We now analyze where the block comes to rest with a constant friction force acting. The first stopping point is at X_1, after a time $t = \pi/\omega$, where

$$X_1 = -(A - 2\alpha_-/\omega^2). \quad (17)$$

We now solve Eq. (6) for the half-cycle return motion of the block up the incline, subject to the conditions

$$\left.\begin{aligned} \dot{x} &= 0 \\ x &= X_1 \end{aligned}\right\} \text{at } t = \pi/\omega.$$

The trial solution $x = C\cos\omega t + D$ satisfies Eq. (6) if we take the constants to be

$$\begin{aligned} \omega^2 &= k/m, \\ D &= -\alpha_+/\omega^2, \\ C &= A - 2\alpha_-/\omega^2 - \alpha_+/\omega^2. \end{aligned} \quad (18)$$

Therefore the solution for the return motion up the incline can be written as

$$x = (A - 2\alpha_-/\omega^2 - \alpha_+/\omega^2)\cos\omega t - \alpha_+/\omega^2. \quad (19)$$

Repetition of the analysis that led to Eqs. (9) and (19) leads to a general expression for the motion during the nth half cycle to be

$$\begin{aligned} x = [A - \alpha_-/\omega^2 - (n-1)(\alpha_- + \alpha_+)/\omega^2]\cos\omega t \\ + [\alpha_-(1-(-)^n) - \alpha_+(1 + (-)^n)]/2\omega^2 \end{aligned} \quad (20)$$

for the time interval $(n-1)/\omega < t < n/\omega$. The block comes to rest when the cosine is ± 1. Therefore the block comes to rest at the locations

$$X_n = \left(A - \frac{n2\mu g\cos\theta}{\omega^2} + \frac{g\sin\theta}{\omega^2}[1-(-)^n]\right)(-)^n. \quad (21)$$

The total elapsed time when the block is at rest at X_n is

$$T = n\pi/\omega. \quad (22)$$

In Eqs. (21)-(22), $n = 0$ corresponds to the block starting its oscillations, $n = 1$ corresponds to when it first comes to rest, etc.

The location where the block comes to rest can also be determined from conservation of energy. Taking the friction force as the only external force doing work on the system of block plus spring, we have

$$\begin{aligned} -\mu mg\cos\theta\,(x - X_n) = (\tfrac{1}{2})m\dot{x}^2 + (\tfrac{1}{2})kx^2 \\ + mgx\sin\theta - E_n. \end{aligned} \quad (23)$$

The minus sign with the frictional work term results from the friction force opposing the motion of the block along the incline as it moves from X_n to the point x. The energy terms on the right-hand side of Eq. (23) are, respectively, the kinetic energy of the block, the spring potential energy, the gravitational potential energy of the block, and finally the energy of the system at the nth stopping point E_n. The spring has been assumed massless in Eq. (23). The energy for the system (E_n) includes the spring potential energy and the block gravitational potential energy at the nth stopping point of the block. This conservation of energy equation can be solved for the location where the block next comes to rest by setting the kinetic energy term to zero, and solving the resulting quadratic equation for x. This solution matches Eq. (21). The quadratic equation provides two roots. It is worth noting they are the prior and next stopping locations of the block.

From the above results, the following simple experiment is proposed for the student to determine a coefficient of kinetic friction. The block is started at a measured position $x_0 = A$, up the incline and allowed to oscillate until the block will no longer move. The time the block was moving should be measured and/or count the number of half cycles undergone by the block. These two measurements are redundant, but can be used to check the spring constant from the relation

$$T = n\pi/\omega. \quad (24)$$

For an even number of half oscillations, the distance d from the initial starting place to the final stopping place is given by Eq. (21) to be

$$d = n2\mu g\cos\theta/\omega^2 = 2\mu g\cos\theta T^2/\pi^2(n). \quad (25)$$

For an odd number of half oscillations, we have

$$d = 2A - 2g\sin\theta/\omega^2 - \mu(n\,2g\cos\theta/\omega^2). \quad (26)$$

The measurements are repeated for various values of n or for various values of $\cos\theta$. The coefficient of kinetic friction can be measured from the slope of a graph of d vs $2ng\cos\theta/\omega^2$. For n an odd number of half cycles, one graphs,

$$2A - 2g\sin\theta/\omega^2 - d \quad \text{vs} \quad 2ng\cos\theta/\omega^2.$$

The slope of this is μ, the coefficient of friction. The independence of the coefficient of friction from the normal force is usually assumed in introductory studies.[2] The validity of this assumption can be assayed from the straightness of the graphs of d obtained experimentally.

