The dynamical equation of the spinning electron

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Abstract
We obtain by invariance arguments the relativistic and non-relativistic invariant
dynamical equations of a classical model of a spinning electron. We apply
the formalism to a particular classical model which satisfies Dirac's equation
when quantized. It is shown that the dynamics can be described in terms
of the evolution of the point charge which satisfies a fourth-order differential
equation or, alternatively, as a system of second-order differential equations
by describing the evolution of both the centre of mass and centre of charge of
the particle. As an application of the found dynamical equations, the Coulomb
interaction between two spinning electrons is considered. We find from the
classical viewpoint that these spinning electrons can form bound states under
suitable initial conditions. Since the classical Coulomb interaction of two
spinless point electrons does not allow for the existence of bound states, it is
the spin structure that gives rise to new physical phenomena not described in
the spinless case. Perhaps the paper may be interesting from the mathematical
point of view but not from the point of view of physics.

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1. Introduction

Wigner defined a quantum elementary particle as a system whose Hilbert space of states
carries an irreducible representation of the Poincaré group [1]. This definition is a group
theoretical one. This led the author to look for a definition of a classical elementary particle
by group theoretical methods, relating its definition to the kinematical group structure. A
classical elementary particle was defined as a Lagrangian system whose kinematical space is
a homogeneous space of the Poincaré group [2, 3]. When quantizing these classical models it
is shown that the wavefunction of the system transforms with a projective unitary irreducible
representation of the kinematical group [4]. The different classical models of spinning particles

produced by this formalism are given in [5]. One of the models, which will be considered in this work, satisfies Dirac's equation when quantized.

The latest LEP experiments at CERN suggest that the electron charge is confined within a region of radius $R_e < 10^{-19}$ m. Nevertheless, the quantum mechanical behaviour of the electron appears at distances of the order of its Compton wavelength $\lambda_C = \hbar/mc \simeq 10^{-13}$ m, which is six orders of magnitude larger. One possibility of reconciling these features in order to obtain a model of the electron is the assumption from the classical viewpoint that the charge of the electron is just a point but at the same time this point is never at rest and it is affected by the so-called *zitterbewegung* and, therefore, it is moving in a confined region of size λ_C.

This is the basic structure of the spinning particle models obtained within the kinematical formalism developed by the author [2–5] and also suggested by Dirac's analysis of the internal motion of the electron [6]. There, the charge of the particle is at a point r, but this point is not the centre of mass of the particle. In general, we obtain that the point charge satisfies a fourth-order differential equation which, as we shall see, is the most general differential equation satisfied by any three-dimensional curve. The charge is moving around the centre of mass in a kind of harmonic or central motion. It is this motion of the charge that gives rise to the spin and dipole structure of the particle. In particular, the classical model that when quantized satisfies Dirac's equation shows, for the centre of mass observer, a charge moving at the speed of light in circles of radius $R = \hbar/2mc$ and contained in a plane orthogonal to the spin direction [4]. It is this classical model of the electron we shall consider in the subsequent analysis and which is reviewed in section 3.

In this paper we shall find the group invariant dynamical equations of these classical systems. The difference between the approach presented here and that of the previous published works is that the dynamical equations are obtained by group theoretical arguments without any appeal to a Lagrangian formalism while there they were obtained by Lagrangian methods. Nevertheless the dynamical equations obtained are the same.

The paper is organized as follows: section 2 is a reminder that the most general differential equation satisfied by a curve in three-dimensional space is of fourth order. In section 3 we introduce the classical model of a spinning electron that has been shown to satisfy Dirac's equation when quantized. Section 4 states the general method for obtaining the group invariant differential equation satisfied by a point and for any arbitrary kinematical group. This method is applied in sections 5 to 7 to the Galilei and Poincaré groups to obtain the non-relativistic and relativistic invariant dynamical equations of a spinless particle and of the spinning model. Finally, as an application of the obtained dynamical equations, the analysis of the Coulomb interaction between two spinning electrons is considered in section 8.

One of the salient features is the classical prediction of the possible existence of bound states for spinning electron–electron interaction. If two electrons have their centres of mass separated by a distance greater than Compton's wavelength they always repel each other. But if two electrons have their centres of mass separated by a distance less than Compton's wavelength they can form bound states provided some conditions on their relative spin orientation and centre of mass position and velocity are fulfilled.

