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Composition law and contractions of the Poincaré group

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Abstract Explicit expressions are given for the Poincaré group composition functions in terms of the physical parameters (space–time displacement, relative velocity and orientation) which relate two inertial observers. By making use of the limiting process of group contraction, the composition functions of the Galilei and Carroll groups are obtained. How these two groups imply some loss of certain relative properties of the Poincaré group is also discussed.

Laburpena Bi inertzi behatzaileri dagozkien parametro fisiko erlatiboak (desplazamendu espazio–denborala, abiadura eta orientazioa) erabiliz, Poincaré-ren taldearen konposaketa-funtzioen adierazpen zehatzak ematen dira. Talde-uzkurduraren bidez, Galileo eta Carroll-en taldeen konposaketa-funtzioak lortzen dira. Talde hauek, Poincaré-renaren erlatibotasunaren propietate batzuren galeraren truke nola lor daitezkeen ere aztertzen da.

1. Introduction

It is well known that a 'relativity principle' is a statement about the existence of a class of equivalent observers, called inertial observers, for whom the laws of physics look the same. Their relative measurements of physical observables are related by some mathematical transformations and it is a fact that these transformations must form a group which makes sense to the transitive character of the equivalence among observers and conversely.

The remarkable fact is that the way two inertial observers relate their measurements of any physical observable is a function only of how they relate their space and time measurements. This is why a relativity principle is always associated with a certain space–time transformation group.

Various transformation groups for defining a relativity principle were elegantly obtained by Bacry and Levy-Leblond (1968) where we find, among others, the Poincaré $\mathscr{P}$, Galilei $\mathscr{G}$ and Carroll $\mathscr{C}$ groups.

In the Galilei relativity principle we deal with the Galilei group, which is usually parametrised in terms of physical observables such as the relative velocity and orientation and the space and time displacements between observers. It has been extensively studied and we just mention here the excellent work (Levy-Leblond 1971) where additional references can be found.

In special relativity the group is the Poincaré group and several parametrisations of it are known, including those given in terms of physical observables. However, in contrast to the Galilei group, no explicit formulae are found in standard textbooks for the most general composition law of the physical parameters that characterise the relative situations of the observers. In fact, frequently only Lorentz boosts or infinitesimal transformations are needed and by making use of the powerful Lie algebra methods and Lie group theory, very general results can be elegantly reached.

On the other hand, however, the example of the Galilei group suggests to us that these explicit formulae can be useful in some contexts. For instance, they can allow a student with little or no knowledge of Lie groups to understand some results which are usually attained by more sophisticated methods.

The aim of this paper is to illustrate the potential pedagogical interest of the general group law in terms of physical parameters of the Poincaré group.

A parametrisation of this group and the corresponding general composition law is given in § 2, and it is mentioned how the Thomas precession can be obtained as a direct consequence of this transformation law.

The usefulness of the explicit expressions for this

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law is explored in § 3, where the Galilei and Carroll groups are derived from the Poincaré group by using the limiting process of group contraction. Section 4 is devoted to some physical comments on these three groups and to what extent the groups $\mathcal{G}$ and $\mathcal{C}$, still being, in principle, possible candidates for implementing a relativity principle, have however lost some relativity properties when compared with $\mathcal{P}$.

2. The Poincaré group

The Poincaré group is usually defined as the group of linear transformations of Minkowski space–time which leave the distance $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ between two space–time events invariant ($g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric tensor).

