[a] Current address: Department of Physics—code PH/DE, Naval Postgraduate School, Monterey, California 93943-5000.

[1]Bruce Denardo, Brad Barber, Chris Folley, and William Wright, "Nonfrictional hysteresis in a rotating U-tube," Am. J. Phys. 57, 1126–1130 (1989).

[2]L. Landau and E. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1987), 2nd ed., Chap. VII.

[3]Reference 2, pp. 51–53.

[4]Fluorescein (Acid Yellow 73), Matheson, Coleman, and Bell, Manufacturing Chemists, Norwood, Ohio.

[5]Photo-Flo 200, Eastman Kodak Co., Rochester, New York.

On dynamical equations and conservation laws in quasistatic electromagnetic systems

J. M. Aguirregabiria, A. Hernández, and M. Rivas

Departamento de Fisica Teórica, Universidad del País Vasco, Apdo. 644, 48080 Bilbao, Spain

(Received 4 May 1989; accepted for publication 6 September 1989)

The necessary attribution of linear and angular momenta to the electromagnetic field, even in quasistatic situations, is illustrated by discussing the dynamical conservation laws of an interacting system composed of two point charges and a magnetic dipole. The evaluation of the trajectories gives an interesting example for numerical computation.

I. INTRODUCTION

Let us consider the static electromagnetic field created by charges at rest and stationary currents. We have at every point in space an electric field **E** as well as a magnetic field **B**, which satisfy

$$\text{div}\,\mathbf{E} = \rho/\epsilon_0, \quad \text{curl}\,\mathbf{E} = 0, \tag{1a}$$

$$\text{div}\,\mathbf{B} = 0, \quad \text{curl}\,\mathbf{B} = \mu_0\,\mathbf{j}, \tag{1b}$$

where ρ and $\mathbf{j}$ are the charge and current density, respectively. Both fields are independent of each other, as follows from the fact that Eqs. (1a) and (1b) are uncoupled.

However, in order to maintain the stationary configuration, it is necessary to act upon the sources with some external forces in such a way that, when these are withdrawn and the system becomes isolated, the fields will no longer be either independent or static and the particles and currents will be affected by electromagnetic forces.

For isolated systems, the linear and angular momenta and energy conservation laws are considered fundamental laws, i.e., independent principles, and since electromagnetic interactions do not satisfy Newton's third law, in order for the conservation principles to still hold in this case, those quantities must be assigned to the electromagnetic fields, giving rise to Poynting's theorems and to the energy-momentum tensor formalism.[1]

When studying the static electromagnetic field, students become puzzled and usually have difficulties accepting that a static field can carry linear and angular momenta. Thus it is useful, at a pedagogical level, to analyze in detail some examples in order to show the coherence of the theory. Angular momentum conservation has been considered in different devices such as cylindrical wires and solenoids[2–5] (their difficulties having been quoted by Romer[6,7]) and spherical conductors and magnetized spheres.[6,8,9] However, less attention has been paid to conservation of linear momentum.[10,11] A related subject is the calculation of the angular momentum of the electron field.[10,12]

In this context, special attention has been paid to the so-called Feynman paradox.[13] A simplified version of it was first analyzed by Aguirregabiria and Hernández.[14] Later, it was studied by Lombardi,[15] Bahder and Sak,[16] and Ma.[17]

The common feature of the above-quoted examples is that one starts with some system in which there is a stored linear (or angular) momentum with no linear (or angular) mechanical momentum. At the end of the process, when the electromagnetic momentum vanishes, the system contains mechanical momentum that clearly matches with the initial electromagnetic one, such that the corresponding conservation law holds. However, no analysis is done on the intermediate situations.

This is precisely the aim of the present work, in which the motion of two charged particles, interacting with each other and with a magnetic dipole moment, is analyzed in detail. It is illustrated how, when the corresponding quantities are attached to the electromagnetic field, the mentioned conservation laws are the first integrals of the dynamical equations of charges and currents. The trajectories are computed numerically, checking at every time the conservation of energy and of linear and angular momenta.