[1] Rudolph E. Langer, *Ordinary Differential Equations* (Wiley, New York, 1956), pp. 163-165.

[2] James A. Richards, Francis Weston Sears, M. Russell Wehr, and Mark W. Zemansky, *Modern College Physics* (Addison-Wesley, Reading, MA, 1962), p. 31.

A relativistic problem: The charge distribution stability on a conductor

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The Lorentz transformation was originally introduced to make the form of the Maxwell's equations invariant in all inertial frames.[1] Subsequently, Einstein deduced the transformation from the postulates of special relativity, making clear the close relationship between relativity and electromagnetism.

On the other hand, if in classical electrodynamics the inertial observers relate their space-time measurements by means of a Galilean transformation, we arrive at contradicting and paradoxical situations with the well-established experimental data. For instance, if we assume the charge invariance principle and that Maxwell's equations have the same form in every inertial frame, we arrive at the surprising result that the magnetic field has the same value for every observer.[2]

But, if according to experience, we accept that the magnetic field created by moving charges of low speed is given by the Biot–Savart law, we still are faced with analogous difficulties. When considering the charge distribution on a conductor, this is stable for an inertial observer standing at rest with respect to the body, but is not stable for a moving one. As we will see below, the charge distribution stability on conductors is made clear if space-time measurements for different observers are related by means of the Lorentz transformation instead of the Galilean one.

Let us consider a conductor at rest in the inertial frame R^*; there exists at every point on its surface a charge density σ^*. The electric field there is orthogonal to the surface and the charge distribution is statically stable as is well known.

Let R be another inertial frame with respect to which R^* is moving with velocity $\mathbf{v}$ along $0X$ axis, and let us assume that space-time measurements in R and R^* are related through a Galilean transformation

$$x = x^* + vt^*, \quad y = y^*, \quad z = z^*, \quad t = t^*. \tag{1}$$

If we assume that the electromagnetism laws, and in particular the Biot–Savart law, are valid in every inertial frame, we will have at every point on the conductor, an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$.

If v is small compared to the speed of light c, then $\mathbf{E} \simeq \mathbf{E}^*$. Let $d\mathbf{E}$ be the electric field at some point due to the charge $\sigma\, ds$ of the surface element ds of the body. Then the magnetic field $d\mathbf{B}$ that this charge produces there, is given by Biot–Savart's law[3]

$$d\mathbf{B} = (1/c^2)\mathbf{v} \times d\mathbf{E}. \tag{2}$$

Thus there will appear at every point on the conductor surface a magnetic field

$$\mathbf{B} = (1/c^2)\mathbf{v} \times \mathbf{E} \simeq (1/c^2)\mathbf{v} \times \mathbf{E}^*. \tag{3}$$

The unit charge on the surface will undergo a total force due to $\mathbf{E}$ and $\mathbf{B}$, which will have in general, a nonvanishing component along the surface giving rise thereby to a charge redistribution until a stable situation is attained. But if charge invariance is assumed, the charge in a surface element must be the same in both frames

$$\sigma^* ds^* = \sigma\, ds \tag{4}$$

and according to (1), $ds = ds^*$, concluding that charge densities must be the same. This is a paradoxical situation since, in the R frame, charge distribution stability implies that σ and σ^* must be different.

An answer to the paradox readily follows if space-time measurements are related by Lorentz transformation, and electric and magnetic fields transform relativistically.