If $x^\mu \equiv (ct, \boldsymbol{x})$ are the coordinates of a space–time point, any Poincaré transformation can be written as:

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu \tag{1}$$

where a^μ represents a space–time translation and Λ a Lorentz transformation which verifies:

$$\Lambda^\mu_\nu g_{\mu\rho} \Lambda^\rho_\sigma = g_{\nu\sigma}. \tag{2}$$

We shall also use the shorthand notation (a, Λ) to describe the Poincaré transformation in equation (1). The composition of two transformations can be obtained by acting twice on x in the form (1), and gives the group law $(a', \Lambda')(a, \Lambda) = (\Lambda' a + a', \Lambda'\Lambda) = (a'', \Lambda'')$, i.e.:

$$a''^\mu = \Lambda'^\mu_\nu a^\nu + a'^\mu \tag{3a}$$

$$\Lambda''^\mu_\nu = \Lambda'^\mu_\sigma \Lambda^\sigma_\nu. \tag{3b}$$

The four-vector $a^\mu \equiv (cb, \boldsymbol{a})$ is parametrised in terms of the real number b (with dimensions of time), which characterises the time translation, and the real three-vector $\boldsymbol{a}$, which is interpreted as the space translation, c being the speed of light.

It is well known (Møller 1972) that every Lorentz transformation can be broken down into a product of a pure Lorentz transformation and a rotation. This can be done in two ways. We shall always understand it in the form $\Lambda = LR$. For completeness, we show in the appendix a possible method for obtaining this result.

L is usually parametrised in terms of the relative velocity between observers (Møller 1972), being $0 \leqslant v < c$, and its action on any space–time point gives:

$$t' = \gamma t + \gamma c^{-2}(\boldsymbol{v} \cdot \boldsymbol{x}) \tag{4a}$$

$$\boldsymbol{x}' = \boldsymbol{x} + \gamma \boldsymbol{v} t + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot \boldsymbol{x})\boldsymbol{v} \tag{4b}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

Every rotation will be characterised by the rotation axis (more precisely, by a unit vector $\boldsymbol{e}$ along it), and by the clockwise rotated angle $\alpha \in [0, \pi]$ when looking along the direction given by $\boldsymbol{e}$. For angles greater than π we take $\boldsymbol{e}$ in the reverse direction. From $\boldsymbol{e}$ and α we define the real adimensional three-vector $\boldsymbol{\mu} = \boldsymbol{e} \tan(\frac{1}{2}\alpha)$ in terms of which every rotation $R \equiv R(\boldsymbol{\mu})$ will be expressed in matrix form (Pars 1968) as:

$$R(\boldsymbol{\mu})^i_j = \frac{1}{1+\mu^2}[(1-\mu^2)\delta^i_j + 2\mu^i\mu_j - 2\varepsilon^i_{jk}\mu^k] \tag{5}$$

where $\mu^i = \mu_i$, δ^i_j is the Kronecker delta and ε^i_{jk} is the completely antisymmetric tensor of order three. When $\alpha = \pi$ the direction of $\boldsymbol{e}$ does not matter since

$$R(\boldsymbol{e}, \pi)^i_j = -\delta^i_j + 2e^i e_j. \tag{6}$$

Consequently every Lorentz transformation is written in terms of the three-vectors $\boldsymbol{v}$ and $\boldsymbol{\mu}$, $\Lambda(\boldsymbol{v}, \boldsymbol{\mu}) = L(\boldsymbol{v})\ R(\boldsymbol{\mu}) \equiv \Lambda(\boldsymbol{v}, \boldsymbol{0})\Lambda(\boldsymbol{0}, \boldsymbol{\mu})$, and a general Poincaré transformation $g \in \mathcal{P}$, depends on the ten real numbers $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ and its action (1) on every space–time point interpreted in the order $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})\ x = (b, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})(0, \boldsymbol{a}, \boldsymbol{0}, \boldsymbol{0})(0, \boldsymbol{0}, \boldsymbol{v}, \boldsymbol{0})(0, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{\mu})\ \ x = x'$ is given by

$$t' = \gamma t + \gamma c^{-2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{x}) + b \tag{7a}$$

$$\boldsymbol{x}' = R(\boldsymbol{\mu})\boldsymbol{x} + \gamma \boldsymbol{v} t + \frac{\gamma^2}{(1+\gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{x})\boldsymbol{v} + \boldsymbol{a}. \tag{7b}$$

So the $(ct', \boldsymbol{x}')$ event is reached by successively applying to $(ct, \boldsymbol{x})$ a rotation given by $\boldsymbol{\mu}$, the boost corresponding to the speed $\boldsymbol{v}$ and the space–time translation $(cb, \boldsymbol{a})$.