The article is organized as follows. First, we consider a system that has electromagnetic angular momentum, consisting of two particles with equal charge that interact with a third one characterized by its magnetic dipole moment. In Sec. III the system is quite similar, except that the charges have opposite values, giving rise to a nonvanishing electromagnetic linear momentum. Finally, the last part is devoted to discussion and comments.

II. STATIC SYSTEM OF PARTICLES WITH ANGULAR MOMENTUM

A. Dynamical equations

Let us consider two equal pointlike charged particles of charge q and mass M placed initially on the X axis at $x_1 = r_0$ and $x_2 = -r_0$, respectively.

At the origin there exists a third particle, characterized by its magnetic dipole moment $\mathbf{m} = m\hat{k}$, with $\hat{k}$ being the unit vector along the Z axis.

The two charged particles are initially held at rest by means of some external forces. At a certain time these forces are simultaneously removed and the charges start moving, describing symmetric trajectories, while the dipole remains at rest at the origin. At a later time, positions and velocities of particles are given by

$$\mathbf{r}_1 = -\mathbf{r}_2 \equiv \mathbf{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \tag{2}$$

$$\mathbf{v}_1 = -\mathbf{v}_2 \equiv \mathbf{v}(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}, \tag{3}$$

where, as usual, the dot means time derivative, and $\hat{\imath}$ and $\hat{\jmath}$ are, respectively, the unit vectors along the X and Y axes.

We assume that velocities are small enough to neglect the magnetic interaction among the charges, radiation, and retarded effects, and to have a quasistationary system. Thus the interaction between charges is given by Coulomb's law. In fact, it is not much more difficult to include the magnetic interaction between charges in the following analysis, but we will not do it in order to keep simpler expressions while retaining a very good level of accuracy.

We also assume that the magnetic moment remains constant in time. This approximation can be justified if, for instance, we imagine that the magnetic moment is due to an electric current in a closed loop with arbitrarily small radius. We can assume that in the limit the magnetic flux through the loop is null and so is its coefficient of inductance.

Due to the symmetry of the system, we shall only analyze the motion of particle 1. The force acting on this particle is due to the electric field of particle 2 and the magnetic field of the magnetic dipole at rest in the origin, i.e.,

$$\mathbf{F} = \frac{q^2}{16\pi\epsilon_0}\frac{x\hat{\imath} + y\hat{\jmath}}{(x^2+y^2)^{3/2}} - \frac{\mu_0 mq}{4\pi}\frac{\dot{y}\hat{\imath} - \dot{x}\hat{\jmath}}{(x^2+y^2)^{3/2}}. \tag{4}$$

Thus the dynamical equations are

$$\ddot{x} = (x^2+y^2)^{-3/2}(C_1 x - C_2\dot{y}), \tag{5a}$$

$$\ddot{y} = (x^2+y^2)^{-3/2}(C_1 y + C_2\dot{x}), \tag{5b}$$

where

$$C_1 = q^2/16\pi\epsilon_0 M, \text{ and } C_2 = \mu_0 mq/4\pi M.$$

Multiplying (5a) by $\dot{x}$ and (5b) by $\dot{y}$ and adding both equations, the following result is reached by integration:

$$\tfrac{1}{2}(\dot{x}^2+\dot{y}^2) + C_1/(x^2+y^2)^{1/2} = E, \tag{6}$$

where E is a constant of the motion. Subtracting Eq. (5a) multiplied by y from (5b) multiplied by x, and integrating, we arrive at

$$x\dot{y} - y\dot{x} + C_2/(x^2+y^2)^{1/2} = L, \tag{7}$$

with L being another constant of the motion.

The mechanical angular momentum of the system with respect to the origin at any arbitrary time is

$$\mathbf{L}_{\text{mech}} = 2\mathbf{r}\times M\mathbf{v} = 2M(x\dot{y} - y\dot{x})\mathbf{k}, \tag{8}$$

and because of (7), it is no longer conserved, even though the system is isolated.