2. Frenet–Serret equations

Let us recall that for an arbitrary three-dimensional curve $r(s)$ when expressed in parametric form in terms of the arc length s, the three orthogonal unit vectors v_i, $i = 1, 2, 3$, called respectively tangent, normal and binormal, satisfy the so-called Frenet–Serret differential

equations:

$$\dot{v}_1(s) = \kappa(s)v_2(s) \qquad \dot{v}_2(s) = -\kappa(s)v_1(s) + \tau(s)v_3(s) \qquad \dot{v}_3(s) = -\tau(s)v_2(s)$$

where κ and τ are respectively the curvature and torsion. Since the unit tangent vector is $v_1 = \dot{r} \equiv r^{(1)}$, taking successive derivatives it yields

$$r^{(1)} = v_1 \qquad r^{(2)} = \kappa v_2$$
$$r^{(3)} = \dot{\kappa}v_2 + \kappa\dot{v}_2 = -\kappa^2 v_1 + \dot{\kappa}v_2 + \kappa\tau v_3$$
$$r^{(4)} = -3\kappa\dot{\kappa}v_1 + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)v_2 + (2\dot{\kappa}\tau + \kappa\dot{\tau})v_3.$$

Then elimination of the v_i between these equations implies that the most general curve in three-dimensional space satisfies the fourth-order differential equation:

$$r^{(4)} - \left(\frac{2\dot{\kappa}}{\kappa} + \frac{\dot{\tau}}{\tau}\right)r^{(3)} + \left(\kappa^2 + \tau^2 + \frac{\dot{\kappa}\dot{\tau}}{\kappa\tau} + \frac{2\dot{\kappa}^2 - \kappa\ddot{\kappa}}{\kappa^2}\right)r^{(2)} + \kappa^2\left(\frac{\dot{\kappa}}{\kappa} - \frac{\dot{\tau}}{\tau}\right)r^{(1)} = 0.$$

All the coefficients in front of the derivatives $r^{(i)}$ can be expressed in terms of the scalar products $r^{(i)} \cdot r^{(j)}$, $i, j = 1, 2, 3$. Let us mention that for helical motions there is a constant relationship $\kappa/\tau = $ constant, and therefore the coefficient of $r^{(1)}$ vanishes.

3. Spinning electron model

We present here the main features of a spinning electron model obtained through the general Lagrangian formalism. The Poincaré group can be parametrized in terms of the variables $\{t, r, v, \alpha\}$, where α is dimensionless and represents the relative orientation of inertial frames and the others have dimensions of time, length and velocity, respectively, and represent the corresponding parameters for time and space translation and the relative velocity between observers. Their range is $t \in \mathbb{R}$, $r \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ but constrained to $v < c$ and $\alpha \in SO(3)$.

One of the important homogeneous spaces of the Poincaré group that defines the kinematical space of a classical elementary particle [3] is spanned by the variables $\{t, r, v, \alpha\}$ with the same range as before but now v is restricted to $v = c$. As variables of a kinematical space of a classical elementary system they are interpreted as the time, position, velocity and orientation observables of the particle, respectively. The Lagrangian associated with this system will be a function of these variables and their next order time derivative. Therefore, the Lagrangian will also depend on the acceleration and angular velocity. It turns out that as far as the position r is concerned dynamical equations will be of fourth order. These are the equations we want to obtain in this work but from a different and non-Lagrangian method. When the invariance of any Lagrangian defined on this manifold under the Poincaré group is analysed, in particular under pure Lorentz transformations and rotations, we get respectively the following Noether constants of the motion which have the form $J^{\mu\nu} = -J^{\nu\mu} = x^\mu P^\nu - x^\nu P^\mu + S^{\mu\nu}$, or its essential components J^{0i} and J^{ij} in three-vector notation [3]:

$$K = \frac{1}{c^2}Hr - Pt - \frac{1}{c^2}S \times v \tag{1}$$
$$J = r \times P + S. \tag{2}$$

Observables H and P are the constant energy and linear momentum of the particle, respectively. The linear momentum is not lying along the velocity v, and S is the spin of the system. It satisfies the dynamical equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = P \times v$$
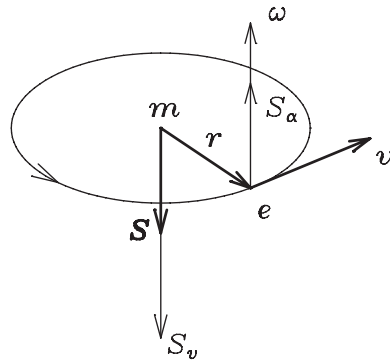
Figure 1. Circular motion at the speed of light of the centre of charge around the centre of mass in the centre of mass frame.

which is the classical equivalent of the dynamical equation satisfied by the Dirac spin observable. For the centre of mass observer it is a constant of the motion. This observer is defined by the requirements $\boldsymbol{P} = 0$ and $\boldsymbol{K} = 0$. The first condition implies that the centre of mass is at rest and the second that it is located at the origin of the frame. The energy of the particle in this frame is $H = mc^2$, so that from (1) we get,

$$mc^2\boldsymbol{r} = \boldsymbol{S} \times \boldsymbol{v}.$$

It turns out that point $\boldsymbol{r}$, which does not represent the centre of mass position, describes the motion depicted in figure 1. It is the position of the charge. It is orthogonal to the constant spin and also to the velocity as it corresponds to a circular motion at the velocity c. The radius and angular velocity of the internal classical motion of the charge are, respectively, $R = S/mc$, and $\omega = c/R = mc^2/S$.

