V. CONCLUSION

Since most problems that the students have encountered in their physics studies during their introductory courses have nice closed-form solutions, many of the students are impressed by the fact that even the "theoretical solution" requires the use of a computer for calculation. They find strong agreement between their finite difference method, which is easy to understand and the Fourier series solution, which at this point in their education seems to come from out of the blue. For all involved, the use of the liquid crystal to explicitly and graphically show the temperature contours supplies the necessary physical basis that makes this learning experience a success, and it is the judgment of this author that this laboratory exercise has elements that appeal to all levels of student competence.

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An example of surface charge distribution on conductors carrying steady currents

J. M. Aguirregabiria, A. Hernández, and M. Rivas

Física Teórica, Facultad de Ciencias, Universidad del País Vasco, Apdo 644, 48080 Bilbao, Spain

(Received 22 October 1990; accepted 15 July 1991)

In order to constrain electrons to move along ohmic conductors carrying steady currents, there must be a surface charge density that is usually very difficult to calculate. An approximate analytic expression for this surface charge density on a conducting square ring is presented here where the only source of emf is a changing external magnetic field. The corresponding electric field is determined and it is checked that the energy balance for this system holds.

I. INTRODUCTION

The hypothesis of null cross section of a closed conductor carrying a steady current implies the vanishing of the outer electric field created by this system.[1] In fact, it can be shown[2] that the electrostatic field created by the positive ions in the conductor lattice is completely cancelled out by the Liénard–Wiechert fields of the moving electrons.

If, on the contrary, the wire cross section is taken into account, a surface charge density necessarily appears on the ohmic conductor in order to constrain the total electric field to lie along the wire. The purpose of this work is to provide a simple example of the lowest-order contribution to this effect.

It is usually accepted[3,4] that steady currents in a circuit are produced by an electric field which consists of two parts: a conservative one with nonzero divergence $\mathbf{E}_c$ and a divergenceless one with nonvanishing curl $\mathbf{E}_n$. (These are the two parts into which every vector field can be decomposed according to Helmholtz's theorem.)

The conservative field $\mathbf{E}_c$ is produced by a certain charge distribution in the circuit, while $\mathbf{E}_n$ is generated by other means, for example by a chemical process in a battery or by some varying magnetic field. Of course, the ultimate origin of both fields involves charges. The field $\mathbf{E}_n$ is usually localized in certain parts of the circuit (batteries, generators, etc.) and vanishes elsewhere.

The charge distributions that give rise to the conservative field $\mathbf{E}_c$ are usually very weak in practice, but, in general, they are needed both to have an electric field in the direction of the current and to constrain the charge carriers to move along the wires.[5–7] In fact, the field $\mathbf{E}_n$ does not usually lie along the wire, and it is precisely the field $\mathbf{E}_c$ which compensates this deviation. The experimental measurements of these charge distributions are discussed in various works.[8–10]

In consequence it is pedagogically interesting to obtain such charge distributions, which in general yield a nonvanishing external electric field, even in stationary situations. However, few explicit analytic samples are found in the literature, mainly due to the fact that the geometry of the circuit leads to very complicated expressions for the surface and volume charge density distributions. Under certain conditions,[11] it turns out that the surface charge density on a straight wire changes linearly along the conductor. A quantitative analysis of currents in closed circular circuits can also be done in two dimensions.[12] The examples of an infinite straight cylindrical conductor has been worked out by several authors; the pertinent papers are cited in Ref. 12.

We consider here a circuit of very simple geometric shape, namely, a squared coil. An approximate analytic expression is given for the surface charge distribution σ on the wire, if the wire's thickness is small compared with the remaining dimensions of the circuit. It should be stressed that in this example the only source of emf is an external varying magnetic field and that the charge distributions are necessary only to constrain the total electric field to lie along the wire.

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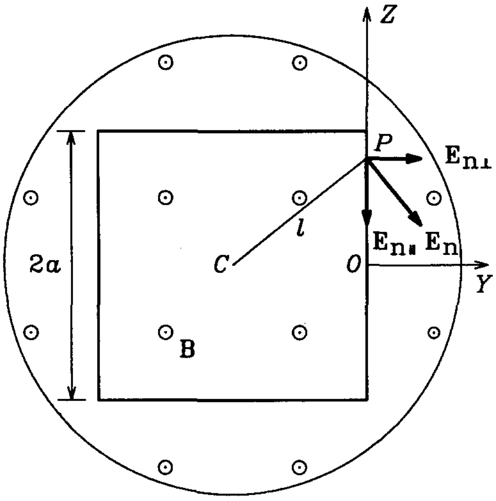

Fig. 1. The squared ring is placed in a uniform magnetic field region. Notice that the origin of coordinates is on the east leg of the circuit, similarly as in Fig. 2.