B. The total angular momentum of the system

In the total energy conservation law (6), besides the mechanical energy, we also consider the electrostatic energy stored in the field. In the same way, in the angular momentum conservation law we must take into account the angular momentum of the static electromagnetic field.

In fact, the general expression for the angular momentum with respect to the origin of an electromagnetic field is given by

$$\mathbf{L}_{\text{em}} = \int_{\mathbb{R}^3} \mathbf{r}\times\epsilon_0(\mathbf{E}\times\mathbf{B})\,dV. \tag{9}$$

If the currents are stationary and the charge distribution is static, then (9) reduces to[14]

$$\mathbf{L}_{\text{em}} = \int_{\mathbb{R}^3} \mathbf{r}\times\rho\mathbf{A}\,dV, \tag{10}$$

where ρ is the charge density at point $\mathbf{r}$ and $\mathbf{A}$ is the vector potential (in the Coulomb gauge, $\nabla\cdot\mathbf{A} = 0$) at the same point. This expression is completely natural in the usual interpretation of $\mathbf{A}$ in field theories and agrees with the discussions (restricted to the analysis of the linear momentum) given by Konopinski.[18] When the charges are pointlike, (10) leads to

$$\mathbf{L}_{\text{em}} = \sum_i \mathbf{r}_i\times q_i\mathbf{A}(\mathbf{r}_i), \tag{11}$$

where $\mathbf{r}_i$ is the position vector of charge q_i and $\mathbf{A}(\mathbf{r}_i)$ is the value of the vector potential at that point. In our example, the system has at the initial time, when $\mathbf{r}_1(0) = -\mathbf{r}_2(0) = \mathbf{r}_0 \equiv r_0\mathbf{i}$, an angular momentum

$$\mathbf{L}_{\text{em}}(0) = 2\mathbf{r}_0\times q\mathbf{A}(\mathbf{r}_0). \tag{12}$$

The vector potential created by a dipole moment at any point $\mathbf{r}$ is given by

$$\mathbf{A}(\mathbf{r}) = (\mu_0/4\pi)\mathbf{m}\times\mathbf{r}/r^3, \tag{13}$$

so that

$$\mathbf{L}_{\text{em}}(0) = (\mu_0/2\pi)(mq/r_0)\mathbf{k}, \tag{14}$$

which agrees with the result obtained by Furry[19] and Lawson.[20]

Since the particles travel with small velocities, the situation can be considered quasistationary, and when the particles are, respectively, at $\mathbf{r}_1 = \mathbf{r}$ and $\mathbf{r}_2 = -\mathbf{r}$, the angular momentum $\mathbf{L}_{\text{em}}(t)$ reads

$$\mathbf{L}_{\text{em}}(t) = 2\mathbf{r}\times q\mathbf{A}(\mathbf{r}) = (\mu_0 mq/2\pi)[\mathbf{k}/(x^2+y^2)^{1/2}]. \tag{15}$$

The mechanical angular momentum is (8), and at any time t the conservation of angular momentum requires that

$$\mathbf{L}_{\text{mech}}(t) + \mathbf{L}_{\text{em}}(t) = \mathbf{L}_{\text{em}}(0), \tag{16}$$

which is equivalent to Eq. (7).

Energy conservation allows us to write

$$E_{\text{mech}}(t) + E_{\text{em}}(t) = E_{\text{em}}(0), \tag{17}$$

where $E_{\text{mech}}(t)$ is the kinetic energy at time t and $E_{\text{em}}(0)$ is the interaction energy at the initial time. Under the accepted conditions for the dynamics, the energy $E_{\text{em}}(t)$ is derived from the Coulomb interaction and (17) can be expressed in the form of the first integral (6).

