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LETTERS AND COMMENTS

Inertial forces and gauge invariance

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Abstract

An elementary mechanical example is discussed in which the appearance of the easiest inertial force leads naturally to two different but equivalent Lagrangians. This provides a family of simple examples to discuss gauge invariance in analytical mechanics. The physical meaning of the gauge in these examples is also analysed.

The equivalence of two Lagrangians differing in a total time derivative is mentioned in many texts about analytical dynamics [1, 2] where it is often proposed as an exercise [3], its proof being elementary. Despite the paramount importance of this result in the Lagrangian formulation of field theories it goes unnoticed by most students of classical mechanics. This may be due to the fact that in courses on analytical mechanics it is only applied (if at all) to the gauge invariance of electrodynamics [3], which is of course its most interesting elementary application but this may be difficult to be fully understood by a student with little background on electromagnetism. The purpose of this note is to discuss a very simple mechanical example in which two natural approaches lead to Lagrangians differing by a gauge transformation.

Let us consider the mathematical pendulum of figure 1. If the suspension point moves with a constant vertical acceleration a, one may directly study the problem in the inertial frame by selecting the angle θ as generalized coordinate. By carefully choosing the axes and the time origin, one can write the Cartesian coordinates of the point mass in the following form

$$x = l\sin\theta \qquad y = \tfrac{1}{2}at^2 - l\cos\theta. \tag{1}$$

Now it is an elementary task to find the Lagrangian

$$L = \tfrac{1}{2}m(l^2\dot\theta^2 + 2alt\dot\theta\sin\theta + a^2t^2) - mg(\tfrac{1}{2}at^2 - l\cos\theta) \tag{2}$$

and the equation of motion

$$\ddot\theta + \frac{g+a}{l}\sin\theta = 0. \tag{3}$$

On the other hand, one may use the reference frame of the suspension point, where

$$x = l\sin\theta \qquad y = -l\cos\theta \tag{4}$$

provided the inertial force $-m\boldsymbol{a}$ is included. (Even students with little knowledge about inertial forces have learned that a linear acceleration of the reference frame produces an artificial

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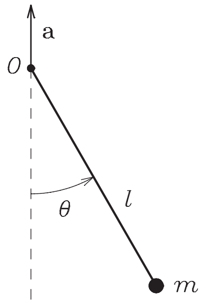

Figure 1. The suspension point of this mathematical pendulum undergoes a constant vertical acceleration.

gravity $\boldsymbol{g}_{\rm art} = -\boldsymbol{a}$.) This force is conservative and supplies the additional potential energy $V_{\rm i} = m\boldsymbol{a}\cdot\boldsymbol{r} = may = -mal\cos\theta$. So, the Lagrangian is written as

$$L' = \tfrac{1}{2}ml^2\dot\theta^2 + m(g+a)l\cos\theta \tag{5}$$

from where one readily recovers (3).

Although Lagrangians (2) and (5) look very different and only the Hamiltonian corresponding to the second one equals the mechanical energy (in the accelerated frame) and is a conserved quantity, they give the same equation of motion because their difference is a total derivative

$$L' - L = \frac{\mathrm{d}}{\mathrm{d}t}\left[malt\cos\theta + \frac{1}{6}ma(g-a)t^3\right]. \tag{6}$$

Similar examples are easily obtained if the acceleration is horizontal (instead of vertical), depends on time or if one considers mechanical systems different from the mathematical pendulum. Extensions to systems of point particles and extended systems (like a rigid solid) are also immediate. In all these cases selecting one of the gauges amounts to choosing the reference frame where the kinetic and potential energies are computed.

To see the meaning of these gauges from a different point of view, let us consider, in a slightly more general context, a reference frame whose position with respect to the inertial system is given by $\boldsymbol{R} = \boldsymbol{R}(t)$. If the relative orientation of both frames is not changing and the Cartesian coordinates of the accelerated frame are used as generalized coordinates for a point particle with potential $V(\boldsymbol{r})$, the Lagrangians obtained in the inertial and accelerated frames of reference are, respectively,

$$L = \tfrac{1}{2}m(\dot{\boldsymbol{R}} + \dot{\boldsymbol{r}})^2 - V(\boldsymbol{r}) \tag{7}$$

$$L' = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - [V(\boldsymbol{r}) + m\ddot{\boldsymbol{R}}\cdot\boldsymbol{r}]. \tag{8}$$

It is easy to check that, of course, these two Lagrangians are equivalent because their difference is a total derivative. But let us reorganize the terms in L as follows

$$L = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - [V(\boldsymbol{r}) - m\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{r}} - \tfrac{1}{2}m\dot{\boldsymbol{R}}^2]. \tag{9}$$

If we drop the last term, which being a total derivative would give no contribution to the equation of motion, we get the equivalent Lagrangian

$$L'' = \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - [V(\boldsymbol{r}) - m\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{r}}]. \tag{10}$$

This can be easily understood as the Lagrangian one would write in the accelerated reference frame if instead of the potential energy $V_{\rm i}(t,\boldsymbol{r}) = m\ddot{\boldsymbol{R}}\cdot\boldsymbol{r}$ one would use the generalized potential energy $U_{\rm i}(t,\dot{\boldsymbol{r}}) = -m\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{r}}$, which gives the same force

$$\boldsymbol{F}_{\rm i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial U_{\rm i}}{\partial\dot{\boldsymbol{r}}} - \frac{\partial U_{\rm i}}{\partial\boldsymbol{r}} = -\frac{\partial V_{\rm i}}{\partial\boldsymbol{r}} = -m\ddot{\boldsymbol{R}} \tag{11}$$

because both energies differ in a total derivative:

$$V_{\rm i} - U_{\rm i} = L'' - L' = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\boldsymbol{R}}\cdot\boldsymbol{r}). \tag{12}$$

From this point of view, the two gauges correspond to two different but equivalent interpretations of the inertial force: either as a generalized conservative force derived from the velocity-dependent $U_{\rm i}(t, \dot{\boldsymbol{r}})$ or as a usual conservative force with potential energy $V_{\rm i}(t, \boldsymbol{r})$.

Finally, let us remark that (10) can also be the Lagrangian of a point charge q moving in an external electromagnetic field with scalar and vector potentials given by $(\phi, \boldsymbol{A}) = (V, m\dot{\boldsymbol{R}})/q$. Since the vector potential does not depend on $\boldsymbol{r}$, there is no magnetic field and $\boldsymbol{A}$ can be made zero by using the gauge transformation defined by the generating function $\Lambda = -(m/q)\dot{\boldsymbol{R}} \cdot \boldsymbol{r}$ as

$$(\phi, \boldsymbol{A}) \longrightarrow \left(\phi - \frac{\partial \Lambda}{\partial t}, \boldsymbol{A} + \nabla \Lambda\right) = \frac{1}{q}(V + m\ddot{\boldsymbol{R}} \cdot \boldsymbol{r}, \boldsymbol{0}). \tag{13}$$

The Lagrangian corresponding to the new potentials in this electromagnetic analogy is precisely (8).

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