Surface charges and energy flow in a ring rotating in a magnetic field

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(Received 21 July 1995; accepted 12 October 1995)

An ohmic ring that rotates with constant angular velocity in an external uniform magnetic field is considered as a simple model for a current generator. Under the assumption that all quantities vary slowly in time, the lowest-order approximation to the surface charge density is found. The flux of the Poynting vector through the loop surface is also computed. Unlike in the examples that are given in textbooks, this flux is not always incoming: It has the outgoing direction around the loop parts where the electrons are moving against electrostatic forces.

I. INTRODUCTION

The surface charge distributions on conductors carrying steady currents have been analyzed from both the theoretical and the experimental point of view.[1,2] In general, these surface charges have a twofold task: They constrain the charge carriers to move along the wires and they contribute to make sure that the current is the same in all sections of the circuit, even in those points where there is no other force acting on the conducting electrons. This last aspect has been extensively analyzed for stationary currents in infinitely long wires.[3–19] The analysis of the surface charge density has also a pedagogical interest because it is the link between electrostatics and electric circuits. Furthermore, the surface charges produced on the circuitry can damage the electronic equipment.[16] Nevertheless, only a few complete circuits have been considered in detail, the reason being probably the computational difficulties that arise in such a study. Heald[20] considered a couple of infinite circuits. Later, the present authors[21] analyzed in a first-order approximation the surface charge density on a squared loop located in a magnetic field varying linearly in time. In that example, the surface charges were needed only to guide the electrons along the loop. More recently, Saslow[22] has considered in detail the surface charges of a circuit in which a spherical battery is embedded in an infinite conducting medium.

In this paper we calculate the lowest-order approximation to the surface charge distribution on a circular loop which is rotating in a uniform external magnetic field. Unlike in the aforementioned examples, in this case the current is (slowly) changing in time and the surface charges are necessary both to guide the electrons and to establish the same current intensity everywhere.

On the other hand, different textbooks[23,24] show that the flux of the Poynting vector entering a part of the circuit exactly balances the power lost there by Joule effect. Heald[25] has analyzed several elementary models in order to clarify the essential features of energy flow in circuits involving electromagnetic inductions. He has stressed the difficulty of calculating the electric field and Poynting vector in circuits.

Since the ultimate source of the power lost in a resistance is a battery or another source of e.m.f., it would be interesting to show explicitly the flow of the electromagnetic energy from the battery to the conductor where it is dissipated. We use the aforementioned system as a simple but plausible model for an alternating current generator, in which it is shown that the electromagnetic energy flows from some parts of the circuit to the remaining ones.

The problem is described in Sec. II while Sec. III is devoted to the explicit calculations of the surface charge densities under the appropriate assumptions. In Sec. IV the Poynting vector field is computed and the energy balance is performed.

II. THE CIRCUIT

Let us consider a circular loop made of ohmic material of resistivity η. We shall assume that the radius a of its cross section is much smaller than the loop radius R.

The loop is rotating inside a uniform magnetic field **B** with a constant angular velocity ω around a diameter perpendicular to the external magnetic field **B**. This diameter is taken along the OZ axis in Fig. 1.

The induced e.m.f. is given by

$$\mathscr{E}=-\frac{d\Phi}{dt}=-\pi R^2K+O(\epsilon),\quad K\equiv\omega B\cos\omega t. \tag{1}$$

All the approximations throughout this paper will be expressed in terms of the small parameter $\epsilon\equiv a/R\ll 1$, and only lowest order contributions will be kept.

The magnetic flux is considered positive if the external magnetic field **B** has a positive projection in the positive direction of the OX axis. According to the usual convention we will say that the current is positive when it flows in the direction of increasing φ.

If the wire resistance is $\mathscr{R}$, the current in the loop is

$$I=\frac{\mathscr{E}}{\mathscr{R}}=-\frac{\pi a^2RK}{2\eta}+O(\epsilon) \tag{2}$$

and when $K>0$ it will be negative and will flow against the positive direction depicted in Fig. 1. To write down Eq. (2) we have assumed that ω is small enough to have

$$R\ll\frac{2\pi c}{\omega}=\lambda, \tag{3}$$

where λ is the wavelength of an electromagnetic wave of angular frequency ω. In this way, we can assume in a first approximation that the system is quasistationary.