C. Particle trajectories

The dynamical equations (5) have bounded and unbounded solutions. Trajectories in which the radial coordinate remains bounded are characterized by (see Appendix)

$$27r_0^2 E \leqslant 2L^2. \tag{18}$$

The particle trajectories can be easily computed by nu-

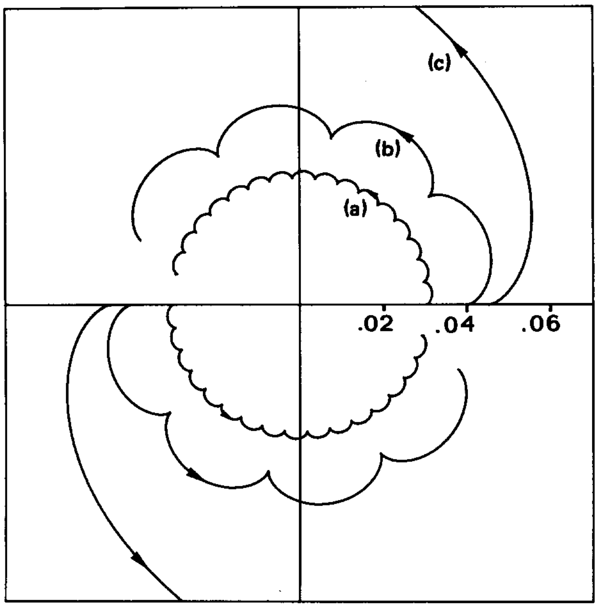

Fig. 1. Trajectories for two protons interacting with a magnetic dipole, for three different initial conditions: (a) $r_0 = 0.03$ m, (b) $r_0 = 0.04$ m, and (c) $r_0 = 0.045$ m.

merical methods. This is an interesting and unusual example that can be used as part of classroom demonstrations on numerical simulation, as we now show.

Let us apply the above results to the particular case in which the charges are protons and the magnetic dipole is a sphere of 1-mm radius with a magnetization of 1.59×10^5 A m^{-1}. Then, the constants appearing in (5) are $C_1 = 0.0345\,\mathrm{m^3\,s^{-2}}$ and $C_2 = 0.0063\,\mathrm{m^3\,s^{-1}}$, and the condition for bounded motion is that $r_0 \leqslant 0.044$ m.

In Fig. 1 the motion of the two particles is depicted for three different initial positions. For trajectory *c*, $r_0 = 0.045$ m, and thus this motion is unbounded and the particles escape. It must be stressed that it can be shown numerically that the velocities always remain rather small (below 2 m s^{-1}), so that one can expect the approximations made so far to be very good.

During the numerical calculations, the addition of the two terms on the left-hand side of both (6) and (7) can be monitored, checking that this sum remains constant, or alternatively, one can use this constancy as a quality test of the numerical integration method.

III. STATIC SYSTEM OF PARTICLES WITH LINEAR MOMENTUM

Let us assume now that particle 1 has charge $+q$, while charge 2 is $-q$ with the magnetic dipole moment at the origin, as before, and with the particles at rest under the action of some external forces at the same initial configuration.

In this situation the angular momentum of the system is zero, but the linear momentum is not, since there exists a linear momentum associated with any electromagnetic field given by

$$\mathbf{P}_{\mathrm{em}} = \int_{\mathbf{R}^3} \epsilon_0(\mathbf{E}\times\mathbf{B})dV, \qquad (19)$$

which, in the stationary case and for point particles, yields[21]

$$\int_{\mathbf{R}^3} \rho\mathbf{A}(\mathbf{r})dV = \sum_i q_i\mathbf{A}(\mathbf{r}_i), \qquad (20)$$

leading, in this particular case, to the expression

$$\mathbf{P}_{\mathrm{em}}(0) = (\mu_0 mq/2\pi r_0^2)\mathbf{j}. \qquad (21)$$

When the external forces are suppressed, the charges move under the action of their mutual Coulomb interaction (constancy of the magnetic dipole moment and low velocities are assumed, and retarded effects and radiation ignored as in the earlier case) and under that of the dipole magnetic field. Now, there exists a nonvanishing magnetic force on this dipole, and in order to hold it at rest at the origin some external force has to be exerted in order to balance the action produced by the charges.

