Eur. J. Phys. 3 (1982) 30–33. Printed in Northern Ireland

A Lewis–Tolman-like paradox

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Received 4 January 1982

Abstract A non-relativistic Lewis–Tolman-like paradox is proposed. It is checked by direct calculation that the paradox disappears if linear and angular momenta are attached to the static electromagnetic field. The storage of linear momentum in the electromagnetic field during the assembling process is also analysed. Finally a naive model of the electron suggested by this system is proposed.

Laburpena Lewis–Tolman-enaren antzeko paradoxa ez-erlatibista bat aurkezten da. Eremu elektromagnetiko estatikoari momentu lineala eta angeluarra egokitzen bazaizkio, paradoxa desagertu egiten dela ikusten da, kalkulu zuzenaren bidez. Gainera dispositiboaren eratze-prozesuan zehar, momentu lineala eta angeluarra nola metatzen diren ere aztertzen da. Azkenik, sistema honek iradokitako elektroiaren eredu bakun bat aurkezten da.

1. Introduction

In systems in which the internal forces do not obey Newton's third law (the action equals reaction principle), it is necessary, in order to maintain the linear and angular momentum and energy conservation laws, to assign these magnitudes to the interaction fields, giving rise to Poynting's theorem in electrodynamics, and in general to the energy–momentum tensor formalism (Møller 1972); otherwise two kinds of apparent paradoxes could appear.

In the first kind we have systems in which the mechanical linear and/or angular momentum of the system can vary with time while the net external force and torque are zero. To this kind belongs the paradox proposed by Feynman (1964). In the second kind, while the mechanical linear and angular momenta remain constant in time, the net external force and/or torque do not necessarily cancel out, giving rise to the well known Lewis–Tolman (1909) paradox, among others.

In the frame of the relativistic theory of continuous media, von Laue (1924) analysed examples of both kinds of paradox. But even in the non-relativistic approach of electromagnetism those paradoxes appear. In this way we have proposed and analysed (Aguirregabiria and Hernández 1981, to be referred to as AH) a simplified model of Feynman's paradox.

In the present work we analyse a paradox of the second kind, based upon the same device as used in AH. It has to be remarked that while this paradox is not strictly relativistic, not only the total torque of external forces is different from zero (as occurs in the Lewis–Tolman case) but also the total external force is non-zero. Another difference with the Lewis–Tolman paradox mentioned is that this paradox appears in its own rest frame without considering a second moving observer or even how the forces transform.

In §2 the problem is exposed. In §3 it will be checked that the paradox disappears by the explicit calculation of the linear momentum associated to the static electromagnetic field. Section 4 is devoted to comments and discussions and in §5 a naive model of the electron, suggested by the analysed device, is proposed.

2. The problem

Let us briefly describe the model. A thin plastic circular disc of radius a is at rest in the XY plane of a coordinate frame with its symmetry axis along $0Z$. At the origin, the disc has a little hole in which a small circular ring is centred, rigidly attached to the disc.

Let us assume that the ring is made up of a superconducting material and that a constant current is flowing in it, generating a magnetic moment of the form $\boldsymbol{m} = m\boldsymbol{k}$, where $\boldsymbol{k}$ is the unit vector along the $0Z$ axis. The ring is assumed to be small enough to be considered point-like.

A charge q is located at the point with coordinates $(a, 0, 0)$ on the edge of the disc. The system is initially static since the fields do not depend on time and there are no forces acting on the charge and the ring.

Let us assume that the magnetic moment starts decreasing at a small rate $\dot{m} = \mathrm{d}m/\mathrm{d}t$, due to, for instance, a small increase of temperature which gives rise to the appearance of a non-zero resistivity in the ring. We shall accept that this process is slow enough to neglect radiation, relativistic and retarded effects.

