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Classical particle systems: I. Galilei free particle

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Abstract. This is the first of a series of papers concerning the group theoretical approach to the classical particle systems, related to a definite special relativity principle and which are explicitly expressed in a Lagrangian formalism. In the Galilei case, the free classical particle can be characterised by its mass, internal energy and spin, similarly in the quantal version. The spin appears as a derived observable related to the orientation and to the internal motion of the system and has in general three parts; one of intrinsic nature, one, like a spherically symmetric body, related to the angular velocity and a third which is the (anti)orbital part of the relative motion of the system around its centre of mass. Galilei photons are massless particles, with spin and linear momentum lying along the direction of motion, travelling at infinite speed and carrying no energy. The Lagrangian of a free particle with intrinsic spin is no longer time-reversal invariant.

1. Introduction

Since Wigner's work on the inhomogeneous Lorentz group (Wigner 1939) an elementary particle is defined as that quantum system whose Hilbert space of states carries a unitary irreducible representation of the Poincaré group.

After Bargmann's definition of projective unitary representations of a group, i.e. unitary up to a phase (Bargmann 1954), the concept of elementarity has to be redefined if various possible relativity principles are going to be considered.

By a principle of special relativity we shall understand the existence of a class of equivalent observers, called inertial observers, for which the laws of physics have the same form, and such that for any two of them, their relative spacetime coordinate measurements y'^{μ} and y^{ν} of the same spacetime event, are related by an element g of a group G, $y'^{\mu}=f^{\mu}(g;y^{\nu})$, called a kinematical group (Bacry and Levy-Leblond 1968), which acts transitively on the spacetime universe Y, as a transformation group.

Then given the group G, a quantum elementary particle is that system whose Hilbert space of states is the representation space of a projective unitary irreducible representation of G.

Projective unitary representations are equivalent to true unitary representations if the group G has no non-trivial central extensions (Bargmann 1954). This is so for the Poincaré group, but not for the Galilei group for which its projective unitary representations are precisely interpreted as the 'physical representations' (Inönu and Wigner 1952, Levy-Leblond 1963), while the unitary ones are, in some sense, related to massless systems.

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In the present work we are concerned with Galilei systems, for which in the quantum version the projective unitary irreducible representations are characterised by the three observables m, U and s. These observables are interpreted as the mass, internal energy and absolute value of the spin respectively and correspond to three independent invariant operators of the extended Galilei Lie algebra (Inönu and Wigner 1952, Levy-Leblond 1963, 1971).

However, this method of characterising the system by the three invariants (m, U, s) is not inherent in its quantum nature, but is also characteristic of other Galilei group realisations, such as the irreducible canonical realisations (Pauri and Prosperi 1968, Levy-Leblond 1971) which are interpreted as representing elementary particles when the canonical coordinates are considered as centre-of-mass position coordinates. What is actually of quantum nature is that observables such as spin or energy have a discrete spectrum.

In order to characterise the particle systems by means of a Lagrangian which depends on the group invariants of the system, we shall follow the Lagrangian formalism developed by Levy-Leblond (Levy-Leblond 1969) which is outlined in § 2. A concept of a classical particle is defined, based upon the condition that the corresponding Lagrangian is necessarily free. If a classical system is characterised by its Lagrangian, quantisation can be obtained, without passing to the canonical formalism, by means of the path integral approach (Feynman and Hibbs 1965). This will be treated in a subsequent paper.

In § 3 different kinds of Galilei particles are considered, according to their possible degrees of freedom and to the nature of their basic kinematical observables, which are defined by their transformation properties under the Galilei group.

We shall see that the appearance of the observable spin is associated with increasing number of degrees of freedom, from three for the point particle to nine in the case of the more general Galilei particle, and is directly related to the orientation of the system and to the non-coincidence of the position of the system with its centre of mass.

Invariance properties of the different Lagrangians are analysed in § 4 under the discrete transformations of space and time reversal, remarking that time-reversal invariance is violated if the system has a spin of a non-rotating nature.

2. Lagrangian formalism

We briefly outline and extend the main features, some of them without proofs, of the ideas developed by Levy-Leblond in his work (Levy-Leblond 1969).