In this low-velocity approximation, the magnetic field created by a pointlike charge at a point **R** is

$$\mathbf{B} = (\mu_0 q/4\pi)(\mathbf{v}\times\mathbf{R}/R^3). \qquad (22)$$

The magnetic field created by the charges at time *t* at the point of coordinates $(0,w)$ is

$$\mathbf{B}(0,w) = (\mu_0 q/2\pi)\{[\dot{x}(w-y) + \dot{y}x]/[x^2 + (w-y)^2]^{3/2}\}\mathbf{k}. \qquad (23)$$

The force of a magnetic field on a magnetic dipole is

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}), \qquad (24)$$

and because of the symmetry of our problem, it is oriented in the *Y* direction. Thus

$$\mathbf{F} = \frac{d}{dw}[\mathbf{m}\cdot\mathbf{B}(0,w)]_{w=0}\hat{j}, \qquad (25)$$

while the external force on the dipole, which must be opposite to this one, is given by

$$\mathbf{F}_{\mathrm{ext}} = -\mathbf{F} = (\mu_0 mq/2\pi) \times\{[(2y^2 - x^2)\dot{x} - 3xy\dot{y}]/(x^2+y^2)^{5/2}\}\hat{j}. \qquad (26)$$

Since Coulomb forces satisfy Newton's third law (action–reaction principle), the dynamical equation of the charges along the *Y* axis is

$$\ddot{y} = (\mu_0 mq/4\pi M)[\dot{x}/(x^2+y^2)^{3/2}]. \qquad (27)$$

It is easily seen that this dynamical equation is equivalent to the linear momentum evolution law:

$$\mathbf{F}_{\mathrm{ext}} = \frac{d\mathbf{P}(t)}{dt}, \qquad (28)$$

where the total linear momentum at any arbitrary time *t* will be the sum of two terms: the mechanical one of the charges plus the electromagnetic one of the field, i.e.,

$$\mathbf{P}(t) = \mathbf{P}_{\mathrm{mech}}(t) + \mathbf{P}_{\mathrm{em}}(t) = \{M\dot{y} + (\mu_0 mq/2\pi)[x/(x^2+y^2)^{3/2}]\}\mathbf{j}. \qquad (29)$$

Figure 2 depicts the particle motion for a proton and an antiproton interacting with the same magnetic dipole as in Sec. II. As in the previous case, we see that in the numerical integration the actual velocities remain small enough to justify the use of the quasistatic approximation.

It has been pointed out by Furry[19] and Coleman and Van Vleck[22] that in some circumstances there is a "hidden momentum" in a magnet. In the case discussed above, we have ignored it because, since the magnetic dipole moment remains constant and is held at rest, its constant contribu-

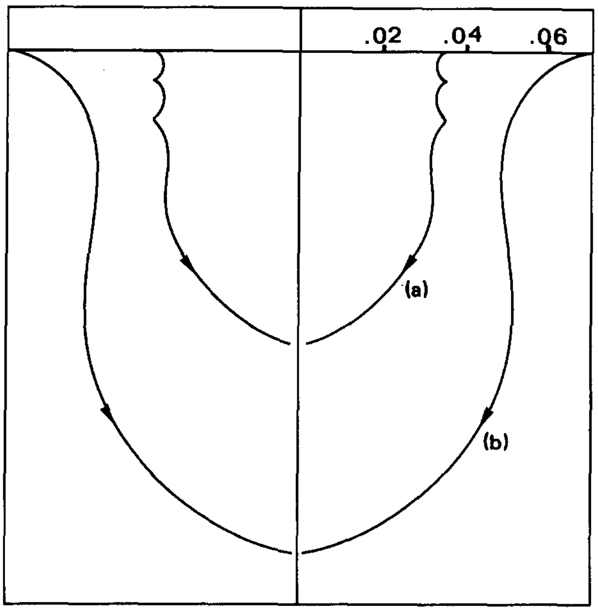

Fig. 2. Proton and antiproton motion interacting with a magnetic dipole for two different initial positions: (a) $r_0 = 0.035$ m and (b) $r_0 = 0.07$ m.