The composition of two Poincaré transformations $g'g = g''$ is given by

$$b'' = \gamma' b + \gamma' c^{-2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a}) + b' \tag{8a}$$

$$\boldsymbol{a}'' = R(\boldsymbol{\mu}')\boldsymbol{a} + \gamma' \boldsymbol{v}' b + \frac{\gamma'^2}{(1+\gamma')c^2}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a})\boldsymbol{v}' + \boldsymbol{a}' \tag{8b}$$

$$\boldsymbol{v}'' = \frac{R(\boldsymbol{\mu}')\boldsymbol{v} + \gamma' \boldsymbol{v}' + \gamma'^2[(1+\gamma')c^2]^{-1}(\boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{v})\boldsymbol{v}'}{\gamma'(1 + \boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{v} c^{-2})} \tag{8c}$$

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu} + \boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}'; \boldsymbol{v}, \boldsymbol{\mu})}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu} + G(\boldsymbol{v}', \boldsymbol{\mu}'; \boldsymbol{v}, \boldsymbol{\mu})} \tag{8d}$$

where $\boldsymbol{F}$ and G are the following functions of $\boldsymbol{v}$ and $\boldsymbol{v}'$:

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{v}', \boldsymbol{\mu}'; \boldsymbol{v}, \boldsymbol{\mu}) = \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}&[\boldsymbol{v} \times \boldsymbol{v}' + \boldsymbol{v}(\boldsymbol{v}' \cdot \boldsymbol{\mu}') \\ &+ \boldsymbol{v}'(\boldsymbol{v} \cdot \boldsymbol{\mu}) + \boldsymbol{v} \times (\boldsymbol{v}' \times \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu}) \times \boldsymbol{v}' \\ &+ (\boldsymbol{v} \cdot \boldsymbol{\mu})(\boldsymbol{v}' \times \boldsymbol{\mu}') + (\boldsymbol{v} \times \boldsymbol{\mu})(\boldsymbol{v}' \cdot \boldsymbol{\mu}') \\ &+ (\boldsymbol{v} \times \boldsymbol{\mu}) \times (\boldsymbol{v}' \times \boldsymbol{\mu}')]\end{aligned} \tag{9a}$$

$$\begin{aligned}G(\boldsymbol{v}', \boldsymbol{\mu}'; \boldsymbol{v}, \boldsymbol{\mu}) = \frac{\gamma\gamma'}{(1+\gamma)(1+\gamma')c^2}&[\boldsymbol{v} \cdot \boldsymbol{v}' + \boldsymbol{v} \cdot (\boldsymbol{v}' \times \boldsymbol{\mu}') \\ &+ \boldsymbol{v}' \cdot (\boldsymbol{v} \times \boldsymbol{\mu}) - (\boldsymbol{v} \cdot \boldsymbol{\mu})(\boldsymbol{v}' \cdot \boldsymbol{\mu}') \\ &+ (\boldsymbol{v} \times \boldsymbol{\mu}) \cdot (\boldsymbol{v}' \times \boldsymbol{\mu}')].\end{aligned} \tag{9b}$$

The (8a) and (8b) transformations come from (3a) taking into account (7), and the relations (8c) and (8d) are calculated in the appendix.