If we take the time derivative of the constant $\boldsymbol{K}$ of (1) and the scalar product with $\boldsymbol{v}$ we get

$$H - \boldsymbol{P} \cdot \boldsymbol{v} - \boldsymbol{S} \cdot \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \times \boldsymbol{v}\right) = 0$$

where we have a linear relationship between H and $\boldsymbol{P}$. The Dirac equation is just the quantum mechanical expression of this Poincaré invariant formula [4].

The spin $\boldsymbol{S} = \boldsymbol{S}_v + \boldsymbol{S}_\alpha$, has two parts: one, $\boldsymbol{S}_v$, related to the orbital motion of the charge and another, $\boldsymbol{S}_\alpha$, due to the rotation of the particle and which is directly related to the angular velocity as it corresponds to a spherically symmetric object. The positive energy particle has the total spin $\boldsymbol{S}$ oriented in the same direction as the $\boldsymbol{S}_v$ part, as shown in the figure. The orientation of the spin is the opposite for the negative energy particle, which corresponds to the time reversed motion. When quantizing the system, the orbital component $\boldsymbol{S}_v$, which is directly related to the magnetic moment, quantizes with integer values while the rotational part $\boldsymbol{S}_\alpha$ quantizes with half-integer values, so that for spin 1/2 particles the total spin is half the value of the S_v part. When expressing the magnetic moment in terms of the total spin we get in this way a pure kinematical interpretation of the $g = 2$ gyromagnetic ratio [7].

For the centre of mass observer this system looks like a system of three degrees of freedom. Two represent the x and y coordinates of the point and the third is the phase of its rotational motion. However, this phase is exactly equal to the phase of the orbital motion of the charge and because the motion is of constant radius at the constant speed c then only one independent degree of freedom remains. Therefore, the system is equivalent to a one-dimensional harmonic

oscillator of frequency ω. When quantizing the system the stationary states of the harmonic oscillator have the energy

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots.$$

But if the system is elementary then it has no excited states and in the C.M. frame it is reduced to the ground state of energy

$$E_0 = \tfrac{1}{2}\hbar\omega = mc^2$$

so that when compared with the classical result $\omega = mc^2/S$, it implies that the constant classical parameter S is necessarily $S = \hbar/2$. The radius of the internal charge motion is half the Compton wavelength. It is this classical model of electron we shall analyse in subsequent sections, and our interest is to obtain the dynamical equation satisfied by the point charge r for any arbitrary relativistic and non-relativistic inertial observer.

To end this section and with the above model in mind let us summarize the main results obtained by Dirac when he analysed the motion of the free electron [8]. Let point r be the position vector on which Dirac's spinor $\psi(t, r)$ is defined. When computing the velocity of point r, Dirac arrived at:

(i) The velocity $v = \mathrm{i}/\hbar[H, r] = c\boldsymbol{\alpha}$, is expressed in terms of $\boldsymbol{\alpha}$ matrices and '... *a measurement of a component of the velocity of a free electron is certain to lead to the result $\pm c$. This conclusion is easily seen to hold also when there is a field present.*'

(ii) The linear momentum does not have the direction of this velocity v, but must be related to some average value of it: ... '*the x_1 component of the velocity, $c\alpha_1$, consists of two parts, a constant part $c^2 p_1 H^{-1}$, connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least $2mc^2/h$, ...*'.

(iii) About the position r: '*The oscillatory part of x_1 is small, ..., which is of order of magnitude $\hbar/mc$, ...*'.

And when analysing, in his original 1928 paper [9], the interaction of the electron with an external electromagnetic field, after performing the square of Dirac's operator, he obtained two new interaction terms:

$$\frac{e\hbar}{2mc}\boldsymbol{\Sigma}\cdot\boldsymbol{B} + \frac{\mathrm{i}e\hbar}{2mc}\boldsymbol{\alpha}\cdot\boldsymbol{E} \tag{3}$$

where the electron spin is written as $S = \hbar\boldsymbol{\Sigma}/2$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

in terms of σ-Pauli matrices and E and B are the external electric and magnetic fields, respectively. Dirac says, '*The electron will therefore behave as though it has a magnetic moment $(e\hbar/2mc)\boldsymbol{\Sigma}$ and an electric moment $(\mathrm{i}e\hbar/2mc)\boldsymbol{\alpha}$. The magnetic moment is just that assumed in the spinning electron model*' (Pauli model). '*The electric moment, being a pure imaginary, we should not expect to appear in the model. It is doubtful whether the electric moment has any physical meaning.*'