The electric field induced by the changing magnetic field, acts on the charge, and in order to leave it at rest, an external force $\boldsymbol{F}_{\mathrm{ext}}$ opposite to that of the field, of value (AH)

$$\boldsymbol{F}_{\mathrm{ext}} = \frac{\mu_0}{4\pi}\frac{\dot{m}q}{a^2}\boldsymbol{j} \tag{1}$$

must be exerted on it; $\boldsymbol{j}$ is a unit vector along the $0Y$ axis. Since during the ring demagnetisation no force is done on the magnetic dipole, $\boldsymbol{F}_{\mathrm{ext}}$ is the total external force acting on the system.

We are facing a situation in which a force and also its torque with respect to the origin are acting on a system and the mechanical linear and angular momenta do not vary with time, in apparent contradiction with classical mechanics theorems.

The total impulse of external force during the dipole annihilation is

$$\boldsymbol{I}_{\mathrm{ext}} = \int \boldsymbol{F}_{\mathrm{ext}}\,\mathrm{d}t = -\frac{\mu_0 q m}{4\pi a^2}\boldsymbol{j} \tag{2}$$

and the angular impulse of the torque of $\boldsymbol{F}_{\mathrm{ext}}$ with respect to the origin is

$$\boldsymbol{J}_{\mathrm{ext}} = \int \boldsymbol{a}\times\boldsymbol{F}_{\mathrm{ext}}\,\mathrm{d}t = -\frac{\mu_0 q m}{4\pi a}\boldsymbol{k}. \tag{3}$$

3. Explanation of the paradox

The linear momentum theorem holds if we add to the mechanical momentum of the system the linear momentum associated to the electromagnetic field.

In the initial stationary situation, this linear momentum of the electromagnetic field is

$$\boldsymbol{P}_{\mathrm{em}} = \int_{\mathbb{R}^3} \varepsilon_0 \boldsymbol{E}\times\boldsymbol{B}\,\mathrm{d}V \tag{4}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields respectively.

In our case,

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}-\boldsymbol{a}}{|\boldsymbol{r}-\boldsymbol{a}|^3} \tag{5}$$

and $\boldsymbol{B} = \boldsymbol{\nabla}\times\mathbf{A}$ where $\mathbf{A}$ is the magnetic vector potential

$$\mathbf{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\boldsymbol{m}\times\frac{\boldsymbol{r}}{r^3}. \tag{6}$$

With the help of the vector identities (Panofsky and Phillips 1975)

$$\boldsymbol{\nabla}\times(\boldsymbol{C}\times\boldsymbol{D}) = \boldsymbol{C}(\boldsymbol{\nabla}\cdot\boldsymbol{D}) - \boldsymbol{D}(\boldsymbol{\nabla}\cdot\boldsymbol{C}) + (\boldsymbol{D}\cdot\boldsymbol{\nabla})\boldsymbol{C} - (\boldsymbol{C}\cdot\boldsymbol{\nabla})\boldsymbol{D} \tag{7}$$

$$\boldsymbol{\nabla}\times(\Phi\boldsymbol{C}) = \Phi\boldsymbol{\nabla}\times\boldsymbol{C} - \boldsymbol{C}\times\boldsymbol{\nabla}\Phi \tag{8}$$

and

$$\frac{\boldsymbol{r}-\boldsymbol{a}}{|\boldsymbol{r}-\boldsymbol{a}|^3} = -\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|}\right) \tag{9}$$

$\boldsymbol{P}_{\mathrm{em}}$ can be expressed as

$$\boldsymbol{P}_{\mathrm{em}} = \frac{\mu_0 q}{4\pi}\int_{\mathbb{R}^3}\delta^3(\boldsymbol{r})\frac{\boldsymbol{r}-\boldsymbol{a}}{|\boldsymbol{r}-\boldsymbol{a}|^3}\times\boldsymbol{m}\,\mathrm{d}V + \frac{\mu_0 q}{(4\pi)^2}\int_{\mathbb{R}^3}\boldsymbol{\nabla}\times\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|}\boldsymbol{\nabla}\left(\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}\right)\right]\mathrm{d}V \tag{10}$$

where $\delta^3(\boldsymbol{r})$ is the Dirac delta.