The composition of two rotations $R(\boldsymbol{\mu}')R(\boldsymbol{\mu}) = R(\boldsymbol{\mu}'')$ is given by

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu}}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu}} \tag{10}$$

and the composition of two pure Lorentz transformations $\Lambda(\boldsymbol{v}', \mathbf{0})\Lambda(\boldsymbol{v}, \mathbf{0}) = \Lambda(\boldsymbol{v}'', \boldsymbol{\mu}'') \equiv \Lambda(\boldsymbol{v}'', \mathbf{0})\Lambda(\mathbf{0}, \boldsymbol{\mu}'')$ involves a rotation, called the Wigner rotation

$$\boldsymbol{\mu}'' = \frac{\boldsymbol{v} \times \boldsymbol{v}'}{(1+\gamma)(1+\gamma')c^2/\gamma\gamma' + \boldsymbol{v}\cdot\boldsymbol{v}'} \tag{11}$$

around an axis orthogonal to $\boldsymbol{v}$ and $\boldsymbol{v}'$, $\boldsymbol{e}'' = \boldsymbol{v}\times\boldsymbol{v}'|\boldsymbol{v}\times\boldsymbol{v}'|^{-1}$ and with angle α'' given by

$$\tan\tfrac{1}{2}\alpha'' = \sin\phi\left[\left(\frac{(\gamma+1)(\gamma'+1)}{(\gamma-1)(\gamma'-1)}\right)^{1/2} + \cos\phi\right]^{-1} \tag{12}$$

ϕ being the angle between $\boldsymbol{v}$ and $\boldsymbol{v}'$, and followed by another pure Lorentz transformation with velocity $\boldsymbol{v}''$ given by the well known relativistic addition of the two velocities $\boldsymbol{v}'$ and $\boldsymbol{v}$ (Møller 1972):

$$\boldsymbol{v}'' = \frac{\boldsymbol{v}\gamma'^{-1} + [\gamma'(1+\gamma')^{-1}\boldsymbol{v}'\cdot\boldsymbol{v}c^{-2} + 1]\boldsymbol{v}'}{1 + \boldsymbol{v}'\cdot\boldsymbol{v}c^{-2}} \tag{13}$$

and only if $\boldsymbol{v}\times\boldsymbol{v}' = 0$ is no rotation involved.

Another method for calculating the Wigner rotation has been recently published (Ben-Menahem 1985).

The calculation of the Thomas precession can be obtained directly by making the substitution $\boldsymbol{v}\to\boldsymbol{v}+\mathrm{d}\boldsymbol{v}$ and $\boldsymbol{v}'\to-\boldsymbol{v}$ in (11) and the result can be compared with van Wyk (1984).

The unit element of the group is $(0, \mathbf{0}, \mathbf{0}, \mathbf{0})$ and the inverse of $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ is:

$$(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})^{-1} = (-\gamma b + \gamma\boldsymbol{v}\cdot\boldsymbol{a}c^{-2}, -R(-\boldsymbol{\mu})\{\boldsymbol{a} - \gamma\boldsymbol{v}b + \gamma^2[(1+\gamma)c^2]^{-1}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}\}, -R(-\boldsymbol{\mu})\boldsymbol{v}, -\boldsymbol{\mu}) \tag{14}$$

where $R(-\boldsymbol{\mu}) = R(\boldsymbol{\mu})^{-1}$.

3. Contractions of $\mathscr{P}$

Inonu and Wigner (1952) developed the concept of group contraction in terms of the Lie algebra of the group. The corresponding version in terms of the composition functions of the group is summarised without proofs in what follows.

Let G be an n-dimensional Lie group, such that every element $a\in G$ can be expressed in terms of n continuous parameters a^i, $i=1,\ldots,n$, e being $(0,0,\ldots,0)$ the unit element. The composition law is given by n differentiable functions ϕ^i such that if $c=ab$ then $c^i = \phi^i(a^1,\ldots,a^n; b^1,\ldots b^n)$, $i=1,\ldots,n$.