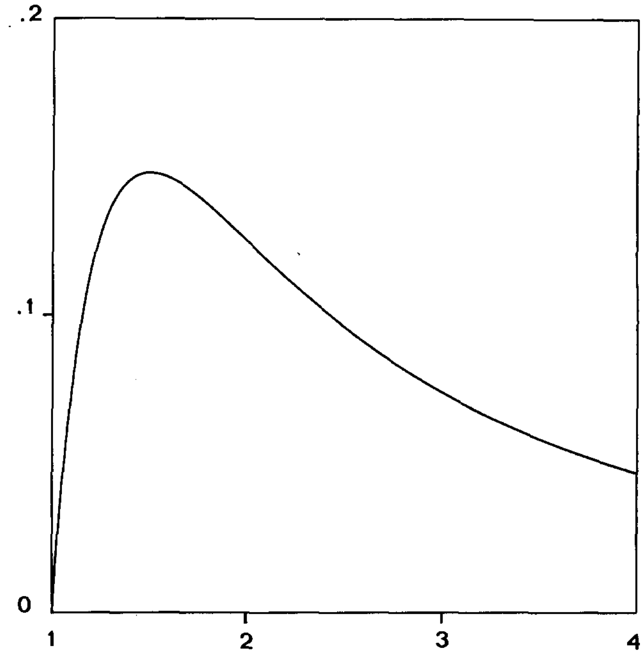

Fig. 3. Graph of function $f(u) = (u-1)/u^3$.

tion does not change the conservation laws. Alternatively, the previous analysis could be seen as a demonstration of the constancy of this "hidden momentum."

IV. DISCUSSION AND COMMENTS

We can also understand the existence of the stored angular momentum $\mathbf{L}_{em}(0)$ of the first system (Sec. II) by paying attention to the way the system can be built through a similar method to the one used by Calkin[21] for the calculation of the linear momentum.

First, we can bring the two charges from infinity to their final points, with a small enough constant velocity to consider just their Coulomb interaction. In this case, because the two forces are opposite and directed toward the origin, no angular momentum is involved.

Once the two charges are held at rest, we can carry along the Y axis the magnetic dipole, with very low velocity. When it is at the point of coordinates $(0,\xi)$, it creates a potential vector $\mathbf{A}(\mathbf{r}_1)$ on the first charge:

$$\mathbf{A}(\mathbf{r}_1) = (\mu_0 m/4\pi)[(\xi\hat{\imath} + r_0\hat{\jmath})/(r_0^2+\xi^2)^{3/2}], \quad (30)$$

and since ξ changes with time, $\mathbf{A}$ does also and a force of $-q\,\partial\mathbf{A}/\partial t$ is operating on particle 1. To hold the particle at rest, another external force opposite to this one must be applied, giving rise to a torque with respect to the origin of value $M_z = qr_0\,\partial A_y/\partial t$. So the following angular momentum is being stored in this process:

$$\mathbf{L}_{ext} = 2\mathbf{k}\int M_z\,dt = 2\mathbf{k}\int_\infty^0 qr_0\frac{\partial A_y}{\partial\xi}d\xi = 2qr_0A_y\mathbf{k} = (\mu_0 mq/2\pi r_0)\mathbf{k}, \quad (31)$$

according to Eq. (12) or (14).

In the case of the system analyzed in Sec. III, the linear momentum stored while building up the system in a similar way will be

$$P_y = 2\int_\infty^0 q\frac{\partial A_y}{\partial\xi}d\xi = \frac{\mu_0 mq}{2\pi r_0^2}, \quad (32)$$

which is, in fact, (20) or (21).