In the last sentence it is difficult to understand why Dirac, who did not reject the negative energy solution and therefore its consideration as the antiparticle states, disliked the existence of this electric dipole which was obtained from his formalism on an equal footing as the magnetic dipole term. In quantum electrodynamics, even in high energy processes, the complete Dirac Hamiltonian contains both terms, perhaps in a rather involved way because the above expression is a first-order expansion in the external fields considered as classical commuting fields. Properly speaking this electric dipole does not represent the existence of

a particular positive and negative charge distribution for the electron. In the classical model, the negative charge of the electron is at a single point but because this point is not the centre of mass, there exists a nonvanishing electric dipole moment with respect to the centre of mass of value $e\boldsymbol{r}$ in the centre of mass frame. Its correspondence with the quantum Dirac electric moment is shown in [5]. I think this is the observable Dirac disliked. It is oscillating at very high frequency and it basically plays no role in low energy electron interactions because its average value vanishes, but it is important in high energy or in very close electron–electron interactions.

4. The invariant dynamical equation

Let us consider the trajectory $\boldsymbol{r}(t), t \in [t_1, t_2]$ followed by a point of a system for an arbitrary inertial observer O. Any other inertial observer O' is related to the previous one by a transformation of a kinematical group such that their relative spacetime measurements of any spacetime event are given by

$$t' = T(t, \boldsymbol{r}; g_1, \ldots, g_r) \qquad \boldsymbol{r}' = \boldsymbol{R}(t, \boldsymbol{r}; g_1, \ldots, g_r)$$

where the functions T and $\boldsymbol{R}$ define the action of the kinematical group G, of parameters $(g_1, \ldots, g_r)$, on spacetime. Then the description of the trajectory of that point for observer O' is obtained from

$$t'(t) = T(t, \boldsymbol{r}(t); g_1, \ldots, g_r) \qquad \boldsymbol{r}'(t) = \boldsymbol{R}(t, \boldsymbol{r}(t); g_1, \ldots, g_r) \qquad \forall t \in [t_1, t_2].$$

If we eliminate t as a function of t' from the first equation and substitute into the second we shall get

$$\boldsymbol{r}'(t') = \boldsymbol{r}'(t'; g_1, \ldots, g_r). \tag{4}$$

Since observer O' is arbitrary, equation (4) represents the complete set of trajectories of the point for all inertial observers. Elimination of the r group parameters among the function $\boldsymbol{r}'(t')$ and their time derivatives will give us the differential equation satisfied by the trajectory of the point. This differential equation is invariant by construction because it is independent of the group parameters and therefore independent of the inertial observer. If G is either the Galilei or Poincaré group it is a ten-parameter group so that we have to work out in general up to the fourth derivative to obtain sufficient equations to eliminate the group parameters. Therefore, the order of the differential equation is dictated by the number of parameters and the structure of the kinematical group.

5. The spinless particle

Let us consider first the case of the spinless point particle. In the non-relativistic case the relationship between inertial observers O and O' is given by the action of the Galilei group:

$$t' = t + b \qquad \boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}. \tag{5}$$

In the relativistic case we have that the Poincaré group action is given by

$$t' = \gamma \left(t + \frac{\boldsymbol{v} \cdot R(\boldsymbol{\alpha})\boldsymbol{r}}{c^2} \right) + b \tag{6}$$

$$\boldsymbol{r}' = R(\boldsymbol{\alpha})\boldsymbol{r} + \gamma \boldsymbol{v}t + \frac{\gamma^2}{(1 + \gamma)c^2}(\boldsymbol{v} \cdot R(\boldsymbol{\alpha})\boldsymbol{r})\boldsymbol{v} + \boldsymbol{a}. \tag{7}$$

For the free spinless point particle it is possible to find a particular observer, the centre of mass observer O^*, such that the trajectory of the particle for this observer reduces to

$$r^*(t^*) \equiv 0 \qquad \forall t^* \in [t_1^*, t_2^*].$$

and therefore its trajectory for any other non-relativistic observer O can be obtained from

$$t(t^*) = t^* + b \qquad r(t^*) = vt^* + a. \tag{8}$$

The trajectory of the point particle for the relativistic observer O will be obtained from

$$t(t^*) = \gamma t^* + b \qquad r(t^*) = \gamma vt^* + a. \tag{9}$$

Elimination of t^* in terms of t from the first equation of both (8) and (9) and substitution into the second yields the trajectory of the point for an arbitrary observer, which in the relativistic and non-relativistic formalism reduces to

$$r(t) = (t - b)v + a.$$

Elimination of group parameters v, b and a by taking succesive derivatives yields the Galilei and Poincaré invariant dynamical equation of a free spinless point particle