If we change the parametrisation such that

$$\alpha^i = a^i \qquad i=1,\ldots,k$$
$$\varepsilon\alpha^j = a^j \qquad j=k+1,\ldots,n \tag{15}$$

and similarly for b^i and c^i in terms of β^i and γ^i respectively, and we take the α^i, β^i, γ^i as the new parameters, the composition law becomes:

$$\gamma^i = \phi^i(\alpha^1,\ldots,\alpha^k,\varepsilon\alpha^{k+1},\ldots,\varepsilon\alpha^n;\beta^1,\ldots,\beta^k,\varepsilon\beta^{k+1},\ldots,\varepsilon\beta^n) \qquad i=1,\ldots,k \tag{16a}$$

$$\gamma^j = \varepsilon^{-1}\phi^j(\alpha^1,\ldots,\alpha^k,\varepsilon\alpha^{k+1},\ldots,\varepsilon\alpha^n;\beta^1,\ldots,\beta^k,\varepsilon\beta^{k+1},\ldots,\varepsilon\beta^n) \qquad j=k+1,\ldots,n \tag{16b}$$

i.e.

$$\gamma^i = \Phi^i(\alpha^1,\ldots,\alpha^n;\beta^1,\ldots,\beta^n) \qquad i=1,\ldots,n \tag{17}$$

and these Φ^i are still the composition functions of the same group but in another parametrisation depending on the nonvanishing real number ε.

What happens if we take the limit $\varepsilon\to0$? In general the $\lim_{\varepsilon\to0}\Phi^i = \Psi^i$ will not define the composition functions of any group. In fact the necessary and sufficient condition that the Ψ^i functions exist and indeed form the transformation law of a group, is that the set of elements $a\in G$ of the form $a\equiv(a^1,\ldots,a^k,0,\ldots,0)$ be a subgroup H of G.

In this case, the Ψ^i functions represent the composition functions of a group G_H, in general different from G, which is called the contraction of G with respect to its subgroup H. If this occurs, the set of elements of the form $(0,\ldots,0,\alpha^{k+1},\ldots,\alpha^n)$ is an abelian invariant subgroup K of the contracted group G_H and H which remains unchanged is isomorphic to the factor group G_H/K.

Following the above considerations, if we take the Poincaré group, we see that the set of elements of the form $(b, \mathbf{0}, \mathbf{0}, \boldsymbol{\mu})$ is the subgroup of $\mathscr{P}$ corresponding to rotations and time translations. If we define the new parameters $b_0 = b$, $\boldsymbol{\mu}_0 = \boldsymbol{\mu}$, $\varepsilon\boldsymbol{a}_0 = \boldsymbol{a}$, $\varepsilon\boldsymbol{v}_0 = \boldsymbol{v}$, in the limit $\varepsilon\to0$, the equations (8) become:

$$b_0'' = b_0' + b_0 \tag{18a}$$
$$\boldsymbol{a}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{a}_0 + \boldsymbol{v}_0'b_0 + \boldsymbol{a}_0' \tag{18b}$$
$$\boldsymbol{v}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{v}_0 + \boldsymbol{v}_0' \tag{18c}$$
$$\boldsymbol{\mu}_0'' = \frac{\boldsymbol{\mu}_0' + \boldsymbol{\mu}_0 + \boldsymbol{\mu}_0'\times\boldsymbol{\mu}_0}{1 - \boldsymbol{\mu}_0'\cdot\boldsymbol{\mu}_0} \quad \text{or} \quad R(\boldsymbol{\mu}_0'') = R(\boldsymbol{\mu}_0')R(\boldsymbol{\mu}_0). \tag{18d}$$

These expressions represent the composition functions of the Galilei group, in a parametrisation in which an element is given in terms of the ten real numbers $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ with the same physical interpretation as before, being now infinite the range of v. $\mathscr{G}$, as a transformation group, acts on the space–time as:

$$t' = t + b \tag{19a}$$
$$\boldsymbol{x}' = R(\boldsymbol{\mu})\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}. \tag{19b}$$

It is in this sense of low velocities and small space translations that the Poincaré group contracts to the Galilei group and it is said that $\mathscr{G}$ is the velocity–space contraction of $\mathscr{P}$.