Let X be a n-dimensional differentiable manifold such that the action of the mechanical system is a real valued continuous and differentiable function $A(x_1, x_2)$ defined on $X \times X$. This can be expressed in terms of a Lagrangian function as

$$A(x_1, x_2) = \int_{\tau_1}^{\tau_2} L(x(\tau), \dot{x}(\tau), \ldots ; \tau)\, \mathrm{d}\tau \qquad (1)$$

where L depends on the derivatives up to a finite order, such that the real trajectory followed by the system $x(\tau)$ in X is obtained by the corresponding variational principle which makes extremal the action functional between the end points $x_1 = x(\tau_1)$ and $x_2 = x(\tau_2)$. For $A(x_1, x_2)$ we understand the value of the integral (1) when the real trajectory $x(\tau)$ is considered, and is only a function of the $2n$ independent variables (x_1, x_2). The evolution is expressed in terms of a parameter τ, and $\dot{x}(\tau) = \mathrm{d}x(\tau)/\mathrm{d}\tau$.

If the function A is known, the Lagrangian L can be obtained by the limiting process:

$$L = \lim_{x' \to x} \left(\frac{\partial A(x, x')}{\partial x'} \right) \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}. \tag{2}$$

Also from (1) by assuming a variable end point $x(\tau_2)$ and taking the derivative of both sides with respect to τ_2,

$$L = \frac{\partial A(x(\tau_1), x(\tau))}{\partial x(\tau)} \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}. \tag{3}$$

We find from this, by taking the corresponding partial derivatives, that L is not an explicit function of τ, it does not depend on the derivatives of order higher than one and is a first degree homogeneous function of the derivatives. Hence, it can be written as

$$L = (\partial L/\partial \dot{x}^i) \cdot \dot{x}^i = F_i(x(\tau), \dot{x}(\tau)) \cdot \dot{x}^i(\tau) \tag{4}$$

where the summation convention on index i is assumed. The observables F_i are zeroth-degree homogeneous functions of the derivatives. Thus, the F_i are functions of the n variables $x^i(\tau)$ and of the $n-1$ quotients $\dot{x}^k(\tau)/\dot{x}^n(\tau)$ obtained by dividing the $\dot{x}^k(\tau)$ by any one $\dot{x}^n(\tau)$.

We shall call the $x^i(\tau)$ the kinematical variables of the system and the manifold they span X, the kinematical space. The $\dot{x}^k/\dot{x}^n = \mathrm{d}x^k/\mathrm{d}x^n$ are called the velocities of the system, giving to the F_i the generic name of momenta. The functions $\dot{x}^i(\tau)$ have no specific name since we do not know at first the physical nature of the evolution parameter τ, so they are simply called the derivatives.

We shall assume the existence of a relativity principle characterised by a group G, which has a realisation on X given by $x' = gx$, or in local coordinates

$$x'^i = f^i(g^\sigma; x^j) \tag{5}$$

x^j and g^σ being the corresponding coordinates of x and g.

The invariance of the dynamical equations for two inertial observers related by a transformation g, implies that the action must transform as follows

$$A(gx_1, gx_2) = A(x_1, x_2) + \alpha(g; x_2) - \alpha(g; x_1). \tag{6}$$

$\alpha(g; x)$ is a real-valued continuous and differentiable function defined on $\mathrm{G} \times X$, called a gauge function for the group G and the kinematical space X. It is not uniquely defined but rather it obeys the identity, for all $g', g \in \mathrm{G}$,

$$\alpha(g'; gx) + \alpha(g; x) - \alpha(g'g; x) = \xi(g', g) \tag{7}$$

where $\xi(g', g)$ is an exponent of G.