In fact, let R and R^* be related by the special Lorentz transformation

$$x = \gamma(x^* + vt^*); \quad y = y^*; \quad z = z^*; \quad t = \gamma(t^* + vx^*/c^2), \tag{5}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2}.$$

The relationship between the fields $\mathbf{E}^*$, $\mathbf{B}^*$ and $\mathbf{E}$, $\mathbf{B}$ at the same point, as measured by R^* and R, respectively, are[4]

$$\begin{aligned} \mathbf{E}_\parallel &= \mathbf{E}_\parallel^*, \quad \mathbf{E}_\perp = \gamma[\mathbf{E}_\perp^* - \mathbf{v} \times \mathbf{B}^*]; \\ \mathbf{B}_\parallel &= \mathbf{B}_\parallel^*, \quad \mathbf{B}_\perp = \gamma[\mathbf{B}_\perp^* + (1/c^2)\mathbf{v} \times \mathbf{E}^*], \end{aligned} \tag{6}$$

where the symbols $\parallel$ and $\perp$, respectively, denote the field components along and perpendicular to $\mathbf{v}$.

For the observer in R, at every point on the conductor surface, there exists an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$, such that the Lorentz force on the unit charge is orthogonal to the surface, because of (5).

Owing to charge invariance, (4) holds and since $ds \neq ds^*$, the charge density in frame R, σ, will be in general, different than the rest-frame charge density σ^*.

Let us apply all this considerations to a charged conducting ellipsoid whose equation in its proper frame is

$$\frac{x^{*2}}{a^2} + \frac{y^{*2}}{b^2} + \frac{z^{*2}}{b^2} = 1 \quad a > b. \tag{7}$$

When the charge distribution attains equilibrium, the surface charge density in R^* is given by[5]

$$\sigma^* = \frac{Q}{4\pi ab^2}\left(\frac{x^{*2}}{a^4} + \frac{y^{*2}}{b^4} + \frac{z^{*2}}{b^4}\right)^{-1/2}, \tag{8}$$

Q being the total charge on it.

If the ellipsoid speed $\mathbf{v}$ with respect to R is such that $\gamma = a/b$, then in R, it has the spherical shape

$$x^2 + y^2 + z^2 = b^2. \tag{9}$$

In R^* the electric field strength and charge density are related by

$$E^* = \sigma^*/\epsilon_0. \tag{10}$$

$\mathbf{E}^*$ being orthogonal to the surface, the field components at any point on the conductor are given by

$$E_x^* = k2x^*/a^2, \quad E_y^* = k2y^*/b^2, \quad E_z^* = k2z^*/b^2, \tag{11}$$

where

$$k = \frac{\sigma^*}{2\epsilon_0}\left(\frac{x^{*2}}{a^4} + \frac{y^{*2}}{b^4} + \frac{z^{*2}}{b^4}\right)^{-1/2}. \tag{12}$$

According to (6) there will be at every point, as measured in R, an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ with components

$$\begin{aligned} E_x &= E_x^*, \quad E_y = \gamma E_y^*, \quad E_z = \gamma E_z^*; \\ B_x &= 0, \quad B_y = -(\gamma v/c^2)E_z^*, \quad B_z = (\gamma v/c^2)E_y^*, \end{aligned} \tag{13}$$

in such a way that after (9) and (11), the Lorentz force on the unit charge on the conductor surface, has the components

$$F_x = \frac{k}{\gamma}\frac{2x}{b^2}, \quad F_y = \frac{k}{\gamma}\frac{2y}{b^2}, \quad F_z = \frac{k}{\gamma}\frac{2z}{b^2}, \tag{14}$$

which has the normal direction to the sphere (9).

It must be observed that the electric field on the sphere, as measured in R, has not the normal direction, as can easily computed by making use of (13).

By using (4) the charge density at every point as measured in R, is

$$\sigma = Q/4\pi b^2, \quad (15)$$

which is the same one would have if the sphere of radius b were at rest in R.

[1] W. G. V. Rosser, *An Introduction to the Theory of Relativity* (Butterworths, London, 1964), pp. 65–66 and 304.

[2] P. G. Bergmann, *Hadbuch der Physik* (Springer-Verlag, Heidelberg, 1962), Vol. IV, p. 111.

[3] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), pp. 133–134.

[4] Reference 1, p. 310.

[5] W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, New York, 1968), pp. 123–124.