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = 0. \tag{10}$$

6. The non-relativistic spinning electron

We take spatial units such that the radius $R = 1$, and time units such that $\omega = 1$ and therefore the velocity $c = 1$. For the centre of mass observer, the trajectory of the charge of the electron is contained in the *XOY* plane and it is expressed in three-vector form as

$$r^*(t^*) = \begin{pmatrix} \cos t^* \\ \sin t^* \\ 0 \end{pmatrix}.$$

For the centre of mass observer O^* we get that

$$\frac{\mathrm{d}^2 r^*(t^*)}{\mathrm{d}t^{*2}} = -r^*(t^*). \tag{11}$$

For any arbitrary inertial observer O we get

$$t(t^*; g) = t^* + b \qquad r(t^*; g) = R(\alpha)r^*(t^*) + t^* v + a.$$

We shall represent the different time derivatives by

$$r^{(k)} \equiv \frac{\mathrm{d}^k r}{\mathrm{d}t^k} = \frac{\mathrm{d}}{\mathrm{d}t^*}\left(\frac{\mathrm{d}^{k-1} r}{\mathrm{d}t^{k-1}} \right) \frac{\mathrm{d}t^*}{\mathrm{d}t}.$$

In this non-relativistic case $\mathrm{d}t^*/\mathrm{d}t = 1$, then, after using (11) in some expressions we get the following derivatives:

$$r^{(1)} = R(\alpha)\frac{\mathrm{d}r^*}{\mathrm{d}t^*} + v \qquad r^{(2)} = R(\alpha)\frac{\mathrm{d}^2 r^*}{\mathrm{d}t^{*2}} = -R(\alpha)r^*$$

$$r^{(3)} = -R(\alpha)\frac{\mathrm{d}r^*}{\mathrm{d}t^*} \qquad r^{(4)} = -R(\alpha)\frac{\mathrm{d}^2 r^*}{\mathrm{d}t^{*2}} = R(\alpha)r^* = -r^{(2)}.$$

Therefore, the differential equation satisfied by the position of the charge of a non-relativistic electron and for any arbitrary inertial observer is

$$r^{(4)} + r^{(2)} = 0. \tag{12}$$

We see that the motion is a helix because there is no $r^{(1)}$ term.

6.1. The centre of mass

The centre of mass position of the electron is defined as

$$q = r + r^{(2)} \tag{13}$$

because it reduces to $q = 0$ and $q^{(1)} = 0$ for the centre of mass observer, so that dynamical equations can be rewritten in terms of the position of the charge and the centre of mass as

$$q^{(2)} = 0 \qquad r^{(2)} = q - r. \tag{14}$$

Our fourth-order dynamical equation (12) can be split into two second-order dynamical equations: a free equation for the centre of mass and a central harmonic motion of the charge position r around the centre of mass q of angular frequency 1 in these natural units.

6.2. Interaction with some external field

The free dynamical equation $q^{(2)} = 0$ is equivalent to $dP/dt = 0$, where $P = mq^{(1)}$ is the linear momentum of the system. Then our free equations should be replaced in the case of an interaction with an external electromagnetic field by

$$mq^{(2)} = e[E + r^{(1)} \times B] \qquad r^{(2)} = q - r \tag{15}$$

where in the Lorentz force the fields are defined at point r and it is the velocity of the charge that gives rise to the magnetic force term, while the second equation is left unchanged since it corresponds to the centre of mass definition.

　　These equations are also obtained in the Lagrangian approach [5] by assuming a minimal coupling interaction and where we get

$$r^{(4)} + r^{(2)} = \frac{e}{m}[E(t, r) + r^{(1)} \times B(t, r)] \tag{16}$$

which reduce to (15) after the centre of mass definition (13).

　　In order to determine the evolution of the system, initial conditions $r(0)$, $r^{(1)}(0)$, $r^{(2)}(0)$ and $r^{(3)}(0)$, i.e., the position of point r and its derivatives up to order 3 evaluated at time $t = 0$, must be given. Alternatively, if we consider our fourth-order differential equation (16) as the set of two second-order differential equations (15), then we have to fix as initial conditions $r(0)$ and $r^{(1)}(0)$ as before and $q(0) = r(0) + r^{(2)}(0)$ and $q^{(1)}(0) = r^{(1)}(0) + r^{(3)}(0)$, compatible with (13), i.e., the position and velocity of both the centre of mass and centre of charge points. The advantage of this method is that we shall be able to analyse the evolution of a two-electron system in section 8 in terms of the centre of mass initial position and velocity.