If we assume that the evolution parameter τ is a group invariant parameter, (5) can be written

$$x'^i(\tau) = f^i(g; x(\tau)) \tag{8}$$

which leads for the derivatives $\dot{x}(\tau)$ to the transformation under the group

$$\dot{x}'^i(\tau) = (\mathrm{d}/\mathrm{d}\tau) f^i(g; x(\tau)) \tag{9}$$

and the Lagrangian, taking into account (2), transforms as follows

$$L(gx(\tau), (\mathrm{d}/\mathrm{d}\tau)gx(\tau)) = L(x(\tau), \dot{x}(\tau)) + (\mathrm{d}/\mathrm{d}\tau)\alpha(g; x(\tau)). \tag{10}$$

Two gauge functions α_1 and α_2 are said to be equivalent if their difference can be expressed as

$$\alpha_1(g;x)-\alpha_2(g;x)=\phi(x)-\phi(gx)+\chi(g) \tag{11}$$

where ϕ and χ are some functions on X and G respectively.

If the Lagrangian L satisfies (10) for the gauge α, then the equivalent Lagrangian $L+\mathrm{d}\phi(x)/\mathrm{d}\tau$, satisfies (10) with the equivalent gauge $\alpha(g;x)-\phi(x)+\phi(gx)$. Then instead of considering all possible gauge function solutions of (7), we shall consider from now on the set of equivalence classes of gauge functions.

With G and X fixed, to every $\alpha(g;x)$, up to an equivalence, there corresponds through (10) a family of Lagrangians. These are in general non-equivalent, representing consequently a family of different mechanical systems with the same kinematical space X, for which the dynamical equations satisfy the relativity principle.

If L_1 and L_2 are two Lagrangians which satisfy (10) for the same gauge $\alpha(g;x)$ then $L_0=L_1-L_2$ is an invariant Lagrangian. Thus the general solution of (10) is constructed by adding to a particular solution of (10) the general solution of

$$L_0(gx,(\mathrm{d}/\mathrm{d}\tau)gx)=L_0(x,\dot{x}). \tag{12}$$

Then every $\alpha(g;x)$ characterises a particular mechanical system which properties are, in some sense, determined by the gauge function. It is in fact the gauge variance of the Lagrangians, and not its invariance, which will give information about the possible different relativistic mechanical systems.

If we consider G as a group of active transformations on the spacetime Y, it contains what are called the inertial or free motions on Y. Y is a homogeneous space of G since G acts transitively on it. A point particle system is that system for which the kinematical space is $X=Y$. It has no internal structure and the only kinematical observables are position and time. Levy-Leblond checks (1969) that the relativity principle in its form (10) leads for that system, in the Galilei and Poincaré case, to a free Lagrangian. There seems to exist an intimate connection between systems for which X is a homogeneous space of G and the fact of these systems being free. If X is a homogeneous space of G, then given two points x_1 and x_2 of X, there always exists a $g\in$ G such that $x_2=gx_1$. This g, considered as an active transformation, is in fact the inertial motion which brings the system from x_1 to x_2. Otherwise if X is not an homogeneous space of G, the equation $x_2=gx_1$ with x_1 and x_2 fixed has not in general a solution, so that the motion from x_1 to x_2 in X space will not be in general a free motion.

If we go to a system for which its kinematical space X, still being a homogeneous space of G, is larger than the spacetime Y, then we increase their degrees of freedom by adding to it some other kinematical observables, which we interpret as internal degrees of freedom. Their physical interpretation will be guided by their transformation properties under the group G. However this process of increasing X ends when we arrive at $X=$ G.

If X is a homogeneous space of G, then the possible gauge functions are

$$\alpha(g;x)=\xi(g,h_x) \tag{13}$$

where $h_x\in G$ is any element of the equivalence class $x\in X$.

We shall verify in what follows, for the Galilei group, that even in the case $X=$ G, the relativity principle (10) implies that the system is free, with some internal structure that will be classically interpreted. Further, if we consider a system for which X is no longer a homogeneous space of G, for instance a two-point particle system, (10) leads to a Lagrangian for which some interaction, depending on the relative distance, is present.

All the above comments allow us to define a classical particle system as that mechanical system for which its kinematical space is a homogeneous space of the group G.

