# Putting things on the energy shell

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In a recent paper Morrison and Feldt[1] (MF) discussed the ambiguities that occur during the practical implementation of scattering theories. They specifically mentioned the sometimes ambiguous implementation of the relationship between the $\hat{S}$ and $\hat{T}$ operators [defined by MF (4) and MF (26), respectively]. The correct relationship is given (in matrix form) by MF (28), which is sometimes written in the form $\hat{S}=\hat{1}-2\pi i\hat{T}$ [MF (31)]. MF write: "This operator form implies that the momentum-space $T$-matrix elements must contain (in the second term) both the delta function $\delta(E_{k'}-E_k)$ and the on-shell restriction."

As an example, we draw attention in this Note to such an ambiguity in a standard reference text by Blum, *Density Matrix Theory and Applications*,[2] which has played a historically significant role as a basic reference for coherence parameter studies in atomic collision physics in the last 25 years. A reader with a general knowledge of scattering theory could follow Blum's prescription for constructing cross sections from his definition of the $T$ operator and reach an untenable formula for the differential cross section as well as the matrix elements of the density operator (matrix) describing the scattered state. On pp. 78–81 and in Appendix E, Blum's definition of the **S** operator [via his Eq. (E1)] and MF's definition of $\hat{S}$ [See MF (5)] are identical. Subsequently Blum[2] defines his **T** operator via the relation $\mathbf{T}=\mathbf{S}-1$ [Blum's Eq. (E3)]. It is at this point where the ambiguity occurs. As pointed out by MF when the matrix elements of such an operator are taken, they will contain the delta function in the energy differences [see MF (32a)]. However, subsequently Blum takes the matrix elements of the above operator and equates them to the scattering amplitudes, Blum's Eq. (3.5.4), whose magnitude squared is then equated to the differential cross section in Blum's Eq. (3.5.5). As a result Blum's most important Eq. (3.5.6) contains some unresolved ambiguities.

The situation can be resolved by proceeding along the lines outlined by MF, if one unites some of the features of MF (5), MF (20), and MF (28). Because we are interested in inelastic processes also, our treatment will be somewhat more general than that of MF. We shall use the notation of Bransden.[3]

For the initial state we have

$$\phi_\alpha^{\text{in}}(E_\alpha)=N_\alpha e^{i\vec{k}_\alpha\cdot\vec{r}}X_\alpha(\vec{x}), \tag{1}$$

where $\vec{r}$ is the space vector of the incident particle (treated as a distinguishable particle from the target particles (electrons), $\vec{k}_\alpha$ is the wave vector of the incident particle, and $X_\alpha(\vec{x})$ refers to the target state of quantum number $\alpha$ with $\vec{x}$ referring to the space and spin coordinates of all target particles. $N_\alpha$ is a normalization constant. This state was denoted as $\phi_\alpha$ by Bransden [see Eq. (4-7) in Bransden].[3] The final state wave vector will be given in the form

$$\phi_\alpha^{\text{out}}(E_\alpha)=\mathbf{S}\phi_\alpha^{\text{in}}(E_\alpha), \tag{2}$$

where **S** is the scattering operator. Following Bransden[3] we will write **S** in the form

$$\mathbf{S}=\mathbf{1}-i\mathbf{T}. \tag{3}$$

[We will interpret this as a simple operator equation, with no implied delta-function factorization, it is quite obvious that this is Bransden's interpretation too; compare his Eq. (4-21a) and (4-21b).] If we introduce now the matrix elements of **T** by the definition (see Ref. 3, p. 127)

$$\mathcal{T}_{\beta,\alpha}=\langle\phi_\beta^{\text{in}}|\mathbf{T}|\phi_\alpha^{\text{in}}\rangle, \tag{4}$$

then Eq. (2) [via using Eqs. (3) and (4)] can be written in the form

$$|\phi_\alpha^{\text{out}}(E_\alpha)\rangle=|\phi_\alpha^{\text{in}}(E_\alpha)\rangle-i\sum_\gamma\mathcal{T}_{\gamma,\alpha}|\phi_\gamma^{\text{in}}(E_\gamma)\rangle. \tag{5}$$

The generalization of this equation for partially coherent states is Blum's Eq. (3.5.6).[2]

Following Ref. 3 (pp. 128–129) we shall introduce the transition operator, $T$, with matrix elements, $T_{\beta\alpha}(E_\alpha)$ via the equation

$$\mathbf{S}_{\beta\alpha}=\delta_{\beta\alpha}-2\pi i\,\delta(E_\beta-E_\alpha)T_{\beta\alpha}(E_\alpha). \tag{6}$$

The $T_{\alpha\beta}$ matrix element is "on the energy shell" (Ref. 3, p. 132), i.e., it is the matrix element of the $T$ operator with wave functions $\phi_\alpha^{\text{in}}$ and $\phi_\beta^{\text{in}}$ whose energies are equal.