7. The relativistic spinning electron

Let us assume the same electron model in the relativistic case. Since the charge is moving at the speed of light for the centre of mass observer O^* it is moving at this speed for every other inertial observer O. Now, the relationship of spacetime measurements between the centre of mass observer and any arbitrary inertial observer is given by

$$t(t^*; g) = \gamma(t^* + v \cdot R(\alpha)r^*(t^*)) + b$$

$$r(t^*; g) = R(\alpha)r^*(t^*) + \gamma v t^* + \frac{\gamma^2}{1 + \gamma}(v \cdot R(\alpha)r^*(t^*))v + a.$$

With the shorthand notation for the following expressions:

$$K(t^*) = R(\alpha)r^*(t^*) \qquad V(t^*) = R(\alpha)\frac{dr^*(t^*)}{dt^*} = \frac{dK}{dt^*} \qquad \frac{dV}{dt^*} = -K$$

$$B(t^*) = v \cdot K \qquad A(t^*) = v \cdot V = \frac{dB}{dt^*} \qquad \frac{dA}{dt^*} = -B$$

we obtain

$$r^{(1)} = \frac{1}{\gamma(1+A)}\left(V + \frac{\gamma}{1+\gamma}(1+\gamma+\gamma A)v\right) \tag{17}$$

$$r^{(2)} = \frac{1}{\gamma^2(1+A)^3}\left(-(1+A)K + BV + \frac{\gamma}{1+\gamma}Bv\right) \tag{18}$$

$$r^{(3)} = \frac{1}{\gamma^3(1+A)^5}\left(-3B(1+A)K - (1+A-3B^2)V + \frac{\gamma}{1+\gamma}(A(1+A)+3B^2)v\right) \tag{19}$$

$$r^{(4)} = \frac{1}{\gamma^4(1+A)^7}\left((1+A)(1-2A-3A^2-15B^2)K - B(7+4A-3A^2-15B^2)V\right.$$
$$\left. - \frac{\gamma}{1+\gamma}(1-8A-9A^2-15B^2)Bv\right). \tag{20}$$

From this we get

$$(r^{(1)} \cdot r^{(1)})^2 = 1 \qquad (r^{(1)} \cdot r^{(2)}) = 0 \tag{21}$$

$$(r^{(2)} \cdot r^{(2)}) = -(r^{(1)} \cdot r^{(3)}) = \frac{1}{\gamma^4(1+A)^4} \tag{22}$$

$$(r^{(2)} \cdot r^{(3)}) = -\frac{1}{3}(r^{(1)} \cdot r^{(4)}) = \frac{2B}{\gamma^5(1+A)^6} \tag{23}$$

$$(r^{(3)} \cdot r^{(3)}) = \frac{1}{\gamma^6(1+A)^8}(1-A^2+3B^2) \tag{24}$$

$$(r^{(2)} \cdot r^{(4)}) = \frac{1}{\gamma^6(1+A)^8}(-1+2A+3A^2+9B^2) \tag{25}$$

$$(r^{(3)} \cdot r^{(4)}) = \frac{1}{\gamma^7(1+A)^{10}}(1+A+3B^2)4B. \tag{26}$$

From equations (22)–(24) we can express the magnitudes A, B and γ in terms of these scalar products between the different time derivatives $(r^{(i)} \cdot r^{(j)})$. The constraint that the velocity is 1 implies that all these and further scalar products for higher derivatives can be expressed in terms of only three of them. If the three equations (17)–(19) are solved in terms of the unknowns v, V and K and substituted into (20), we obtain the differential equation satisfied by the charge position

$$r^{(4)} - \frac{3(r^{(2)} \cdot r^{(3)})}{(r^{(2)} \cdot r^{(2)})}r^{(3)} + \left(\frac{2(r^{(3)} \cdot r^{(3)})}{(r^{(2)} \cdot r^{(2)})} - \frac{3(r^{(2)} \cdot r^{(3)})^2}{4(r^{(2)} \cdot r^{(2)})^2} - (r^{(2)} \cdot r^{(2)})^{1/2}\right)r^{(2)} = 0. \tag{27}$$

It is a fourth-order ordinary differential equation which contains as solutions motions at the speed of light. In fact, if $(r^{(1)} \cdot r^{(1)}) = 1$, then by derivation we have $(r^{(1)} \cdot r^{(2)}) = 0$ and the next derivative leads to $(r^{(2)} \cdot r^{(2)}) + (r^{(1)} \cdot r^{(3)}) = 0$. If we take this into account and make the scalar product of (27) with $r^{(1)}$, we get $(r^{(1)} \cdot r^{(4)}) + 3(r^{(2)} \cdot r^{(3)}) = 0$, which is another relationship between the derivatives as a consequence of $|r^{(1)}| = 1$. It corresponds to a helical motion since the term in the first derivative $r^{(1)}$ is missing.