3. Galilei particles

3.1. *The Galilei group*

Let G be the Galilei group $\mathcal{G}$ with elements $g \equiv (b, \boldsymbol{a}, \boldsymbol{v}, R)$ where we follow Levy-Leblond's notation (1971). Rotations are parametrised in terms of a 3-vector $\boldsymbol{\mu} = \tan(\alpha/2)\boldsymbol{e}$, $\boldsymbol{e}$ being a unit vector along the rotation axis, and $\alpha \in [0, \pi]$ the clockwise rotated angle when looking along the direction given by $\boldsymbol{e}$. The orthogonal rotation matrix $R(\boldsymbol{\mu})$ is given by

$$R(\boldsymbol{\mu})^i_j = \frac{1}{1+\boldsymbol{\mu}^2}[(1-\boldsymbol{\mu}^2)\delta^i_j + 2\mu^i\mu_j - 2\varepsilon^i{}_{jk}\mu^k] \tag{14}$$

where $\mu^i = \mu_i$, δ^i_j is the Kroenecker delta and $\varepsilon^i{}_{jk}$ is the completely antisymmetric symbol.

Then an element g is parametrised by $(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu})$ and its action on the spacetime Y is

$$t' = t + b \tag{15}$$

$$\boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}. \tag{16}$$

The group law

$$(b', \boldsymbol{a}', \boldsymbol{v}', \boldsymbol{\mu}')(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu}) = \left(b' + b, \boldsymbol{a}' + \boldsymbol{v}'b + R(\boldsymbol{\mu}')\boldsymbol{a}, \boldsymbol{v}' + R(\boldsymbol{\mu}')\boldsymbol{v}, \frac{\boldsymbol{\mu}' + \boldsymbol{\mu} + \boldsymbol{\mu}' \times \boldsymbol{\mu}}{1 - \boldsymbol{\mu}' \cdot \boldsymbol{\mu}}\right). \tag{17}$$

Some remarkable subgroups are

$\mathcal{T} = \{(b, 0, 0, 0)/b \in \mathbb{R}\}$	time-translation group
$\mathcal{S} = \{(0, \boldsymbol{a}, 0, 0)/\boldsymbol{a} \in \mathbb{R}^3\}$	space-translation group
$\mathcal{V} = \{(0, 0, \boldsymbol{v}, 0)/\boldsymbol{v} \in \mathbb{R}^3\}$	pure Galilei transformation group
$\mathcal{R} = \{(0, 0, 0, \boldsymbol{\mu})/\boldsymbol{\mu} \in \mathbb{R}^3_c\}$	rotation group.

By $\mathbb{R}^3_c$ we mean the compact $\mathbb{R}^3$ where compactification has been obtained by adding to $\mathbb{R}^3$ the end points, considered to be the same point, of every straight line passing through the origin.

The homogeneous Galilei group is the semidirect product $\mathcal{V}\square\mathcal{R}$. The Newtonian spacetime Y is just $Y = \mathcal{G}/(\mathcal{V}\square\mathcal{R})$. This homogeneous space corresponds to the kinematical space of a point particle of mass m (Levy-Leblond 1969).

We shall consider next the following two examples: the case $X = \mathcal{G}/\mathcal{V}$ and the most general case $X = \mathcal{G}$.

3.2. Spinning particle

Instead of considering first the general case, let us consider that mechanical system for which the kinematical space $X = \mathscr{G}/\mathscr{V}$. An element of X is given by the seven real numbers $x \equiv (t, \boldsymbol{r}, \boldsymbol{\rho})$ which under $\mathscr{G}$ transform as follows

$$t' = t + b, \qquad \boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a} \qquad \boldsymbol{\rho}' = \frac{\boldsymbol{\mu} + \boldsymbol{\rho} + \boldsymbol{\mu} \times \boldsymbol{\rho}}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}}. \tag{18}$$