If we use now the assumption that the $|\phi_\gamma^{\text{in}}(E_\gamma)\rangle$ states form a complete set, then Eqs. (1), (2), and (4) give

$$|\phi_\alpha^{\text{out}}(E_\alpha)\rangle=|\phi_\alpha^{\text{in}}(E_\alpha)\rangle-2\pi i\sum_{\gamma'}\int dE_\gamma\delta(E_\gamma-E_\alpha)T_{\gamma\alpha}(E_\gamma)\times|\phi_\gamma^{\text{in}}(E_\gamma)\rangle, \tag{7}$$

where $\gamma'$ refers to all the other quantum numbers incorporated into $\gamma$ besides the energy $E_\gamma$, i.e., we introduced the notation

  

$$\gamma = (\gamma', E_\gamma). \tag{8}$$

This is an important point which can be easily lost in some 'hand-waving arguments.' **S**, **T**, and $T$ are operators and the quantum numbers used for their matrix representation: $\alpha, \beta, \gamma$ are to some extent arbitrary as long as they completely describe the states defined by Eq. (1). But now, we will choose the total energy of the states $E_\alpha, E_\beta, E_\gamma, \ldots$. as one of the quantum numbers and the rest of the quantum numbers: $\alpha', \beta', \gamma', \ldots$. will give a complete description of the state along with the total energy.

Equation (5) can be immediately reduced to the form

$$|\phi_\alpha^{\text{out}}(E_\alpha)\rangle = |\phi_\alpha^{\text{in}}(E_\alpha)\rangle - 2\pi i \sum_{\gamma'} T_{\gamma,\alpha}(E_\gamma = E_\alpha)|\phi_\gamma^{\text{in}}(E_\gamma = E_\alpha)\rangle. \tag{9}$$

If we now introduce the notation

$$t_{\gamma',\alpha'}(E_\alpha) = 2\pi T_{\gamma,\alpha}(E_\gamma = E_\alpha), \tag{10}$$

then Eq. (7) has the form

$$|\phi_\alpha^{\text{out}}(E_\alpha)\rangle = |\phi_\alpha^{\text{in}}(E_\alpha)\rangle - i \sum_{\gamma'} t_{\gamma',\alpha'}|\phi_\gamma^{\text{in}}(E_\gamma = E_\alpha)\rangle. \tag{11}$$

Now, we can immediately see that Eq. (11) is exactly of the same form as Eq. (5) except the $t_{\gamma,\alpha}$ matrix elements do not contain the ambiguous delta functions, and they are "on the energy shell." These steps were summarized succinctly by Coester and Jauch[4] by saying: "To simplify the notation it will be convenient to drop the $\delta$-function with respect to the energies and have it understood that the matrix elements are functions of the energy." Unfortunately, even such a succinct statement is missing from Blum's book.[2] The comment by Coester and Jauch indicates that the procedure described here was already well known in the 1950s, however, the exact details of it were lost as demonstrated by the comments of MF and by the documented procedure of Blum.[2]

Finally, we want to point out that an equation analogous to our Eq. (11) for elementary particle reactions *was* derived by Saenger and Schmidt,[5] [Eq. (5.26)] which they called Von Neumann's formula.


[1] M. A. Morrison and A. N. Feldt, Am. J. Phys. **75**, 67 (2007). In this Note the equations of Morrison and Feldt will be quoted by their equation number preceded by MF and this reference will be referred to as MF.
[2] K. Blum, *Density Matrix Theory and Applications* (Plenum, New York, 1981).
[3] B. H. Bransden, *Atomic Collision Theory* (W.A. Benjamin, New York, 1970).
[4] F. Coester and J. M. Jauch, "Theory of angular correlations," Helv. Phys. Acta **26**, 1–16 (1953).
[5] R. Saenger and W. Schmidt, "Polarization measurements at high energy," Ann. Phys. (N.Y.) **54**, 307–349 (1969).


# A simple model for inelastic collisions


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## I. INTRODUCTION

In introductory physics courses students are taught that inelastic collisions between two bodies are analyzed by means of conservation of linear momentum, rather than energy conservation, because some kinetic energy is transferred to internal energy[1] in the form of vibrational energy of the bodies' constituents.[2] Some models for inelastic collisions have been discussed and compared to experimental results.[3–7] The purpose of this note is to discuss a simpler model for the transfer of energy that can be used in introductory physics courses. Our goal is to help students gain insight into the loss of kinetic energy in inelastic collisions by studying a simple physical system which can be analyzed with straightforward mathematics and for which the energy of the elastic modes can be an explicitly evaluated.