II. THE CIRCUIT

Let us consider a cylindrical region (see Fig. 1) where there is a uniform magnetic field $\mathbf{B}_{\text{ext}}$ parallel to the cylinder axis. We shall assume that the field intensity is increasing slowly at a constant rate dB_{ext}/dt. This field can be produced inside a very large solenoid of circular cross section, and it is compatible with Maxwell's equations.[13] Because of the symmetry of the system it is easy to see[14] that the electric field $\mathbf{E}_n$ induced at a point P by the varying magnetic field is orthogonal to the radius vector **CP** and has an intensity given by

$$E_n = \frac{l}{2}\frac{dB_{\text{ext}}}{dt} = kl, \tag{1}$$

where $k = 1/2\ (dB_{\text{ext}}/dt)$ is constant.

We now place inside that region a squared circuit of side $2a$, orthogonal to the magnetic field $\mathbf{B}_{\text{ext}}$. The circuit has its center on the cylinder axis and is made up of a cylindrical wire of radius $c \ll a$ and ohmic material of constant resistivity η, so that its total resistance is

$$R = \eta(8a/\pi c^2). \tag{2}$$

Since dB_{ext}/dt is constant, the induced emf in the circuit gives rise to a constant current:

$$I = \frac{4a^2\, dB_{\text{ext}}/dt}{R} = \frac{ka\pi c^2}{\eta}. \tag{3}$$

The field lines of the current density vector **J** are confined within the wire, lying parallel to its axis. Under the hypothesis of a very thin conductor we can assume that **J** is constant over a cross section and, because of charge conservation, that it has the same modulus in every cross section:

$$J = ka/\eta. \tag{4}$$

Since the material of the circuit is ohmic, the resulting electric field everywhere points in the direction of **J**. Thus there necessarily appears a charge distribution on the coil that generates an electric field $\mathbf{E}_c$ at every point such that the compound field $\mathbf{E} = \mathbf{E}_c + \mathbf{E}_n$ will lie along the wire in the direction of **J**.

By using results (1) and (4) and Ohm's law, $\mathbf{E} = \eta\mathbf{J}$, we can see that, under the approximation $c \ll a$, the components of both fields in the directions parallel and orthogonal to the current are

$$E_{n\parallel} = ka, \quad E_{n\perp} = kz, \tag{5}$$

$$E_{c\parallel} = 0, \quad E_{c\perp} = -kz. \tag{6}$$

Obviously Eqs. (5) and (6) hold only for the East leg of the circuit, but analogous expressions are valid for the remaining parts. It goes without saying that the whole analysis is not valid near the four corners of the square, where fields and charge distributions presumably have a complicated structure. Moreover, as we shall see in the next section, the charge distribution on each of the circuit legs generates a field outside the conductor. The effect of this field on the other sides of the coil has not been taken into account since it is negligible except in the neighborhood of the corners if, as assumed, $c \ll a$. See the Appendix. Note also that in our particular case $E_{c\parallel} = 0$, but that in general the electrostatic field has a nonvanishing component in the current's direction.

In the previous discussion, the magnetic force on the charge carriers has not been taken into consideration, i.e., $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$ has been ignored. If it is considered, there appears inside the wire another transverse electric field[15] that is orthogonal to the current direction and has the value $E_m = vB$. It will give rise to additional surface and volume charge distributions,[6] but for a copper conductor, with charge carrier density $n \approx 8\times10^{28}\ \text{m}^{-3}$ and resistivity $\eta \approx 1.75\times10^{-18}\ \Omega\text{m}$, and a magnetic field as high as 0.1 T we have that $E_m/E_c = (B/ne\eta)a/z \approx 4.5\times10^{-4}a/z$. So this magnetic contribution may be neglected save in the immediate vicinity of $z = 0$. For simplicity, we will ignore in the following this minute and independent Hall effect. Consequently, there is no volume charge density as the current is stationary and therefore

$$\text{div}\,\mathbf{E} = \eta\,\text{div}\,\mathbf{J} = 0. \tag{7}$$

III. THE SURFACE CHARGE DENSITY

To obtain the charge distribution that yields a field orthogonal to the wire of value $E_c = -kz$, we first recall the result of a somewhat similar but simpler problem. The surface charge density on an infinite conducting cylinder, initially discharged, under the effect of an external uniform electric field $\mathbf{E}_0$ orthogonal to its axis is given by[16]

$$\sigma = 2\epsilon_0 E_0 \sin\varphi, \tag{8}$$

the whole cylinder being equipotential and having in its interior a constant electric field produced by σ and of value $-\mathbf{E}_0$ that cancels out the external field. We have taken $\mathbf{E}_0 = E_0\mathbf{j}$ in the coordinate system depicted in Fig. 2.