7.1. The centre of mass

The centre of mass position is defined by

$$q = r + \frac{2(r^{(2)} \cdot r^{(2)})r^{(2)}}{(r^{(2)} \cdot r^{(2)})^{3/2} + (r^{(3)} \cdot r^{(3)}) - \frac{3(r^{(2)} \cdot r^{(3)})^2}{4(r^{(2)} \cdot r^{(2)})}}. \tag{28}$$

We can check that both q and $q^{(1)}$ vanish for the centre of mass observer. Then, the fourth-order dynamical equation for the position of the charge can also be rewritten here as a system of two second-order differential equations for the positions q and r

$$q^{(2)} = 0 \qquad r^{(2)} = \frac{1 - q^{(1)} \cdot r^{(1)}}{(q - r)^2} (q - r) \tag{29}$$

a free motion for the centre of mass and a kind of central motion for the charge around the centre of mass.

For the non-relativistic electron we get in the low velocity case $q^{(1)} \to 0$ and $|q - r| = 1$, the equations of the Galilei case

$$q^{(2)} = 0 \qquad r^{(2)} = q - r \tag{30}$$

a free motion for the centre of mass and a harmonic motion around q for the position of the charge.

7.2. Interaction with some external field

The free equation for the centre of mass motion $q^{(2)} = 0$, represents the conservation of the linear momentum $\mathrm{d}P/\mathrm{d}t = 0$. But the linear momentum is written in terms of the centre of mass velocity as $P = m\gamma(q^{(1)})q^{(1)}$, so that the free dynamical equations (29) in the presence of an external field should be replaced by

$$P^{(1)} = F \qquad r^{(2)} = \frac{1 - q^{(1)} \cdot r^{(1)}}{(q - r)^2} (q - r) \tag{31}$$

where F is the external force and the second equation is left unchanged because we consider, even with interaction, the same definition of the centre of mass position:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = m\gamma(q^{(1)})q^{(2)} + m\gamma(q^{(1)})^3(q^{(1)} \cdot q^{(2)})q^{(1)}$$

we get

$$m\gamma(q^{(1)})^3(q^{(1)} \cdot q^{(2)}) = F \cdot q^{(1)}$$

and by leaving the highest derivative $q^{(2)}$ on the left-hand side we finally get the differential equations which describe the evolution of a relativistic spinning electron in the presence of an external electromagnetic field:

$$mq^{(2)} = \frac{e}{\gamma(q^{(1)})}[E + r^{(1)} \times B - q^{(1)}([E + r^{(1)} \times B] \cdot q^{(1)})] \tag{32}$$

$$r^{(2)} = \frac{1 - q^{(1)} \cdot r^{(1)}}{(q - r)^2}(q - r). \tag{33}$$
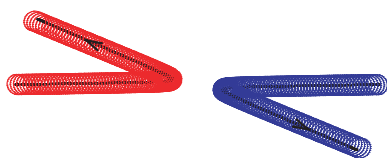
Figure 2. Scattering of two spinning electrons with the spins parallel, in their centre of mass frame. The scattering of two spinless electrons with the same energy and linear momentum is also depicted.
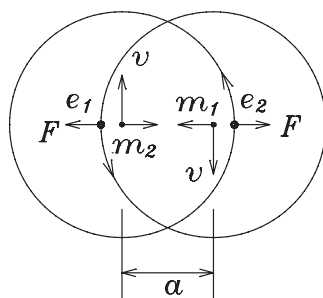

Figure 3. Initial position and velocity of the centre of mass and charges for a bound motion of a two-electron system with parallel spins. The circles would correspond to the trajectories of the charges if considered free. The interacting Coulomb force F is computed in terms of the separation distance between the charges.

8. Two-electron system

If we have the relativistic and non-relativistic differential equations satisfied by the charge of the spinning electrons we can analyse as an example, the interaction among them by assuming a Coulomb interaction between their charges. In this way we have a system of differential equations of the form (15) for each particle. For instance, the external field acting on charge e_1 is replaced by the Coulomb field created by the other charge e_2 at the position of e_1, and similarly for the other particle. The integration is performed numerically by means of the numerical integration program *Dynamics Solver* [10].

In figure 2 we represent the scattering of two spinning electrons analysed in their centre of mass frame. We send the particles with their spins parallel and with a nonvanishing impact parameter. In addition to the curly motion of their charges we can also depict the trajectories of their centres of mass. If we compare this motion with the Coulomb interaction of two spinless electrons coming from the same initial position and with the same velocity as the centre of mass of the spinning electrons we obtain the solid trajectory marked with an arrow. Basically this corresponds to the trajectory of the centre of mass of each spinning particle provided the two particles do not approach each other below the Compton wavelength. This can be understood because the average position of the centre of charge of each particle approximately coincides with its centre of mass and as long as they do not approach each other too closely the average Coulomb force is the same. The difference comes out when we consider a very deep interaction or very close initial positions.

