This problem illustrates a rather unusual way for a system of interacting particles to gain more and more linear momentum under an external force.

III. A SOFT SPRING

The analysis of the elastic bar can be done with elementary physics using the first example of a wave equation.[12] In our experiment we needed bigger deformations and slower propagation velocities, so we considered using a spring. The study of the latter is more difficult because the deformations are no longer small, and the elastic properties when stretched and under compression are qualitatively different. Also, when hanging from an end it stretches and develops non-negligible torsion, which changes when moving. However, we expect that at least some qualitative results would be the same as in the elastic bar.

Instead of an elastic bar we released a colorful plastic spring[3] with black tags stuck on every third loop. We used a digital camera to shoot a short video sequence at 30 frames/s. The resulting animation is displayed (at two different speeds) in Ref. 4. We can clearly see there that the tags and the lower end remain at rest for a while. To further explore the animation, we extracted consecutive frames,[5] which are shown in Fig. 1.

At first sight we might conclude that the tagged points start moving with the same constant velocity only when the elastic wave reaches them, but a simple calculation shows disagreement between theory and experiment: the upper coils quickly become completely compressed. The solution calculated for the bar is approximately applicable to the spring and was calculated in this context by a somewhat more advanced mathematical method in Ref. 6 (see also Ref. 13); we have only to replace c by $L\sqrt{k/m}$, where m and k are respectively the mass and the elastic constant of the spring. But, unlike in a metallic bar, in a soft spring the velocity gL/c of the coils above the wavefront $x=ct$ quickly becomes larger than the velocity of the wavefront:

$$\frac{d}{dt}[ct + u(t,ct)] = \frac{gL}{c} + c - gt. \tag{18}$$

From $t=c/g$ onward (a bit earlier due to the finite thickness of the coils) an increasing number of upper coils touch one another and drop onto the coils below before there is time for the tension to change there. We have a kind of matter wave that moves faster than the elastic wave created when the spring was released. This dynamical problem is completely different and was analyzed by Calkin.[6] We can check his solution in our case, where both the unstretched length and the minimum compressed length of the spring are $L \approx 6.5$ cm, the quotient of the spring constant and mass is $k/m \approx 4\ \mathrm{s}^{-2}$, and there is no contact between the coils when the spring is hanging at rest. Good numerical agreement is shown in Fig. 3. This problem is interesting, but because we are more interested in elasticity, we turned to the study of the problem discussed in Sec. IV.

From another point of view we can see from the velocity in Eq. (18) that the solution (14) breaks down at $t=c/g$ provided that the latter value is less than L/c, which would never happen for an actual metallic bar. At that moment the velocity of a point at $x+u(t,x)$ is

$$\frac{\partial}{\partial t}[x + u(t,x)] = 1 - \frac{gx}{c^2}, \tag{19}$$

and becomes negative when reached by the wavefront at $x = ct$, which is impossible, because it would mean an inversion of the spatial order of coils.

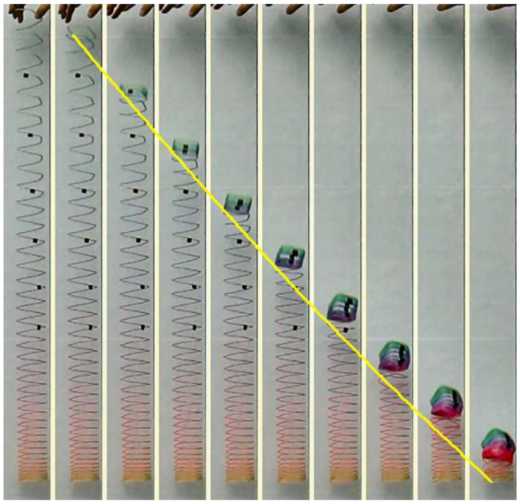

Fig. 3. Same frames as in Fig. 1 with the solution in Ref. 6 superimposed (continuous line).

Fig. 4. Consecutive frames with a thin slab fixed at the upper end and a block on top of it. The slab and block immediately separate because they move with different accelerations.

IV. A FALLING BAR WITH AN ATTACHED POINTLIKE MASS

Let us consider again the bar of Fig. 2, but let us assume that a pointlike mass M is attached at the upper end A. The analysis of Sec. II changes only beginning with Eq. (12). We now have to take into account the contribution of M to the zero velocity of the center of mass in the freely falling reference frame:

$$\frac{1}{L}\int_0^L \frac{\partial u^*(t,x)}{\partial t}\,dx + \mu\frac{\partial u^*}{\partial t}(t,0) = 0, \tag{20}$$

where we have written $\mu = M/m$, m being the bar mass. If we substitute Eqs. (8) and (9) into Eq. (20), we obtain the condition

$$\mu L f'(x) - f(x) = \frac{g}{4c^2}[2\mu L^2 + 2(1+\mu)Lx + x^2]. \tag{21}$$

Equation (21) is a first-order linear equation with constant coefficients, which can be solved, along with the initial condition $f(0)=0$ (for example, by integrating with respect to x after multiplying it by $e^{-x/\mu L}$ or by using computer algebra), to give

$$f(x) = -\frac{g}{4c^2}[x(2L+x) - 2q_\mu(x)], \tag{22}$$

$$q_\mu(x) \equiv 2\mu L[(1+\mu)L(e^{x/\mu L} - 1) - x]. \tag{23}$$

If we use Eqs. (5), (9), (22), and (23), we obtain the deformation field in the laboratory frame:

$$u(t,x) = \frac{g}{2c^2}\begin{cases} 2Lx - x^2, & x > ct, \\ 2Lct - x^2 + q_\mu(x-ct), & x < ct < L. \end{cases} \tag{24}$$

Equation (24) reduces to Eq. (14) in the limit $\mu \to 0$, because $\lim_{\mu\to 0} q_\mu(x-ct) = 0$ for $x < ct$. At time t the velocity of the point labeled x is

$$\frac{\partial u}{\partial t} = \frac{g(1+\mu)L}{c}(1 - e^{(x-ct)/\mu L})\begin{cases} 0, & x > ct, \\ 1, & x < ct < L. \end{cases} \tag{25}$$

This velocity is now, unlike in Eq. (16), continuous through the wavefront $x = ct$ for any mass ratio $\mu > 0$.