The second term on the right-hand side of (10) can be transformed in the surface integral

$$\frac{\mu_0 q}{(4\pi)^2}\lim_{R\to\infty}\int_{S(R)}\mathrm{d}\boldsymbol{S}\times\frac{1}{|\boldsymbol{r}-\boldsymbol{a}|}\boldsymbol{\nabla}\left(\frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}\right) \tag{11}$$

where $S(R)$ is a sphere of radius R centred at the origin. The functional structure of this integrand guarantees the vanishing of the integral, and finally

$$\boldsymbol{P}_{\mathrm{em}} = \frac{\mu_0 q}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{a}}{a^3} = q\mathbf{A}(\boldsymbol{a}). \tag{12}$$

In static situations the linear momentum associated with the electromagnetic field of point charges and currents can be written according to Calkin (1966) in the form

$$\boldsymbol{P}_{\mathrm{em}} = \sum_i q_i \mathbf{A}_i$$

with $\mathbf{A}_i$ the vector potential at the location of q_i, in agreement with the result (12).

Furry (1969), making use of a different method from ours, and assuming a magnetic dipole with internal structure, obtains the same formula.

The angular electromagnetic momentum has been obtained before (AH) and is

$$\boldsymbol{L}_{\mathrm{em}} = q\boldsymbol{a}\times\mathbf{A}(\boldsymbol{a}). \tag{13}$$

In the final situation when the dipole has vanished, we have a stationary system with a point charge at rest and an electromagnetic field without linear and angular momenta. Consequently the change in the electromagnetic linear momentum in the demagnetisation process is $-\boldsymbol{P}_{\mathrm{em}}$ which is just the impulse (2). Similarly the change in the electromagnetic angular momentum is $-\boldsymbol{L}_{\mathrm{em}}$ given by (3) (see AH). Remarking that the mechanical linear and angular momenta are zero, before, during and after the demagnetisation process, we see explicitly that assigning momenta to the electromagnetic field alllows the classical theorems on the impulse of external forces and torques still to hold.

4. Comments and discussions

One intuitive way of checking the existence of linear and angular momenta stored in the static electromagnetic field is to consider a possible assembling process of the system, starting from a situation in which charge and dipole are far enough apart and the density of the linear electromagnetic momentum, given by Poynting's vector, is zero.

Let us assume that the point charge is taken from infinity along the $0X$ axis to a point in the neighbourhood of the dipole, at a constant speed $\boldsymbol{v} = v\boldsymbol{i}$ $(v < 0)$ sufficiently small to accept the non-relativistic approach.

The charge creates a magnetic field which is zero at every point on the $0X$ axis, and in particular at the point where the dipole is, giving no torque on it but the force is non-zero due to a non-vanishing gradient of the magnetic field.

In this non-relativistic approach, the magnetic field created by a point charge q located at the point $x\boldsymbol{i}$ and moving with the constant speed $\boldsymbol{v}$ is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 q}{4\pi} \frac{\boldsymbol{v} \times (\boldsymbol{r} - x\boldsymbol{i})}{|\boldsymbol{r} - x\boldsymbol{i}|^3} \tag{14}$$

and the force $\boldsymbol{F}_m$ on the magnetic dipole will be according to Jackson (1975)

$$\boldsymbol{F}_m = \boldsymbol{\nabla}(\boldsymbol{m} \cdot \boldsymbol{B}) = \frac{\mu_0 q m v}{4\pi x^3} \boldsymbol{j} \tag{15}$$

where $v = \mathrm{d}x/\mathrm{d}t$.

This force is equal to the force $\boldsymbol{F}_q$ exerted on the charge by the magnetic field of the dipole (AH). We have in this situation a very clear violation of Newton's third law.