The lowest order of the current density inside the wire is uniform and equal to $j=-KR/2\eta+O(\epsilon)$. According to Ohm's law, the net force per unit charge acting on every charge carrier is

$$\frac{F}{q}=\eta j=-\frac{1}{2}KR+O(\epsilon) \tag{4}$$

and points along the wire direction. On the other hand, by using the electric field **E** inside the conductor and the exter-

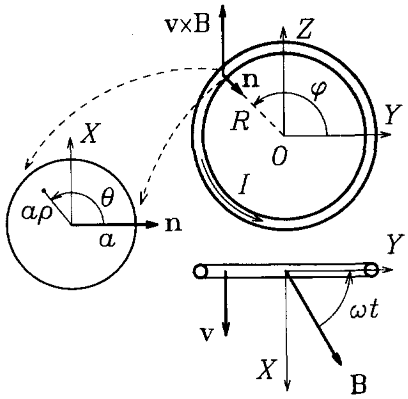

Fig. 1. An ohmic ring is rotating inside a uniform external magnetic field **B** with constant angular velocity ω around the OZ axis. Diagrams on the right display the ring and a top view showing the relative position of the magnetic field with respect to the ring plane. The left picture represents its cross section, at a different scale. The two set of coordinates, (x,y,z) and (ρ,θ,φ), are also shown, where **n** is the unit vector toward the center of the ring.

nal magnetic field **B**, we can express this force as follows:

$$\frac{\mathbf{F}}{q}=\mathbf{E}+\mathbf{v}\times\mathbf{B}. \tag{5}$$

In our first-order approximation, we neglect in expression (5) the magnetic field created by the induced current and consider that the velocity **v** is due exclusively to the ring motion. To go beyond this approximation we should take into account the motion of the electrons inside the wires, which would give new contributions to both factors in the last term of (5) through the Hall effect and through the (small) mean velocity of the electrons.

III. SURFACE CHARGE DENSITY

To simplify the analysis we shall consider a given but arbitrary instant of time and locate the laboratory frame in such a way that the ring is contained on the YZ plane at that instant of time. The Z axis is chosen in the direction of the angular velocity and the magnetic field **B** lies parallel to the XY plane (see Fig. 1). We shall also use a second set of orthogonal coordinates, (ρ,θ,φ), defined as follows:

$$\begin{aligned} x&=r\sin\theta=\epsilon R\rho\sin\theta,\\ y&=(R-r\cos\theta)\cos\varphi=R(1-\epsilon\rho\cos\theta)\cos\varphi,\\ z&=(R-r\cos\theta)\sin\varphi=R(1-\epsilon\rho\cos\theta)\sin\varphi. \end{aligned}\tag{6}$$

These equations define a pure mathematical transformation and not a change of reference frame. Angle φ characterizes each cross section of the ring and the points of a given cross section are described by their polar coordinates (r,θ) (see the left most picture of Fig. 1). The radial coordinate r is written in terms of the wire radius a and the dimensionless variable ρ, $r=a\rho$. Vector **n** is the unit vector directed toward the ring center O.

The ohmic material lies inside the region $0\leq\rho\leq1$, while $\rho>1$ represents the outer part of the ring. The above transformations (6) are not globally invertible. For instance, all points of the ring's symmetry axis belong to every cross section and thus φ is not defined for them. Even more, cross sections corresponding to the values φ and $\varphi+\pi$ are exactly on the same plane, so that every point of this plane has a double representation in terms of these coordinates. This difficulty is easily avoided by restricting transformations (6) to a tube surrounding the ring, of radius less than R.

To have a self-consistent perturbation scheme we will further restrict the transformation equations to the neighborhood of the tube: We shall always assume that ρ is of the order of the unity in an expansion in powers of ϵ. Notice that, in particular, points close to the ring center O correspond to values $\rho\simeq1/\epsilon$ and lie, in consequence, outside the domain of validity of coordinates (ρ,θ,φ). The analysis will be valid only near (or inside) the ring.