The fact that the behavior is qualitatively different might seem counterintuitive, but there is a clear physical reason for it. By applying Hooke's law to Eq. (24) we obtain the stress

$$\tau(t,x) = E\frac{\partial u}{\partial x} = \rho g\begin{cases} L - x, & x > ct \\ -\mu L - x + (1+\mu)Le^{(x-ct)/\mu L}, & x < ct < L, \end{cases} \tag{26}$$

which does not become instantaneously zero at $x=0$ when the spring is released at $t=0$, but retains its previous value, $\tau(0,0) = \rho g L$, because of the attached mass, however small. For the same reason the points above the wavefront are now accelerated:

$$\frac{\partial^2 u}{\partial t^2} = g\frac{1+\mu}{\mu}e^{(x-ct)/\mu L}, \quad x < ct < L. \tag{27}$$

According to Eq. (25), the velocity of points at the wavefront is zero. However, the solution breaks down when

$$t = \frac{\mu L}{c}\log\frac{(1+\mu)L}{\mu L + x - c^2/g} + \frac{x}{c}, \quad x < ct < L, \tag{28}$$

provided t is real and less than L/c, as we can see by repeating the calculation in Eq. (19). We can check that for our spring this value increases with μ, so that we would expect the analytical solution (24) to be valid for longer intervals with heavier masses.

Notice that when the elastic wave reaches a point its acceleration is $g(1+\mu)/\mu$, which is greater than g; the same behavior initially occurs at the upper end A.

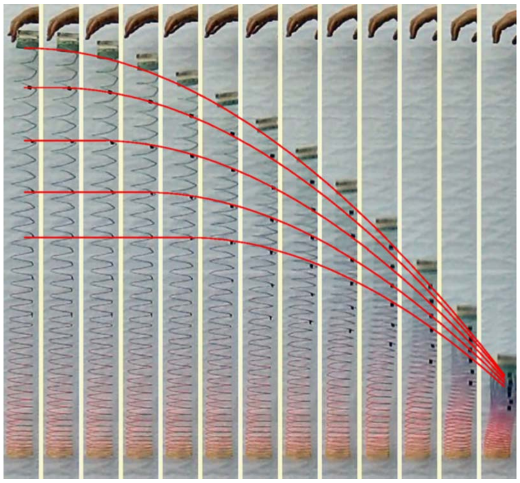

Fig. 5. Consecutive frames with a wooden block ($\mu \approx 2.35$) fixed at the top end. The superimposed continuous curves are the trajectories of the upper end and four coils with black tags, as given by Eq. (24).

V. A SOFT SPRING WITH AN ATTACHED MASS

The results of Sec. IV can be clearly seen in our second experiment[4] where a thin wooden slab is fixed at the top end A. A thicker block is then put on top of it. When the spring is released, the acceleration of A is greater than that of the block, which immediately separates from the slab and follows the familiar free fall trajectory, as displayed in Fig. 4. We can see that the small mass ($\mu \approx 0.22$) is enough to ensure that the upper coils are stretched for a while. Thus the early breakdown predicted by our continuous theoretical model is avoided by making sure the coils do not touch one another for a longer time.

This time interval is even greater in our third experiment,[4] where the thin slab is replaced by a thicker block ($\mu \approx 2.35$). We can see in Fig. 5 that each black tag starts moving only when the stretching begins to change, that is, when the elastic wave reaches it. We have also plotted the trajectories of the top most coil and the first four tags as calculated using Eq. (24). We obtain good agreement, despite the simple experiment and the differences between a bar and a spring.

VI. CONCLUSIONS

We have analyzed by using elementary physical concepts several problems in elasticity with results that would be difficult to anticipate. To help students understand elastic waves we checked the most striking aspects of the analytical results in an experiment in which the elastic bar is substituted by a spring, which allows much larger deformations and slower wave propagation. The similarities and differences between the bar and spring can be used in an illustrative discussion in introductory physics courses.

Because we need only a digital camera (or a video camera) and some freeware to process the video sequences, the experiments can be easily performed in the classroom and repeated at home by interested students.

We can take advantage of the widespread availability of digital cameras to visually check the solution of other problems in mechanics. A well-known problem is the motion of the free end of an articulated arm released from an angle less than $\arcsin 1/\sqrt{3} \approx 35°$. In this case the acceleration of the end is always greater than g.[14] The use of digital cameras as measuring devices in other kinds of problems has been recently discussed.[15]

ACKNOWLEDGMENT

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[2] Reference 1, p. 185.

[3] "Rainbow magic spring," ⟨www.physlink.com⟩.

[4] See EPAPS Document No. E-AJPIAS-75-001707 for animations and full color figures of our experiments. This document can be reached via a direct link in the online article's HTML reference section or via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html). They can also be seen at ⟨tp.lc.ehu.es/jma/mekanika/jarraitua/spring.html⟩.

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