In order that the charge and the dipole do not accelerate, two forces $-\boldsymbol{F}_q = -\boldsymbol{F}_m$ must be applied respectively on them. In this way the total impulse exerted on the system will be

$$\boldsymbol{I} = 2\int -\boldsymbol{F}_q \, \mathrm{d}t = \frac{\mu_0 q m}{4\pi a^2} \boldsymbol{j} \tag{16}$$

which is independent of $\boldsymbol{v}$ and also of the path followed, as can easily be proved. According to the classical theorem on the impulse of external forces, and knowing that when the charge and the dipole were far apart the electromagnetic linear momentum was zero, at the end of the process a linear momentum given by (16) will be stored in the electromagnetic field, i.e. precisely the $\boldsymbol{P}_{\mathrm{em}}$ of (12).

Similarly, the same conclusion is obtained about the impulse of the external torque and the angular momentum (AH).

If a vector of value $\boldsymbol{P}_{\mathrm{em}}$ is supposed to be located at the point charge, the electromagnetic angular momentum can be written

$$\boldsymbol{L}_{\mathrm{em}} = \boldsymbol{a} \times \boldsymbol{P}_{\mathrm{em}}. \tag{17}$$

5. A naive model of the electron

The system analysed above suggests a naive model of a particle just by taking the zero limit of the parameter a; this gives rise to a point-like particle with a charge and a magnetic moment as is the case of the electron, for instance.

However, as usually happens, the electromagnetic linear and angular momenta diverge because of the point-like character of the model. In order to avoid this difficulty one usually assumes that the particle is a small sphere of radius R_0. Let us imagine that it has a surface charge density $\sigma = q/4\pi R_0^2$ and in its centre a point-like magnetic dipole $\boldsymbol{m} = m\boldsymbol{k}$. The electric and magnetic fields are

$$\begin{aligned} \boldsymbol{E} &= 0 && \text{for} \quad r < R_0 \\ \boldsymbol{E} &= \frac{1}{4\pi\varepsilon_0} \frac{q\boldsymbol{r}}{r^3} && \text{for} \quad r > R_0 \end{aligned} \tag{18}$$

$$\boldsymbol{B} = \frac{\mu_0}{4\pi} \left(-\frac{\boldsymbol{m}}{r^3} + \frac{3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r}}{r^5} \right) \qquad \text{for} \quad r \neq 0. \tag{19}$$

The electromagnetic angular momentum with respect to the origin will be given by

$$\boldsymbol{L}_{\mathrm{em}} = \varepsilon_0 \int \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) \, \mathrm{d}V = \frac{\mu_0 q}{(4\pi)^2} \int \left(\frac{\boldsymbol{m}}{r^4} - \frac{(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r}}{r^6} \right) \mathrm{d}V \tag{20}$$

where the integral is extended to the outer region of the sphere since the electric field is zero inside. By symmetry considerations we see that the angular momentum will be along the direction of $\boldsymbol{m}$. In fact, computing the integral of (20) in spherical coordinates we arrive at

$$\boldsymbol{L}_{\mathrm{em}} = \frac{\mu_0 q \boldsymbol{m}}{6\pi R_0}. \tag{21}$$

It is also easy to see by symmetry considerations that the total electromagnetic linear momentum calculated according to (4) is zero, concluding that the angular momentum does not depend on the reference point.

For an elementary particle, the magnetic moment $\boldsymbol{m}$ is related to its intrinsic angular momentum (spin) $\boldsymbol{S}$ by

$$\boldsymbol{m} = g \frac{q}{2M} \boldsymbol{S} \tag{22}$$

where M is the mass and g the gyromagnetic ratio which depends on the internal structure of the particle.

If we identify the spin $\mathbf{S}$ with the electromagnetic angular momentum (21), and take for R_0 the well known classical radius of the electron (Rohrlich 1965)

$$R_0 = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{Mc^2} \qquad (23)$$

we see that the gyromagnetic ratio takes the value $g = 3$. If another internal structure is assumed, for instance if the charge and magnetic moment are uniformly distributed inside the sphere we get the value $g = 2.5$.

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