As depicted in Fig. 1 the $\mathbf{v}\times\mathbf{B}$ vector lies on the YZ plane, and its modulus is $vB\cos\omega t$, where $v=\omega R\cos\varphi$ is the velocity of the corresponding cross section. In coordinates (ρ,θ,φ) the $\mathbf{v}\times\mathbf{B}$ force per unit charge has the following components:

$$\begin{aligned} (\mathbf{v}\times\mathbf{B})_\varphi&=-KR\cos^2\varphi+O(\epsilon),\\ (\mathbf{v}\times\mathbf{B})_\rho&=KR\cos\varphi\sin\varphi\cos\theta+O(\epsilon),\\ (\mathbf{v}\times\mathbf{B})_\theta&=-KR\cos\varphi\sin\varphi\sin\theta+O(\epsilon). \end{aligned}\tag{7}$$

According to (4), (5), and (7), the surface charges must necessarily create inside the conductor the following electric field:

$$\begin{aligned} E_\varphi&=KR(\cos^2\varphi-\tfrac{1}{2})+O(\epsilon),\\ E_\rho&=-KR\cos\varphi\sin\varphi\cos\theta+O(\epsilon),\\ E_\theta&=KR\cos\varphi\sin\varphi\sin\theta+O(\epsilon). \end{aligned}\tag{8}$$

Notice that the longitudinal component of the electric field is opposite to the current when $\cos^2\varphi>1/2$ and that the magnitude of the field has a constant value $KR/2$.

In these coordinates the divergence operator is

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon R\rho(1-\epsilon\rho\cos\theta)}\left\{\frac{\partial}{\partial\rho}(\rho(1-\epsilon\rho\cos\theta)E_\rho)+\frac{\partial}{\partial\theta}((1-\epsilon\rho\cos\theta)E_\theta)+\epsilon\frac{\partial}{\partial\varphi}(\rho E_\varphi)\right\},\tag{9}$$

and to lowest order in ϵ is given by

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon R\rho}\left(\frac{\partial}{\partial\rho}(\rho E_\rho)+\frac{\partial}{\partial\theta}E_\theta+O(\epsilon)\right),\tag{10}$$

In this lowest-order quantity there is no φ contribution.

Calculation using Eqs. (8) shows that the divergence of **E** vanishes inside the conductor, to order ϵ, so that no volume charge distribution appears inside the conductor at this lowest order and thus it turns out that this internal electric field will be produced by the surface charge distribution. On the other hand, if we take into account the explicit form of the gradient operator in these coordinates, the electrostatic field should be derived from a potential V satisfying the following conditions:

$$\begin{aligned} E_\varphi&=-\frac{1}{R(1-\epsilon\rho\cos\theta)}\frac{\partial V}{\partial\varphi}=-\frac{1}{R}\frac{\partial V}{\partial\varphi}+O(\epsilon),\\ E_\rho&=-\frac{1}{\epsilon R}\frac{\partial V}{\partial\rho},\\ E_\theta&=-\frac{1}{\epsilon R\rho}\frac{\partial V}{\partial\theta}. \end{aligned}\tag{11}$$

By putting $V=V_0+\epsilon V_1+O(\epsilon^2)$, it is easy to conclude from (8) and (11) that the potential inside the conductor, $\rho\leqslant 1$, is

$$V=-\tfrac{1}{4}KR^2 \sin 2\varphi(1-2\epsilon\rho \cos \theta+\epsilon U(\varphi))+O(\epsilon^2). \tag{12}$$

To determine the function $U(\varphi)$ we should go beyond the lowest order approximation. Fortunately, we do not need it to compute the lowest order of the surface charge density.

To find the electrostatic potential outside the conductor, we note that Laplace's equation reads as follows:

$$\nabla^2 V=\frac{1}{\epsilon^2 R^2}[\nabla_0^2 V+\epsilon\nabla_1^2 V+O(\epsilon^2)]=0, \tag{13}$$

where

$$\nabla_0^2 V=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\theta^2}, \tag{14}$$

and

$$\nabla_1^2 V=\frac{\sin\theta}{\rho}\frac{\partial V}{\partial\theta}-\cos\theta\frac{\partial V}{\partial\rho}. \tag{15}$$