## II. THE MACROSCOPIC VIEW OF THE COLLISION

We consider two identical deformable bodies with internal structure. Each body has internal components of mass $m_1 = m$ and $m_2 = \alpha m$, which interact through a massless spring of stiffness $k$ and natural length $L$. Initially the springs are unstretched and the two bodies move on a smooth surface with opposite velocities $\dot{x}_1 = \dot{x}_2 = -v$, as depicted in Fig. 1. To take into account the symmetry of the system and to simplify the notation the positive direction of coordinates and velocities is chosen to the left for the left-hand body and its components and to the right for the body on the right. We neglect air resistance and energy loss by acoustic waves.

The collision between the two components of mass $m$ is assumed to be elastic, as is the case for collisions between the microscopic components of macroscopic bodies. Just after the collision these components move with velocity $\dot{x}_1 = v$ while the components of mass $\alpha m$ still have velocity $\dot{x}_2 = -v$. As a consequence the center of mass of each body moves after the collision with the constant velocity

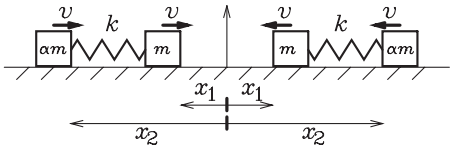


Fig. 1. Coordinate definitions and initial motion of the two bodies with internal structure. Note the choice of positive direction for each body.

$$V = \frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1 + m_2} = \frac{1-\alpha}{1+\alpha}v, \tag{1}$$

as long as no further collision occurs (see the following). Because each body initially moves with velocity $-v$, the coefficient of restitution is given by

$$e = \left|\frac{V}{v}\right| = \left|\frac{1-\alpha}{1+\alpha}\right|. \tag{2}$$

As expected, the collision is completely inelastic when all masses are equal ($\alpha=1$) because in this case the entire kinetic energy is transferred to the internal mode: Each center of mass remains at rest from now on, and there is only oscillations about the center of mass (until a new collision happens). The final kinetic energy of the center of mass of each body increases monotonically for decreasing $\alpha$, so that an elastic collision is recovered when $\alpha=0$ and the components of mass $\alpha m$ effectively disappear from the model. The energy of each body transferred from the initial translational mode to the elastic mode excited by the collision is

$$T - T' = \frac{1}{2}(1+\alpha)mv^2 - \frac{1}{2}(1+\alpha)mV^2 = \frac{2\alpha m v^2}{1+\alpha}, \tag{3}$$

where $T \equiv 1/2(m_1+m_2)v^2$ and $T' \equiv 1/2(m_1+m_2)V^2$ are the kinetic energies of the center of mass of each body before and after the collision.

## III. THE INTERNAL MODE

The analysis in Sec. II is independent of the details of the interaction between the two components of each body. We now use the model to account for the internal energy of each body. To study the internal motion after the collision we can either use a reference frame moving with the center of mass of each body or analyze the relative motion of the two components. We choose the latter approach because the calculation is somewhat more direct and provides an example of the usefulness of the reduced mass.

After the collision the equations of motion for the components of each body are given by Newton's and Hooke's laws

$$m\ddot{x}_1 = k(x_2 - x_1 - L), \tag{4a}$$

$$\alpha m\ddot{x}_2 = -k(x_2 - x_1 - L). \tag{4b}$$

We divide Eq. (4b) by $1+\alpha$, multiply Eq. (4a) by $\alpha/(1+\alpha)$ and subtract the results to obtain the equation of motion for the difference $r=x_2-x_1-L$ between the relative position of the body's components and the natural length of each spring

$$\mu\ddot{r} = -kr, \tag{5}$$

where the reduced mass is

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{\alpha m}{1+\alpha}. \tag{6}$$

Equation (5) is equivalent to the equation of motion in one dimension of a single body of mass $\mu$ attached to a spring of constant $k$. Therefore, the frequency of the oscillations is $\omega \equiv \sqrt{k/\mu}$. We see that the relative motion of the components of each body can be interpreted as oscillations whose conserved mechanical energy (which is the energy of the internal vibrational mode) is given by

$$E_{\text{int}} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}kr^2 = 2\mu v^2 = \frac{2\alpha m v^2}{1+\alpha}, \tag{7}$$

taking into account the initial conditions immediately after the collision: $r=0$ and $\dot{r}=\dot{x}_2-\dot{x}_1=-2v$. By using Eq. (7) we can write Eq. (3) as $T=T'+E_{\text{int}}$, which shows explicitly how the total energy of each body is conserved. Initially it equals the kinetic energy $T$. After the collision it is divided into the translational kinetic energy $T'$ corresponding to the motion of each body as a whole and the internal energy $E_{\text{int}}$ given by Eq. (7). We obtain the oscillation amplitude by setting $\dot{r}=0$ in Eq. (7),

$$A = \frac{2v}{\omega} = 2v\sqrt{\frac{\alpha m}{(1+\alpha)k}}. \tag{8}$$

We can also obtain the evolution of $x_1$ and $x_2$ after the collision by integrating Eq. (4) with the initial conditions $x_1=0$, $x_2=L$, $\dot{x}_1=v$, and $\dot{x}_2=-v$.