Now, if we tentatively substitute kz for E_0 in (8), the resulting charge distribution produces a field $\mathbf{E}_c$ that satisfies the requirements in Eq. (6). This will be directly proven in the Appendix, but let us give now a more systematic way of finding the surface charge density.

First of all, we realize that $-kz\mathbf{j}$ is not a conservative field. In fact, within the wire the conservative field is more precisely given by $\mathbf{E}_c = -kz\mathbf{j} - ky\mathbf{k}$, which is conservative. Everywhere within the wire $|y| < c$, so we may choose to absorb the $-ky\mathbf{k}$ term into the other term of order (c/a) which we have been ignoring to this point and write

$$\begin{aligned}\mathbf{E}_c &= -kz\mathbf{j} + O(c/a)\\ &= -kz(\sin\varphi\,\mathbf{u}_\rho + \cos\varphi\,\mathbf{u}_\varphi) + O(c/a).\end{aligned} \tag{9}$$

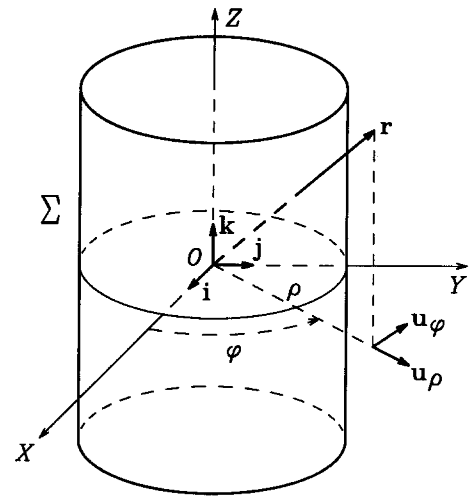

Fig. 2. Cartesian and cylindrical coordinates in the wire's East leg.

In order to elucidate the order of each quantity, we shall use [instead of the polar coordinates (ρ,φ,z)] the dimensionless variables $(\bar{\rho},\varphi,\bar{z})$, where

$$\bar{\rho}=\frac{\rho}{c}=\frac{\rho}{\epsilon a},\quad \bar{z}=\frac{z}{a},\quad \epsilon=\frac{c}{a}, \tag{10}$$

so that the interior of the cylinder is $-1<\bar{\rho},\bar{z}<1$, and our conservative field

$$\mathbf{E}_c=-ka\bar{z}(\sin\varphi\mathbf{u}_\rho+\cos\varphi\mathbf{u}_\varphi)+O(\epsilon) \tag{11}$$

can be written as

$$\begin{aligned}\mathbf{E}_c&=-\operatorname{grad}V\\ &=-\frac{1}{\epsilon a}\left(\frac{\partial V}{\partial\bar{\rho}}\mathbf{u}_\rho+\frac{1}{\bar{\rho}}\frac{\partial V}{\partial\varphi}\mathbf{u}_\varphi+\epsilon\frac{\partial V}{\partial\bar{z}}\mathbf{k}\right),\end{aligned} \tag{12}$$

where

$$V=\epsilon ka^2\bar{z}\bar{\rho}\sin\varphi+O(\epsilon^2). \tag{13}$$

Obviously, this internal potential satisfies Laplace's equation:

$$\nabla^2V=\frac{1}{\epsilon^2a^2}\left[\frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}\left(\bar{\rho}\frac{\partial V}{\partial\bar{\rho}}\right)+\frac{1}{\bar{\rho}^2}\frac{\partial^2V}{\partial\varphi^2}+\epsilon^2\frac{\partial^2V}{\partial\bar{z}^2}\right]=0. \tag{14}$$

The lowest order of the potential outside the conductor must satisfy

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial\bar{\rho}}\left(\bar{\rho}\frac{\partial V_0}{\partial\bar{\rho}}\right)+\frac{1}{\bar{\rho}^2}\frac{\partial^2V_0}{\partial\varphi^2}=0. \tag{15}$$