In consequence the two lowest orders of the outer potential must satisfy the following equations:

$$\nabla_0^2 V_0=0,\quad \nabla_1^2 V_0=0,\quad \nabla_0^2 V_1=0. \tag{16}$$

Since $\nabla_0^2 V$ in (13) is precisely the two-dimensional Laplacian operator in polar coordinates, the general solution of $\nabla_0^2 V=0$ can be written as follows:[26]

$$V=(a_0+b_0 \ln\rho)+\sum_{n=1}^{\infty}(a_n\rho^{-n}+b_n\rho^n)\times(c_n \cos n\theta+d_n \sin n\theta), \tag{17}$$

where a_n, b_n, c_n, and d_n are arbitrary functions of the absent variable φ. By assuming that the fields decrease with distance outside the conductor, we conclude that $b_n=0$. On the other hand, by using the continuity of the potential through the conductor surface $\rho=1$ and the expression (12) for the inner potential, it is very easy to see that the outer potential is, for $\rho\geqslant 1$,

$$V=-\tfrac{1}{4}KR^2 \sin 2\varphi[1-2(\epsilon/\rho)\cos\theta+\epsilon U(\varphi)]+O(\epsilon^2). \tag{18}$$

It can be easily checked that the condition $\nabla_1^2 V_0=0$ is satisfied by both inner and outer potentials. Of course, expression (18) for the potential is valid only for $\epsilon\ll 1$ at points outside but close to the ring.

The surface charge density will be given by the jump of the radial component of the electrostatic field:

$$\sigma=\varepsilon_0 \lim_{\delta\to 0}(E_\rho|_{\rho=1+\delta}-E_\rho|_{\rho=1-\delta}) = -\frac{\varepsilon_0}{\epsilon R}\lim_{\delta\to 0}\left(\left.\frac{\partial V}{\partial\rho}\right|_{\rho=1+\delta}-\left.\frac{\partial V}{\partial\rho}\right|_{\rho=1-\delta}\right). \tag{19}$$

The result is

$$\sigma=\varepsilon_0 KR \sin 2\varphi \cos\theta+O(\epsilon). \tag{20}$$

Notice that σ and the electric field E in this lowest order approximation are independent of the radius a of the cross section and of the resistivity of the material, which appears only in the induced current. It is easy to see by integrating expression (20) over the ring surface that the total charge on the conductor vanishes.

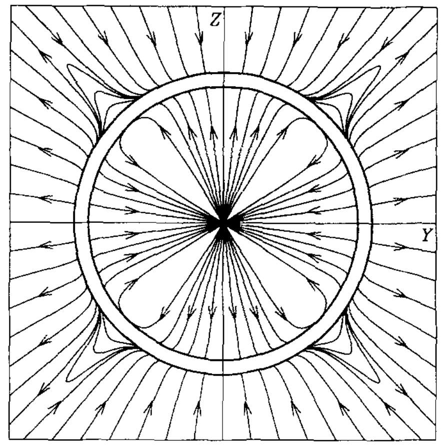

Fig. 2. Field lines when $\epsilon=0.05$, of the extrapolated Poynting field $\mathbf{N}_2$ on the YZ plane. The Poynting field far from the ring, and especially near the center, may be quite different from that shown here; see text.

IV. ENERGY ANALYSIS

The e.m.f. can also be computed from a microscopic point of view, as the line integral of the force per unit charge acting on the charge carriers:

$$\mathscr{E}=\oint_C \frac{\mathbf{F}}{q}\,d\mathbf{l}=\oint_C(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}=\oint_C(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}, \tag{21}$$

because the curl of the electric field vanishes in our approximation. This calculation leads to the expression (1) for the e.m.f..

From (1) we see that the total power supplied to the moving charges and dissipated by Joule effect is

$$\mathscr{E}I=I^2\mathscr{R}=\pi KR^2 I. \tag{22}$$

Let us consider now a ring element of length $dl=Rd\varphi$. The external magnetic field exerts on it a force $d\mathbf{F}=Id\mathbf{l}\times\mathbf{B}$, so that to maintain its motion with constant angular velocity, the total mechanical force in the direction perpendicular to the ring is $d\mathbf{F}_\perp=-d\mathbf{F}$ and since this current element is moving at the speed $\mathbf{v}$ the external power supplied to this element is

$$d\mathbf{F}_\perp\cdot\mathbf{v}=I(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}=KR^2 I \cos^2\varphi d\varphi+O(\epsilon). \tag{23}$$

This value is, of course, equal to $Id\mathscr{E}$ where $d\mathscr{E}=\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}$ is the e.m.f. corresponding to this element. The contribution of the conservative electric field to Eq. (21), $\mathbf{E}\cdot d\mathbf{l}$, is responsible for the change in the electrostatic potential of the charge carriers when they traverse this element.