APPENDIX

We can obtain the conditions obeyed by the bounded orbits mentioned in Sec. II by making use of the conservation laws (6) and (7) by means of a method similar to the one used when discussing the orbits of the Kepler problem.[23]

In polar coordinates, the constants of the motion of (6) and (7) are

$$E = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + C_1/r, \quad (A1)$$

$$L = r^2\dot{\theta} + C_2/r. \quad (A2)$$

Since at time $t=0$, $\dot{r}_0 = \dot{\theta}_0 = 0$, we get $C_1 = Er_0$ and $C_2 = Lr_0$, and that $u \equiv r/r_0 \geqslant 1$, because from (A1), we have $r \geqslant r_0$. So we get from (A1) and (A2) that

$$E = \frac{1}{2}r_0^2\dot{u}^2\frac{u}{u-1} + \frac{L^2}{2r_0^2}\frac{u-1}{u^3}. \quad (A3)$$

The first term of the right-hand side is always positive, and if E has to remain positive, the following must hold:

$$2r_0^2E/L^2 \geqslant (u-1)/u^3 \geqslant 0. \quad (A4)$$

The function $f(u) = (u-1)/u^3$ has a maximum of value $f(\frac{3}{2}) = \frac{4}{27}$ for $u = \frac{3}{2}$ (see Fig. 3). Thus bounded motions will take place when

$$0 \leqslant r_0^2E/L^2 \leqslant 2/27. \quad (A5)$$

In terms of the constants C_1 and C_2, the above condition reduces to

$$r_0 \leqslant (2C_2^2/27C_1)^{1/3}. \quad (A6)$$

If this condition does not hold, particles will necessarily separate indefinitely from each other.

[1] C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1972), 2nd ed., pp. 144–168.

[2] R. H. Romer, "Angular momentum of static electromagnetic fields," Am. J. Phys. **34**, 772–777 (1966).

[3] E. Corinaldesi, "Angular momentum of a static electromagnetic field," Am. J. Phys. **48**, 83–84 (1980).

[4] F. L. Boos, Jr., "More on the Feynman's disk paradox," Am. J. Phys. **52**, 756–757 (1984).
[5] H. S. T. Driver, "Angular momentum in static electric and magnetic fields: A simple case," Am. J. Phys. **55**, 755–757 (1987).
[6] R. H. Romer, "Electromagnetic angular momentum," Am. J. Phys. **35**, 445–446 (1967).
[7] R. H. Romer, "Electromagnetic angular momentum," Am. J. Phys. **53**, 15–16 (1985).
[8] E. M. Pugh and G. E. Pugh, "Physical significance of the Poynting vector in static fields," Am. J. Phys. **35**, 153–156 (1967).
[9] N. L. Sharma, "Fields versus action-at-a-distance in a static situation," Am. J. Phys. **56**, 420–423 (1988).
[10] J. M. Aguirregabiria, A. Hernández, and M. Rivas, "A Lewis–Tolman-like paradox," Eur. J. Phys. **3**, 30–33 (1982).
[11] A. Butoli, "Action et reaction en electrodynamique," Eur. J. Phys. **10**, 59–60 (1989).
[12] J. Higbie, "Angular momentum in the field of an electron," Am. J. Phys. **56**, 378–379 (1988).
[13] R. P. Feynman, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II, pp. 17.5–17.6.
[14] J. M. Aguirregabiria and A. Hernández, "The Feynman paradox revisited," Eur. J. Phys. **2**, 168–170 (1981).
[15] G. G. Lombardi, "Feynman's disk paradox," Am. J. Phys. **51**, 213–214 (1983).
[16] T. Bahder and J. Sak, "Elementary solution to Feynman's disk paradox," Am. J. Phys. **53**, 495–497 (1985).
[17] Tzu-Chiang Ernest Ma, "Field angular momentum in Feynman's disk paradox," Am. J. Phys. **54**, 949–950 (1986).
[18] E. J. Konopinski, "What the electromagnetic vector potential describes," Am. J. Phys. **46**, 499–502 (1978).
[19] W. H. Furry, "Examples of momentum distributions in the electromagnetic field and in matter," Am. J. Phys. **37**, 621–636 (1969).
[20] A. C. Lawson, "Field angular momentum of an electric charge interacting with a magnetic dipole," Am. J. Phys. **50**, 946–948 (1982).
[21] M. G. Calkin, "Linear momentum of quasistatic electromagnetic fields," Am. J. Phys. **34**, 921–925 (1966).
[22] S. Coleman and J. H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys. Rev. **171**, 1370–1375 (1968).
[23] H. G. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980), 2nd ed., pp. 70–105.