Similarly, the set of elements of the form $(0, \boldsymbol{a}, \mathbf{0}, \boldsymbol{\mu})$ is another subgroup of $\mathscr{P}$, and by defining $\boldsymbol{a}_0 = \boldsymbol{a}$, $\boldsymbol{\mu}_0 = \boldsymbol{\mu}$, $\varepsilon b_0 = b$, $\varepsilon\boldsymbol{v}_0 = \boldsymbol{v}$ and taking the $\varepsilon\to0$ limit we get from (8):

$$b_0'' = b_0' + b_0 + \boldsymbol{v}_0'c^{-2}\cdot R(\boldsymbol{\mu}_0')\boldsymbol{a}_0 \tag{20a}$$
$$\boldsymbol{a}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{a}_0 + \boldsymbol{a}_0' \tag{20b}$$
$$\boldsymbol{v}_0'' = R(\boldsymbol{\mu}_0')\boldsymbol{v}_0 + \boldsymbol{v}_0' \tag{20c}$$
$$\boldsymbol{\mu}_0'' = \frac{\boldsymbol{\mu}_0' + \boldsymbol{\mu}_0 + \boldsymbol{\mu}_0'\times\boldsymbol{\mu}_0}{1 - \boldsymbol{\mu}_0'\cdot\boldsymbol{\mu}} \quad \text{or} \quad R(\boldsymbol{\mu}_0'') = R(\boldsymbol{\mu}_0')R(\boldsymbol{\mu}_0) \tag{20d}$$

which are the composition functions of the Carroll group (Levy-Leblond 1965), which acts on space–time as a transformation group:

$$t' = t + \boldsymbol{v}c^{-2} \cdot R(\boldsymbol{\mu})\boldsymbol{x} + b \tag{21a}$$

$$\boldsymbol{x}' = R(\boldsymbol{\mu})\boldsymbol{x} + \boldsymbol{a}. \tag{21b}$$

$\mathscr{C}$ is said to be the velocity–time contraction of $\mathscr{P}$. That (18) and (20) are respectively the composition functions of the space–time transformation groups (19) and (21) can be easily checked by twice applying these transformations to a space–time point.

4. Final comments

We have derived the composition functions of $\mathscr{G}$ and $\mathscr{C}$ groups from those of $\mathscr{P}$ by the limiting process of group contraction, but to what extent can the action of these groups on space–time (19) and (21) respectively, be obtained from that of $\mathscr{P}$?

All these three groups are ten-parameter transformation groups acting on space–time. Their difference lies in the different conception of space–time they supply. What is a different conception of the space–time at the physical level is something which is related to how and how much any two observers disagree about concepts such as simultaneity and isotropy of pairs of events.

Let us consider two events which we call 1 and 2, with space–time coordinates x_1 and x_2 as measured by the inertial observer O and x_1' and x_2' respectively for the observer O′. $\Delta t = t_1 - t_2$ and $\Delta\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ are the time and space intervals between the two events for O and similarly $\Delta t'$ and $\Delta\boldsymbol{x}'$ for O′.

If the two events are simultaneous for O ($\Delta t = 0$), for O′, $\Delta t'$ will be in general non-zero and hence there exists a discrepancy between them about simultaneity. We can measure this discrepancy by defining a magnitude that represents the loss of simultaneity per unit length. For O′ this magnitude will be related to things like $|\Delta t'|/|\Delta\boldsymbol{x}'|$, but it can happen that simultaneous events for O with the same value of $|\Delta\boldsymbol{x}'|$ can have different $\Delta t'$ values. We call the maximum value of this ratio d'.

Taking into account (14) we see for $\mathscr{P}$ that $\Delta t = \gamma\Delta t' - \gamma c^{-2}(R(-\boldsymbol{\mu})\boldsymbol{v}) \cdot (R(-\boldsymbol{\mu})\Delta\boldsymbol{x}')$ and $\Delta t = 0$ implies $\Delta t' = \boldsymbol{v}c^{-2} \cdot \Delta\boldsymbol{x}'$, so that $d' = vc^{-2}$. For $\mathscr{G}$ $\Delta t = \Delta t'$ and $d' = 0$. In the Carroll group $d' = vc^{-2}$. Similarly d is defined for the observer O in terms of events which are simultaneous for O′. In all three groups considered $d = d'$.