According to (22) the power dissipated by the Joule effect in this element is

$$I^2 d\mathscr{R}=I^2\mathscr{R}\frac{d\varphi}{2\pi}=\frac{1}{2}KR^2 Id\varphi+O(\epsilon). \tag{24}$$

The Poynting vector

$$\mathbf{N}=\mathbf{N}_1+\mathbf{N}_2,\quad \mathbf{N}_1\equiv\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B},\quad \mathbf{N}_2\equiv\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}_I, \tag{25}$$

can be easily computed on the conductor surface. In our lowest order approximation, the magnetic field is the sum of the external field **B** and the usual azimuthal field $B_I=\mu_0 I/2\pi r$ created by a cylindrical current. The contribution to the Poynting vector due to the external magnetic field, $\mathbf{N}_1$, gives no net contribution to the power entering any closed surface, because outside the currents creating **B** the vector field $\mathbf{N}_1$ is divergenceless.

As for the electric field, only its tangent component at $\rho=1$ is necessary and it can be obtained from (18). The fields contributing to the $\mathbf{N}_2$ vector are $E_\varphi=KR\cos 2\varphi/2$, $E_\rho=-KR\cos 2\varphi\cos\theta/2$ and $B_\theta\equiv B_I=\mu_0 I/2\pi a$. The contribution to the net flux of the term $E_\rho B_\theta$ in the ring direction vanishes. In consequence, the Poynting vector flux through the closed surface $d\Sigma$ of this current element is given by the $E_\varphi B_\theta$ part:

$$\oint_{d\Sigma}\mathbf{N}\cdot d\mathbf{S}=\oint_{d\Sigma}\mathbf{N}_2\cdot d\mathbf{S}=\frac{1}{2}KR^2 I\cos 2\varphi\, d\varphi+O(\epsilon). \tag{26}$$

It should be stressed that according to this result the energy is flowing out of the conductor for $-\pi/4<\varphi<\pi/4$ and $3\pi/4<\varphi<5\pi/4$, while it is entering the remaining parts of the conductor at exactly the same rate.

Comparing (23), (24), and (26), we see that the energy conservation law holds locally: The external power supplied to the element exactly balances the sum of the power dissipated by Joule effect and the (positive or negative) electromagnetic power leaving it.

It would be interesting to draw the current lines of the Poynting vector. Since the current lines of $\mathbf{N}_1$ correspond to a divergenceless vector field they are closed and give no net flux through closed surfaces. In consequence, to ease the understanding of the figure, we have omitted them. Unfortunately we only know the vector $\mathbf{N}_2$ in the surroundings of the ring. To have a qualitative idea of the energy flow we have constructed a new vector field by extrapolating the expression of $\mathbf{N}_2$ to the whole plane. The field lines of this new vector are then readily obtained by using a numerical program[27] to integrate their differential equations and are displayed in Fig. 2. They will approximate the field lines of the vector $\mathbf{N}_2$ at points close to the ring. Far from the ring (and especially near the ring center) the true field lines might happen to be very different from those depicted in the figure.

ACKNOWLEDGMENTS

This work has been partially supported by the Universidad del País Vasco/Euskal Herriko Unibertsitatea under Contract No. UPV/EHU 172.310 EA034/94.

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ANTEDILUVIAN SCIENCE

In view of the levels of precision which have been achieved over the intervening years by others with the use of sophisticated pseudo-potentials, clever expansion methods, modern computers and much ingenuity, I fully appreciate the fact that our original work now seems antediluvian. That, however, is the way of science.

Frederick Seitz, "The Princeton Years and Beyond: 1930–1940," paper delivered at the March, 1992 meeting of the American Physical Society.