The solutions of this equation which are periodic in φ can be written as[17]

$$\begin{aligned}V_0=(a_0+b_0\ln\bar{\rho})&+\sum_{n=1}^{\infty}(a_n\bar{\rho}^{-n}+b_n\bar{\rho}^n)\\ &\times(c_n\cos n\varphi+d_n\sin n\varphi),\end{aligned} \tag{16}$$

where a_n, b_n, c_n, and d_n are arbitrary functions of $\bar{z}$. Since we expect the potential to decrease as $\bar{\rho}$ increases, we must take $b_n=0$. Furthermore, the continuity of the potential at the surface of the conductor $(\bar{\rho}=1,|\bar{z}|<1)$ gives

$$V=V_0+O(\epsilon^2)=\epsilon(ka^2\bar{z}/\bar{\rho})\sin\varphi+O(\epsilon^2). \tag{17}$$

Therefore, for $|z|<a$, the external conservative field will be

$$\mathbf{E}_c=(kzc^2/\rho^2)(\sin\varphi\mathbf{u}_\varphi-\cos\varphi\mathbf{u}_\rho)+O(c/a), \tag{18}$$

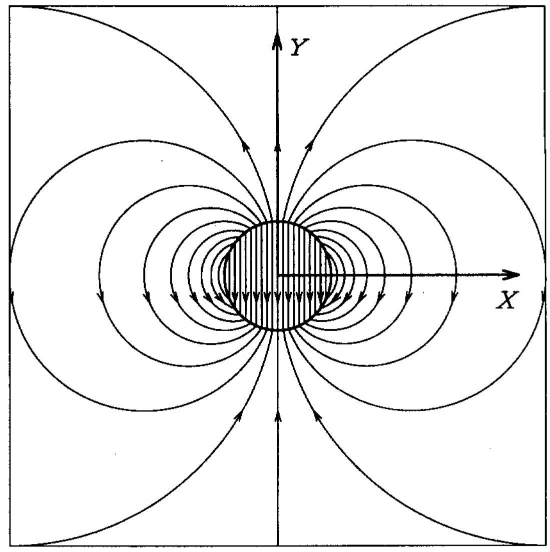

Fig. 3. Field lines of the electrostatic field created by the surface charge distribution on the east leg. A plane $z=\text{const}>0$ is displayed.

that vanishes in the limit $c\to0$ as expected.[1,2] The field lines of the partial field $\mathbf{E}_c$ given by (9) and (18) can be easily computed and are depicted in Fig. 3 in a plane $z=\text{const}>0$.

Comparing (9) and (18) we see that the parallel component of the electric field is continuous at the surface, while the discontinuity in the orthogonal component gives the expected surface charge density:

$$\sigma=2\epsilon_0kz\sin\varphi+O(c/a). \tag{19}$$

IV. POYNTING THEOREM

It follows from (5) and (18) that the lowest-order approximation to the total electric field $\mathbf{E}=\mathbf{E}_c+\mathbf{E}_n$ at any exterior point, but very close to the conductor surface, is

$$\mathbf{E}=2kz\sin\varphi\mathbf{u}_\rho-ka\mathbf{k}, \tag{20}$$

while the magnetic field is given, to the same approximation, by

$$\begin{aligned}\mathbf{B}&=(\mu_0I/2\pi c)\mathbf{u}_\varphi+B_{\text{ext}}\mathbf{i}\\ &=-(\mu_0kac/2\eta)\mathbf{u}_\varphi+B_{\text{ext}}\mathbf{i}.\end{aligned} \tag{21}$$

Let us consider a cylinder Σ of unit length having the same axis and radius as the wire. The outgoing flux of the Poynting vector through Σ is

$$\oint_\Sigma\mathbf{S}\cdot d\mathbf{s}=\frac{1}{\mu_0}\oint_\Sigma(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{s}=-\frac{\pi k^2a^2c^2}{\eta}, \tag{22}$$

which, as required by the Poynting theorem on the energy, is opposite to the value RI^2, where R is the ohmic resistance of the piece of wire enclosed by Σ.

ACKNOWLEDGMENTS

We thank the anonymous referees for their valuable criticisms and suggestions. The work of JMA and MR was supported by the Universidad del País Vasco under Contract No. UPV 172.310-E023/90.