The way these variables transform allows us to say that $t(\tau)$ is the *time*, $\boldsymbol{r}(\tau)$ the *position* and $\boldsymbol{\rho}(\tau)$ the *orientation* of the system. If τ is assumed to be invariant under $\mathscr{G}$, the derivatives transform:

$$\dot{t}'(\tau) = \dot{t}(\tau), \qquad \dot{\boldsymbol{r}}'(\tau) = R(\boldsymbol{\mu})\dot{\boldsymbol{r}}(\tau) + \boldsymbol{v}\dot{t}(\tau) \tag{19a, b}$$

$$\dot{\boldsymbol{\rho}}'(\tau) = \frac{(\dot{\boldsymbol{\rho}} + \boldsymbol{\mu} \times \dot{\boldsymbol{\rho}})(1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}) + (\boldsymbol{\rho} + \boldsymbol{\mu} + \boldsymbol{\mu} \times \boldsymbol{\rho})\boldsymbol{\mu} \cdot \dot{\boldsymbol{\rho}}}{(1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho})^2} \tag{19c}$$

and we shall replace $\dot{\boldsymbol{\rho}}(\tau)$ by the linear function of it

$$\boldsymbol{\omega}(\tau) = 2(\dot{\boldsymbol{\rho}} + \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}})/(1 + \boldsymbol{\rho}^2) \tag{20}$$

with inverse $\dot{\boldsymbol{\rho}} = \frac{1}{2}(\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\rho} + \boldsymbol{\rho}(\boldsymbol{\rho} \cdot \boldsymbol{\omega}))$ such that the Jacobian $\partial(\boldsymbol{\rho}, \dot{\boldsymbol{\rho}})/\partial(\boldsymbol{\rho}, \boldsymbol{\omega}) \neq 0$. If τ would be the time, $\boldsymbol{\rho}(\tau)$ would mean the total rotation undergone by the system since the instant $\tau = 0$, and $\boldsymbol{\omega}(\tau)$ would be the instantaneous angular velocity at time τ. Instead of (19c) we have for $\boldsymbol{\omega}(\tau)$:

$$\boldsymbol{\omega}(\tau)' = R(\boldsymbol{\mu})\boldsymbol{\omega}(\tau). \tag{19d}$$

According to (4) the momenta will be functions of the seven variables t, $\boldsymbol{r}$, $\boldsymbol{\rho}$ and of six possible quotients among the seven $\dot{t}$, $\dot{\boldsymbol{r}}$, $\boldsymbol{\omega}$.

It is not possible to divide by the $\dot{r}_i$ and the ω_j since by a pure Galilei transformation and a rotation respectively we can make them vanish, leaving the quotient undefined. So we are just left with $\dot{t}$ whenever possible, i.e. when $\dot{t} \neq 0$. Let us consider this situation.

If $\dot{t}$ is different from zero this will allow us to invert the function $t(\tau)$ and obtain $\tau(t)$ and the system evolution will be given in terms of $\boldsymbol{r}(t)$ and $\boldsymbol{\rho}(t)$, establishing the dynamics constraint between the kinematical variables. If $\dot{t}(\tau) > 0$ we say that the system goes forward in time, and backward in time when $\dot{t}(\tau) < 0$.

We call the *velocity* to $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and the *angular velocity* to $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$ and they transform under $\mathscr{G}$ as

$$\begin{aligned} \boldsymbol{u}(\tau)' &= R(\boldsymbol{\mu})\boldsymbol{u}(\tau) + \boldsymbol{v} \\ \boldsymbol{\Omega}(\tau)' &= R(\boldsymbol{\mu})\boldsymbol{\Omega}(\tau). \end{aligned} \tag{21}$$

We write for the momenta $H = -\partial L/\partial \dot{t}$, $p_i = \partial L/\partial \dot{r}^i$ and $S_i = \partial L/\partial \omega^i$ and they are functions of $(t, \boldsymbol{r}, \boldsymbol{\rho}, \boldsymbol{u}, \boldsymbol{\Omega})$, so that the most general form for the Lagrangian will be:

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{r}} + \boldsymbol{S} \cdot \boldsymbol{\omega} - H\dot{t}. \tag{22}$$

The Galilei group has only one family of equivalence classes of exponents $\xi_m(g', g)$ characterised by a real parameter m such that one of them is

$$\xi_m(g', g) = m(\boldsymbol{v}'^2 b/2 + \boldsymbol{v}' \cdot R(\boldsymbol{\mu}')\boldsymbol{a}) \tag{23}$$

so that the equivalence classes of gauge functions are

$$\alpha_m(g;x)=\xi_m(g,h_x)=m(\boldsymbol{v}^2t/2+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{r}) \tag{24}$$

and

$$(\mathrm{d}/\mathrm{d}\tau)\alpha_m(g;x(\tau))=m(\boldsymbol{v}^2\dot{t}/2+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\dot{\boldsymbol{r}}). \tag{25}$$

Choosing a particular $\alpha_m(g;x)$, the real parameter m that characterises it, is called the *mass* of the system.