Dielectric screening by cavities and the method of images

W. M. Saslow
Department of Physics and Center for Theoretical Physics, Texas A&M University, College Station, Texas 77843-4242

(Received 8 June 1989; accepted for publication 29 September 1989)

A bare multipole potential $\Phi(\mathbf{r}) = (A/r^{l+1})P_l(\cos\theta)$, concentric with a spherical cavity embedded within a dielectric, has its potential within the dielectric changed, not by the factor ϵ^{-1}, but rather by the factor $F(l) = (2l+1)/[l+(l+1)\epsilon]$. Moreover, a multipole source potential in the dielectric $\Phi(\mathbf{r}) = (Ar^l)P_l(\cos\theta)$ has its potential within a concentric spherical cavity changed by a factor $\epsilon F(l)$. The result is that the interaction between two multipoles in widely separated cavities is modified by a factor ϵ^{-1} times a factor $\epsilon F(l)$ for each multipole. By decomposing the bare potential of a point monopole source into its cavity multipole components, screening each component separately, and then resumming, it is shown why the method of images works both for point charges with semi-infinite dielectrics and for line charges with dielectric circular cylinders—but *not* for point changes with dielectric spheres.

I. ON DIELECTRIC SCREENING AND THE METHOD OF IMAGES

When charge is placed within a cavity, the resulting potential depends both upon the dielectric properties of the system and upon the geometry. For point sources and simple cavity geometries, it is possible to solve for the potential exactly. The results are not always obvious. Consider the following.

Since a point charge in a dielectric has its electric field decreased by a factor ϵ^{-1}, and since all multipoles consist of a sum of point charges, it appears reasonable to assume that all multipoles have their electric fields decreased by the same factor. However, if this were so, then a point charge placed off-center within a spherical cavity embedded in a dielectric would be solvable by the method of images, whereas Landau and Lifshitz remark that there is no closed-form solution for this case.[1] This dielectric screening effect should not be confused with local field effects,[2] where one breaks up the response of a dielectric into the far part (which is treated macroscopically) and the near part (which is treated microscopically).[3,4] It is solely a macroscopic question that is under consideration.

Consider a spherical cavity of radius R within an infinite dielectric of dielectric constant ϵ. If a multipole l is at the center of the cavity, with bare potential

$$\Phi_0(\mathbf{r}) = (A/r^{l+1})P_l(\cos\theta), \tag{1}$$

then the effect of the dielectric is to respond with a surface charge density $\sigma(\hat{\Omega})$ on $r = R$, having the symmetry of the multipole. This causes the net potential to take the form

$$(r<R) \quad \Phi(\mathbf{r}) = (A/r^{l+1} + Br^l)P_l(\cos\theta), \tag{2}$$

$$(r>R) \quad \Phi(\mathbf{r}) = (C/r^{l+1})P_l(\cos\theta), \tag{3}$$

where B and $C-A$ are nonzero because of the effect of $\sigma(\hat{\Omega})$. Matching the boundary condition that the tangential component of the electric field $\mathbf{E} = -\nabla\Phi$ is contin-