If the two events are isotropic for O (i.e. $\Delta\boldsymbol{x} = 0$), in general $\Delta\boldsymbol{x}'$ will be different from zero and consequently there exists a discrepancy about isotropy. We can measure this discrepancy by defining a magnitude related to the loss of isotropy per unit time. For observer O′ we define $D' = |\Delta\boldsymbol{x}'|/|\Delta t'|$, and similarly the magnitude D for the observer O in terms of events which are isotropic for O′. For all three groups $D = D'$.

We thus see that the relative space–time interval measurements are characterised by the two discrepancy parameters (d, D). For the Poincaré group we get $(v/c^2, v)$, $(0, v)$ for the Galilei group and $(v/c^2, 0)$ for $\mathscr{C}$. Thus the three groups supply a different pair of discrepancies between observers.

When passing from $\mathscr{P}$ to $\mathscr{G}$, $(v/c^2, v)$ becomes $(0, v)$ and this can be achieved as usual by assuming the limit $c \to \infty$ while v remains finite. It is in this sense that the transformation group equations (7) for $\mathscr{P}$ become those of $\mathscr{G}$, (19).

Similarly, when going from $\mathscr{P}$ to $\mathscr{C}$ we pass from $(v/c^2, v)$ to $(v/c^2, 0)$. This can be obtained by the limiting process $v \to 0$, $c \to 0$, such that $v/c \to 0$ while v/c^2 remains finite, i.e. v goes to zero as fast as c^2 does. By applying this limit to (7) we get the transformation group equations (21) of $\mathscr{C}$.

One possible physical interpretation of these limiting processes is that in the Galilei limit light is assumed to travel at infinite speed, so that the relative velocity interval $[0, c)$ is enlarged to $[0, \infty)$, thus leaving for description all events which travel at velocities below c, including the relative motions among observers. In the Carroll limit the $[0, c)$ interval is contracted to zero, thus transforming $[c, \infty)$ in the $[0, \infty)$ interval, such that light and observers are all at rest, the space is absolute but time is not since observers have different clocks, and only events with velocities above c are considered for description.

In other words, the Poincaré group describes bradyons, light and tachyons, while the Galilei group describes only bradyons and the Carroll group only tachyons.

Appendix

The object of this appendix is twofold: to provide a method for the breaking down of a Lorentz transformation in terms of a rotation and a pure Lorentz transformation, and to obtain the group law. This can be achieved directly from (3) and (4) but it is a rather cumbersome method and we have preferred to present it using spinor calculus.

In the spinor representations of rotations we obtain (Misner *et al* 1973) that every rotation around axis $\boldsymbol{e}$ and angle α is given by the 2×2 unitary matrix

$$R(\boldsymbol{e}, \alpha) = \sigma_0 \cos \tfrac{1}{2}\alpha - \mathrm{i}(\boldsymbol{\sigma} \cdot \boldsymbol{e}) \sin \tfrac{1}{2}\alpha \tag{22}$$

and by using $\boldsymbol{\mu} = \boldsymbol{e} \tan \frac{1}{2}\alpha$ we get

$$R(\boldsymbol{\mu}) = (1 + \boldsymbol{\mu}^2)^{-1/2}[\sigma_0 - \mathrm{i}(\boldsymbol{\sigma} \cdot \boldsymbol{\mu})]. \tag{23}$$

From this we easily obtain that $R(\boldsymbol{\mu}')R(\boldsymbol{\mu}) = R(\boldsymbol{\mu}'')$ where $\boldsymbol{\mu}''$ is given by (10).