The relativity principle in the form (10) with $\alpha(g;x)$ given in (24), leads for the momenta to the transformation properties under $\mathscr{G}$:

$$H(\tau)'=H(\tau)+\tfrac{1}{2}mv^2+\boldsymbol{v}\cdot R(\boldsymbol{\mu})\boldsymbol{p}(\tau) \tag{26}$$

$$\boldsymbol{p}(\tau)'=R(\boldsymbol{\mu})\boldsymbol{p}(\tau)+m\boldsymbol{v} \tag{27}$$

$$\boldsymbol{S}(\tau)'=R(\boldsymbol{\mu})\boldsymbol{S}(\tau). \tag{28}$$

Under a general infinitesimal Galilei transformation with parameters δg^σ we obtain:

$$\delta A=F_\sigma\delta g^\sigma \tag{29}$$

$$\delta x^i=\delta x^i_\sigma\delta g^\sigma \tag{30}$$

and Noether's theorem defines the constants of the motion:

$$C_\sigma=F_\sigma-(\partial L/\partial\dot{x}^i)\delta x^i_\sigma \tag{31}$$

which are named as follows.

Under time translation we get the *energy*

$$H=-\partial L/\partial\dot{t} \tag{32}$$

under space translation the *linear momentum*

$$p_i=\partial L/\partial\dot{r}^i \tag{33}$$

under a pure Galilei transformation the *Galilei momentum*

$$g_i=mr_i-t\,\partial L/\partial\dot{r}^i$$

or

$$\boldsymbol{g}=m\boldsymbol{r}-\boldsymbol{p}t \tag{34}$$

and finally under a rotation around axis i, $\delta\mu_i=\tfrac{1}{2}\delta\alpha_i$ the *angular momentum*

$$J_i=(\partial L/\partial\dot{r}^k)\delta r^k_i+(\partial L/\partial\dot{\rho}^k)\delta\rho^k_i \tag{35}$$

where

$$\delta r^k_i=\varepsilon_{ij}{}^k r^j,\qquad \delta\rho^k_i=\tfrac{1}{2}(\delta^k_i+\varepsilon^k{}_{ij}\rho^j+\rho^k\rho_i)$$

and

$$\frac{\partial L}{\partial\dot{\rho}^k}\delta\rho^k_i=\frac{\partial L}{\partial\omega^j}\frac{\partial\omega^j}{\partial\dot{\rho}^k}\delta\rho^k_i=\frac{\partial L}{\partial\omega^i} \tag{36}$$

and thus

$$\boldsymbol{J}=\boldsymbol{r}\times\boldsymbol{p}+\boldsymbol{S}. \tag{37}$$

Since $\dot{g}=0=m\dot{r}-p\dot{t}$ this implies that $p=mu$ and being $\dot{p}=0$, the system moves with constant velocity u.

Because of (26) and (27) $H-p^2/2m=U$ is a group invariant and we call it the *internal energy*, which due to (32) and (33) is also a constant of the motion, so that it does not depend on the variables t, r and u. When $p=0$, $J=S$ so we call S the *spin* which is also a constant of the motion and from (28) we see that S^2 is a group invariant and S depends only on the variables ρ and Ω.

The most general Lagrangian for the gauge function $\alpha_m(g;x)$ is of the form

$$L=\tfrac{1}{2}m\dot{r}^2/\dot{t}-U(\rho,\Omega)\dot{t}+S(\rho,\Omega)\cdot\omega \tag{38}$$

and S and U verify the system of differential equations:

$$2\varepsilon^{i}{}_{lk}\Omega^k\,\partial U/\partial\Omega^i+(\delta_{jl}+\rho_j\rho_l+\varepsilon_{jl}{}^{k}\rho_k)\partial U/\partial\rho_j=0 \tag{39}$$

$$2\varepsilon^{j}{}_{lk}\Omega^k\,\partial S_i/\partial\Omega^j+(\delta_{jl}+\rho_j\rho_l+\varepsilon_{jl}{}^{k}\rho_k)\,\partial S_i/\partial\rho_j=2\varepsilon_{il}{}^{k}S_k \tag{40}$$

$$\partial U/\partial\Omega_i=\Omega^j\,\partial S_j/\partial\Omega^i \tag{41}$$

where (39) comes from the U group invariance, (40) is (28) in differential form and (41) is obtained from the spin definition.