## IV. COMMENTS

We have implicitly assumed that the spring deformations are perfectly elastic. Of greater consequence is the assumption that the internal components collide only once. This assumption is not necessarily valid as can be seen when $\alpha=1$. Just after the first collision takes place the springs are unstretched and the velocities of the inner and outer components are $\dot{x}_1=v$ and $\dot{x}_2=-v$, respectively. Each body oscillates in place and when the springs return to their natural length, the components move with velocities $\dot{x}_1=-v$ and $\dot{x}_2=v$. Then the inner components collide again and all the components start moving with constant velocity $\dot{x}_1=\dot{x}_2=v$ while the springs remain unstretched. As a result, the entire energy is transferred in the second collision back from the (internal) oscillatory mode to the translational mode. Although the first collision is fully inelastic (in our macroscopic interpretation), the entire process is completely elastic.

For other values of $\alpha$ the condition $x_1=0$ for secondary collisions leads to a transcendental equation, so that we have to use numerical methods to solve it and to show that for $\alpha>0.697\ldots$, the inner components collide at least twice, and

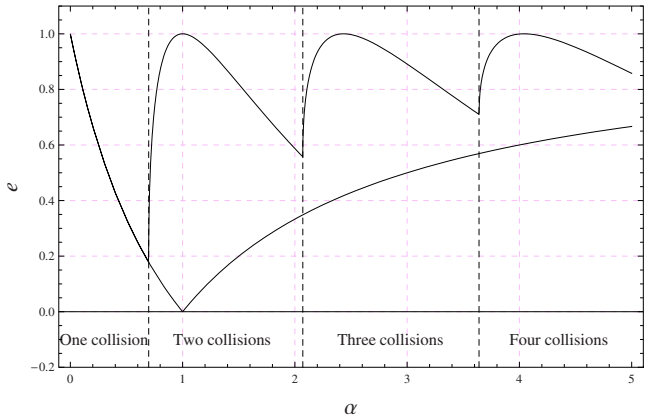


Fig. 2. Coefficient of restitution $e$. The lower curve is the value of $e$ after the first collision, as given by Eq. (2). For $\alpha>0.697\ldots$, additional collisions occur between the components of mass $m$; the final value of $e$ is the upper curve. The dashed vertical lines indicate the values of $\alpha$ where the number of those collisions increases by one. $e$ reaches a minimum at these values of $\alpha$ and then rises sharply.

the solution we have discussed is valid only until the second collision. The motion after the second collision is an interesting problem on its own and may be simulated numerically.[8,9] (It could be an oversimplified model for two identical cars of mass $\alpha m$ with spring-mounted bumpers of mass $m$.) The final value of the coefficient of restitution is depicted in Fig. 2, along with the value given by Eq. (2) and the number of collisions, which increases with $\alpha$ (more bumper collisions are required to reverse the motion of heavier cars).

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[1] M. Alonso and E. J. Finn, *Physics* (Addison-Wesley, Reading, MA, 1992), Sec. 14.9.
[2] R. M. Eisberg and L. S. Lerner, *Physics: Foundations and Applications* (McGraw-Hill, New York, 1981), Sec. 8–3.
[3] R. Gruebel, J. Dennis, and L. Choate, "A variable coefficient of restitution experiment on a linear air track," Am. J. Phys. **39**, 447–449 (1971).
[4] B. F. Bayman, "Model of the behavior of solid objects during collision," Am. J. Phys. **44**(7), 671–676 (1976).
[5] B. Leroy, "Collision between two balls accompanied by deformation: A qualitative approach to Hertz's theory," Am. J. Phys. **53**(4), 346–349 (1985).
[6] D. Auerbach, "Colliding rods: Dynamics and relevance to colliding balls," Am. J. Phys. **62**(6), 522–525 (1994).
[7] R. Cross, "Vertical bounce of two vertically aligned balls," Am. J. Phys. **75**(11), 1009–1016 (2007).
[8] The web page ⟨tp.lc.ehu.es/jma/mekanika/inelastic/inelastic.html⟩ has an interactive simulation written using the Easy Java Simulations framework (⟨www.um.es/fem/Ejs/⟩) by F. Esquembre. The Mathematica notebook at ⟨tp.lc.ehu.es/jma/mekanika/inelastic/restn.nb⟩ was used to compute the number of collisions and the coefficient of restitution.
[9] See EPAPS Document No. E-AJPIAS-76-015810 for the interactive Java simulation and the Mathematica notebook of Ref. 8. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.