Similarly a pure Lorentz transformation with velocity $\boldsymbol{v}$ is given by the 2×2 Hermitean matrix

$$L(\boldsymbol{v}) = \sigma_0 \cosh \tfrac{1}{2}\theta - (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \sinh \tfrac{1}{2}\theta \tag{24}$$

where $\boldsymbol{n} = \boldsymbol{v}/v$ and $\tanh \theta = v/c$ being

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and making $\boldsymbol{u}=\gamma\boldsymbol{v}/c$:

$$L(\boldsymbol{u})=(\tfrac{1}{2}+\gamma/2)^{1/2}\left(\sigma_0-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{u}}{1+\gamma}\right). \tag{25}$$

Finally if every Lorentz transformation is interpreted as the product of a pure Lorentz transformation and a rotation:

$$\Lambda(\boldsymbol{u},\boldsymbol{\mu})=L(\boldsymbol{u})R(\boldsymbol{\mu})=\left(\frac{1+\gamma}{2(1+\mu^2)}\right)^{1/2} \times\left[\sigma_0\left(1-\mathrm{i}\frac{\boldsymbol{u}\cdot\boldsymbol{\mu}}{1+\gamma}\right)+\boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{u}+\boldsymbol{u}\times\boldsymbol{\mu}}{1+\gamma}-\mathrm{i}\boldsymbol{\mu}\right)\right] \tag{26}$$

where use has been made of the identity:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=(\boldsymbol{a}\cdot\boldsymbol{b})\sigma_0+\mathrm{i}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{\sigma}. \tag{27}$$

Thus $\Lambda(\boldsymbol{u},\boldsymbol{\mu})=a^\nu\sigma_\nu$ with a^ν a complex four-vector which verifies $a^\nu a_\nu=1$, and is:

$$a^0=\left(\frac{1+\gamma}{2(1+\mu^2)}\right)^{1/2}\left(1-\mathrm{i}\frac{\boldsymbol{u}\cdot\boldsymbol{\mu}}{1+\gamma}\right) \tag{28a}$$

$$\boldsymbol{a}=\left(\frac{1+\gamma}{2(1+\mu^2)}\right)^{1/2}\left(\frac{\boldsymbol{u}+\boldsymbol{u}\times\boldsymbol{\mu}}{1+\gamma}-\mathrm{i}\boldsymbol{\mu}\right). \tag{28b}$$

Conversely, given any 2×2 matrix with determinant $+1$ can always be written in the form $A=a^\nu\sigma_\nu$ and $\det A=a^\nu a_\nu=+1$, thus (28) can be inverted and we get:

$$\boldsymbol{\mu}=-\frac{\operatorname{Im}\boldsymbol{a}}{\operatorname{Re}a^0} \tag{29a}$$

$$\boldsymbol{u}=2(\operatorname{Im}a^0\operatorname{Im}\boldsymbol{a}+\operatorname{Re}\boldsymbol{a}\times\operatorname{Im}\boldsymbol{a}). \tag{29b}$$

If $\Lambda(\boldsymbol{u},\boldsymbol{\mu})$ were interpreted in the reverse order, i.e. as a product of a rotation and a pure Lorentz transformation, then only in (26), (28b) and (29b) the sign of the cross product is changed. We thus see that in any case the rotation involved is always the same (29a).

If we multiply two Lorentz transformations of the form (26), $\Lambda(\boldsymbol{u}',\boldsymbol{\mu}')\Lambda(\boldsymbol{u},\boldsymbol{\mu})=\Lambda(\boldsymbol{u}'',\boldsymbol{\mu}'')$, by applying (29) we obtain $\boldsymbol{u}''$ and $\boldsymbol{\mu}''$ in terms of $\boldsymbol{u}'$, $\boldsymbol{\mu}'$, $\boldsymbol{u}$ and $\boldsymbol{\mu}$ and expressing the result in terms of $\boldsymbol{v}$, $\boldsymbol{v}'$ and $\boldsymbol{v}''$ since

$$\gamma''=\gamma\gamma'\left(1+\frac{\boldsymbol{v}'\cdot R(\boldsymbol{\mu}')\boldsymbol{v}}{c^2}\right) \tag{30}$$

we get the result (8c) and (8d).

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