In figure 3 we represent the initial positions for a pair of particles with the spins parallel. The initial separation a of their centres of mass is a distance below the Compton wavelength. We also consider that initially the centre of mass of each particle is moving with a velocity v
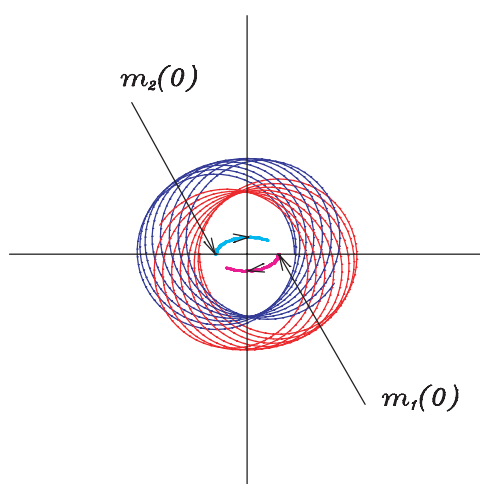
Figure 4. Bound motion of two electrons with parallel spins during a short period of time.

as depicted. That the spins are parallel is reflected by the fact that the internal motions of the charges, represented by the oriented circles that surround the corresponding centre of mass, have the same orientation. It must be remarked that the charge motion around its centre of mass can be characterized by a phase. The phases of each particle are chosen opposite to each other. We also represent the repulsive Coulomb force F computed in terms of the separation of the charges. This interacting force F has also been attached to the corresponding centre of mass, so that the net force acting on point m_2 is directed towards point m_1, and conversely. We thus see there that a repulsive force between the charges represents an attractive force between their centres of mass when located at such a short distance.

In figure 4 we depict the evolution of the charges and masses of this two-electron system for $a = 0.4\lambda_C$ and $v = 0.004c$ during a short time interval. Figure 5 represents only the motions of the centres of mass of both particles for a longer time. It shows that the centre of mass of each particle remains in a bound region.

The evolution of the charges is not shown in this figure because it blurs the picture but it can be inferred from the previous figure. We have found bound motions at least for the range $0 \leqslant a \leqslant 0.8\lambda_C$ and velocity $0 \leqslant v \leqslant 0.01c$. We can also obtain similar bound motions if the initial velocity v has a component along the OX axis. The bound motion is also obtained for different initial charge positions such as those depicted in figure 3. This range for the relative phase depends on a and v but in general the bound motion is more likely if the initial phases of the charges are opposite to each other.

We thus see that if the separation between the centre of mass and centre of charge of a particle (*zitterbewegung*) is responsible for its spin structure, as has been shown in the formalism developed by the author, then this attractive effect and also a spin polarized tunnelling effect can be easily interpreted [11].

A bound motion for classical spinless electrons is not possible. We can conclude that one of the salient features of this example is the existence from the classical viewpoint of bound states for spinning electron–electron interaction. It is the spin structure which contributes to the prediction of new physical phenomena. If two electrons have their centres of mass separated by a distance greater than Compton's wavelength they always repel each other as
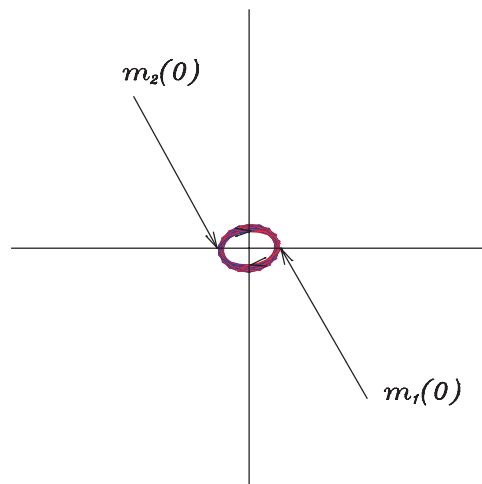
Figure 5. Evolution of the centre of mass of both particles for a longer time.

in the spinless case. But if two electrons have their centres of mass separated by a distance less than the Compton wavelength they can form from the classical viewpoint bound states provided some initial conditions on their relative initial spin orientation, position of the charges and centre of mass velocity are fulfilled.

The example analysed gives just a classical prediction, not a quantum one, associated with a model that satisfies Dirac's equation when quantized. The possible quantum mechanical bound states, if they exist, must be obtained from the corresponding analysis of two interacting quantum Dirac particles, bearing in mind that the classical bound states are not forbidden from the classical viewpoint. Bound states for a hydrogen atom can exist from the classical viewpoint for any negative energy and arbitrary angular momentum. It is the quantum analysis of the atom that gives the correct answer to the allowed bound states. The quantum mechanical analysis of a two-electron system is left to a subsequent paper.

Acknowledgments

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