Some remarks on a generalized heat conduction equation

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The classical theory of heat conduction in solids is based on Fourier's law of heat conduction

$$\mathbf{q}(\mathbf{x},t) = -\kappa \mathbf{g}(\mathbf{x},t), \quad (1)$$

where $\mathbf{q}$ is the heat flux vector, $\mathbf{g}[=\nabla\theta(\mathbf{x},t)]$ is the gradient of temperature θ, and $\kappa(>0)$ is a constant known as thermal conductivity. It is further assumed that

$$e - e_0 = c(\theta - \theta_0), \quad (2)$$

where e and e_0 are the internal energy densities at temperature θ and θ_0, respectively, and c (>0) is the heat capacity. In the absence of any heat sources or sinks; (1), (2), and the balance of energy leads to

$$\left(\frac{\kappa}{c}\right)\nabla^2\theta = \frac{\partial\theta}{\partial t}, \quad (3)$$

where ∇^2 is the Laplacian. Equation (3) is a parabolic partial differential equation for θ in $\mathbf{x}$ $[0,\infty)$, and therefore gives the result that the speed of propagation of a thermal pulse is infinite. That is, a thermal disturbance created at any place in a body is felt instantly at any other point. This result is unacceptable on a physical basis; also, some experiments with helium[1,2] at a low temperature (2.2°K) have shown that the thermal disturbances propagate as progressive waves. This has led many researchers to seek a theory of heat conduction with finite propagation speed. R. J. Swenson[3,4] has recently reviewed and discussed some of the theories and has proposed one himself in this Journal. In both of these papers a substantial literature in continuum thermodynamics bearing directly on the problem of heat conduction has escaped attention. In order to provide to the readers of this Journal a more encompassing view of this subject, we would like to draw attention to Refs. 5–12, where the point of view and approach taken in developing theories is that of modern continuum thermodynamics. As a general reference Coleman and Noll's paper[13] on the thermodynamics of elastic materials with heat conduction and viscosity is very instructive.

A thermodynamic theory must be consistent with the requirements of the second law of thermodynamics. In continuum mechanics the appropriate form of the second law has been taken to be the Clausius–Duhem inequality, although there is some controversy on what final form this inequality should take. Some of the theories mentioned above are based on the exploitation of a particular form of Clausius–Duhem inequality. The interested reader would reap benefits from consulting Müller,[14] Green and Laws,[15] Gurtin and Williams,[16] and a very recent paper by Green and Naghdi[17] on the subject of the Clausius–Duhem inequality.

A comparative assessment of various theories mentioned above is not yet available, mainly because the theories are very new. The author[18,19] has recently considered the propagation of surface and plane waves in two important theories—that of Lord and Shulman and of Green and Lindsay. The former theory uses a modification of Fourier's law and the latter modifies the energy equation but keeps the Fourier law. Insofar as the results from Refs. 18 and 19 are concerned, the theory of Green and Lindsay seems to be somewhat more general in the sense that all the results of the theory of Lord and Shulman are obtained as a special case and some new results are also obtained. Equally interesting are the approaches that these authors have taken in formulating their theories.

By taking the approach of continuum thermodynamics, Gurtin and Pipkin,[9] and Coleman and Gurtin[20] have developed consistent general nonlinear theories of heat conduction for rigid materials with memory. These theories have been developed systematically and are more general than the theory proposed by Swenson. Nunziato[21] has established a linearized form of a theory which is slightly more general than that of Coleman and Gurtin. The expressions for internal energy density and heat flux are given as[21]

$$e = c\theta + \int_0^\infty \alpha(s)\theta(t-s)ds, \quad (4)$$

$$\mathbf{q} = -\kappa\nabla\theta - \int_0^\infty \beta(s)\nabla\theta(t-s)ds, \quad (5)$$

where c and κ are, respectively, the instantaneous heat capacity and the thermal conductivity, and α and β denote the energy and the heat flux relaxation functions:

$\beta(s) = 0$ in (5) recovers Fourier's law, and

$\alpha(s) = 0$ in (4) gives (2).

In a physical (linear) theory it is expected that the solution exists, is unique, and depends continuously on the initial-boundary values. These conditions also provide a means of

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