APPENDIX: ELECTRIC FIELD PRODUCED BY THE SURFACE CHARGE DENSITY

In this appendix we shall perform a direct computation of the electric field produced by the surface charge distribution given by (19). In this way we shall be able to recover results (9) and (18) and to prove that for $|z| > a$, $\mathbf{E}_c = O(c/a)$. This result is beyond the scope of the method used in Sec. III. It shows that the electrostatic field produced by the charge distribution in a circuit's leg satisfies $E_{cz} = O(c/a)$ everywhere outside it. This fact gives support to the approximation in which we neglect the action on each leg of the electrostatic field produced by the others.

The electric field created at a point $\mathbf{r}$ by a surface charge distribution σ is

$$\mathbf{E}_c(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}')(\mathbf{r}-\mathbf{r}')ds'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{A1}$$

For a cylinder of radius c we have $\mathbf{r}' = c\cos\varphi\mathbf{i} + c\sin\varphi\mathbf{j} + z'\mathbf{k}$ and $ds' = c\,d\varphi\,dz'$. In the dimensionless variables $\bar{x} = x/c$, $\bar{y} = y/c$, and $\bar{z} = z/a$, the inner region of the cylinder is given by the conditions $0 \leqslant \bar{x}^2 + \bar{y}^2 < 1$, $|\bar{z}| < 1$, and expression (A1) takes the form

$$\mathbf{E}_c = \frac{ka\epsilon}{2\pi}\int_0^{2\pi} d\varphi \sin\varphi \int_{-1}^{1} \frac{\bar{z}'[\epsilon(\bar{x}-\cos\varphi)\mathbf{i} + \epsilon(\bar{y}-\sin\varphi)\mathbf{j} + (\bar{z}-\bar{z}')\mathbf{k}]}{\{\epsilon^2[(\bar{x}-\cos\varphi)^2 + (\bar{y}-\sin\varphi)^2] + (\bar{z}-\bar{z}')^2\}^{3/2}}\,d\bar{z}'. \tag{A2}$$

For $\alpha > 0$ and $|\bar{z}| \neq 1$ we have

$$\lim_{\epsilon\to 0}\int_{-1}^{1} \frac{\epsilon^2\bar{z}'\,d\bar{z}'}{[\epsilon^2\alpha + (\bar{z}-\bar{z}')^2]^{3/2}} = \theta(1-|\bar{z}|)\frac{2\bar{z}}{\alpha},$$
$$\lim_{\epsilon\to 0}\int_{-1}^{1} \frac{\epsilon\bar{z}'(\bar{z}-\bar{z}')d\bar{z}'}{[\epsilon^2\alpha + (\bar{z}-\bar{z}')^2]^{3/2}} = 0, \tag{A3}$$

where θ stands for the Heaviside step function. Thus

$$\mathbf{E}_c = \theta(1-|\bar{z}|)\frac{ka\bar{z}}{\pi} \times \int_0^{2\pi} \frac{\sin\varphi(\bar{x}-\cos\varphi)\mathbf{i} + \sin\varphi(\bar{y}-\sin\varphi)\mathbf{j}}{(\bar{x}-\cos\varphi)^2 + (\bar{y}-\sin\varphi)^2} \times d\varphi + O(\epsilon). \tag{A4}$$

The following two integrals can be readily computed in the complex plane:

$$\int_0^{2\pi} \frac{\sin\varphi(\bar{x}-\cos\varphi)d\varphi}{(\bar{x}-\cos\varphi)^2 + (\bar{y}-\sin\varphi)^2} = \begin{cases} 0, & \text{if } \bar{x}^2+\bar{y}^2 < 1; \\ 2\pi\bar{x}\bar{y}/(\bar{x}^2+\bar{y}^2)^2, & \text{if } \bar{x}^2+\bar{y}^2 > 1, \end{cases}$$

$$\int_0^{2\pi} \frac{\sin\varphi(\bar{y}-\sin\varphi)d\varphi}{(\bar{x}-\cos\varphi)^2 + (\bar{y}-\sin\varphi)^2} = \begin{cases} -\pi, & \text{if } \bar{x}^2+\bar{y}^2 < 1; \\ -\pi(\bar{x}^2-\bar{y}^2)/(\bar{x}^2+\bar{y}^2)^2, & \text{if } \bar{x}^2+\bar{y}^2 > 1, \end{cases} \tag{A5}$$

and one obtains expressions (9) and (18) for the electric field created by the charge distribution, and $\mathbf{E}_c = O(c/a)$, for $|z| > a$.

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