We see that the Lagrangian (38), similarly as in the quantum situation, depends on the three invariants m, U, S and U being a constant of the motion, the term $U\dot{t}=\mathrm{d}(Ut)/\mathrm{d}\tau$ is a total derivative and can be neglected to obtain the dynamical equations, which is equivalent to considering the particular case $U=0$. This is to be compared with the irreducible unitary projective representations of $\mathcal{G}$ (Inönu and Wigner 1952, Levy-Leblond 1963) that are characterised by the invariants (m, U, S) such that the representations with fixed m and S are all equivalent to the $(m, 0, S)$ representation. The internal energy for a free particle has no dynamical influence.

The difference between two Lagrangians (38) with the same gauge function $\alpha_m(g;x)$ is the invariant Lagrangian:

$$L_0=S\cdot\omega-U\dot{t} \tag{42}$$

where S and U are functions of ρ and Ω and satisfy the system of differential equations (39)-(41). The gauge function characterises the mass of the particle but not its internal structure which is associated with the invariance of the Lagrangian. Two free particles with the same mass can differ by their spin and internal energy.

The general solution of (39) is an arbitrary function of Ω^2 and β where

$$\beta=\Omega_i R(\rho)^i_k C^k \tag{43}$$

and C^k, $k=1,2,3$ are three arbitrary real numbers.

The solution of (40) depends on the seven arbitrary invariant functions $I(\Omega^2,\lambda)$, $D^k(\Omega^2,\gamma_k)$, $N^k(\Omega^2,\zeta_k)$, $k=1,2,3$ in the form:

$$S^i=I(\Omega^2,\lambda)\Omega^i+R(\rho)^i_k D^k(\Omega^2,\gamma_k)+\varepsilon^{i}{}_{jl}\Omega^j R(\rho)^l_k N^k(\Omega^2,\zeta_k) \tag{44}$$

where the variables λ, γ_k, ζ_k, $k=1,2,3$ are of the form (43) with, in general, different C^k's and such that the invariant observable S^2 must be just a function of Ω^2 and η, where this

$$\eta=\Omega_i R(\rho)^i_k G^k \tag{45}$$

depends only on the three arbitrary real numbers G^k.

Finally, equations (41) establish another relationship among the invariant functions U, I, D^k and N^k.

Roughly speaking, the spin is of the form

$$S = I\mathbf{\Omega} + \mathbf{\Sigma} + N\mathbf{\Omega} \times \mathbf{\Sigma} \tag{46}$$

which enables us to define I as the *moment of inertia* and $\mathbf{\Sigma}$ as the *intrinsic spin*, to distinguish it from the $I\mathbf{\Omega}$ term which reflects its spherically symmetric rotating nature. If $I \neq 0$ we can define a characteristic size of the system, the *gyration radius* r_0 by

$$mr_0^2 = I \tag{47}$$

which is not in general a constant of the motion. If $r_0(\tau)$ is indeed a constant we shall say that the system is a rigid particle.

A particular model of a Galilei particle consists of fixing the arbitrary functions U, I, D^k and N^k in a compatible way. For instance, if we take for I and D^k constant real numbers and $N^k = 0$, we get from (39) and (41)

$$U = \tfrac{1}{2}I\Omega^2 + E \tag{48}$$

where E is a real number that can be interpreted as the *excitation energy*, and the Lagrangian takes the form, in a time evolution description

$$L = \tfrac{1}{2}m(\mathrm{d}\boldsymbol{r}/\mathrm{d}t)^2 + \tfrac{1}{2}I\Omega^2 + \mathbf{\Sigma} \cdot \mathbf{\Omega} - E \tag{49}$$

where $\mathbf{\Sigma}^i = R(\boldsymbol{\rho})^i{}_k D^k$.

In this model $\mathrm{d}\boldsymbol{S}/\mathrm{d}t = \mathrm{d}\mathbf{\Sigma}/\mathrm{d}t + I\, \mathrm{d}\mathbf{\Omega}/\mathrm{d}t$ and $\mathrm{d}\mathbf{\Sigma}/\mathrm{d}t = \mathbf{\Omega} \times \mathbf{\Sigma}$ and $\boldsymbol{S}$ being a constant of the motion

$$\mathrm{d}(I\mathbf{\Omega})/\mathrm{d}t = I^{-1}\boldsymbol{S} \times (I\mathbf{\Omega}). \tag{50}$$

We see that the kinematical variable $\boldsymbol{\rho}(t)$ is just the actual orientation at time t, of the intrinsic spin $\mathbf{\Sigma}(t)$, if we have chosen as initial conditions $\boldsymbol{\rho}(0) = 0$ and $\mathbf{\Sigma}_k(0) = D_k$, and it rotates with angular velocity $\mathbf{\Omega}$. The rotating term $I\mathbf{\Omega}$ precesses around $\boldsymbol{S}$ with angular velocity $\boldsymbol{S}/I$, such that the sum $\mathbf{\Sigma} + I\mathbf{\Omega} = \boldsymbol{S}$ remains a constant vector.

3.3. *Photons*

Following with the same kinematical space $X = \mathscr{G}/\mathscr{V}$, and thus with the same class of gauge functions (24) let us consider the situation in which $\dot{t} = 0$. Hence no velocities can be defined and the most general Lagrangian will be

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{r}} + \boldsymbol{S} \cdot \boldsymbol{\omega} \tag{51}$$

where $\boldsymbol{p}$ and $\boldsymbol{S}$ are only functions of $\boldsymbol{\rho}$, which transform under $\mathscr{G}$:

$$\boldsymbol{p}' = R(\boldsymbol{\mu})\boldsymbol{p} + m\boldsymbol{v} \tag{52}$$

$$\boldsymbol{S}' = R(\boldsymbol{\mu})\boldsymbol{S}. \tag{53}$$

Since $\boldsymbol{p}$, being only a function of $\boldsymbol{\rho}$, must be invariant under a pure Galilei transformation this implies that $m = 0$. The particle is massless and the general solutions of (52) and (53) are:

$$p^i(\boldsymbol{\rho}) = R(\boldsymbol{\rho})^i{}_k A^k \tag{54}$$

$$S^i(\boldsymbol{\rho}) = R(\boldsymbol{\rho})^i_k C^k \tag{55}$$

where A^k, C^k, $k = 1, 2, 3$ are arbitrary real numbers.

Constants of the motion are

Energy	$H = 0$
Linear momentum	$\boldsymbol{p}$
Galilei momentum	$\boldsymbol{g} = \boldsymbol{p}t$
Angular momentum	$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{S}.$

Because $\dot{\boldsymbol{p}} = 0 = \boldsymbol{\omega} \times \boldsymbol{p}$, then $\boldsymbol{\omega}$ and $\boldsymbol{p}$ are parallel vectors and similarly $\dot{\boldsymbol{S}} = \boldsymbol{\omega} \times \boldsymbol{S}$, then $\dot{\boldsymbol{S}} \cdot \boldsymbol{S} = 0 = \mathrm{d}S^2/\mathrm{d}\tau$, the modulus of $\boldsymbol{S}$ is a constant of the motion. From $\dot{\boldsymbol{J}} = 0 = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \dot{\boldsymbol{S}}$, by taking the dot product with $\boldsymbol{S}$, we get $\boldsymbol{S} \cdot (\dot{\boldsymbol{r}} \times \boldsymbol{p}) = 0$, and $\boldsymbol{S}$ lies on the plane determined by $\dot{\boldsymbol{r}}$ and $\boldsymbol{p}$, with a constant projection on $\boldsymbol{p}$ since $\boldsymbol{S}$ rotates with $\boldsymbol{\omega}$ which is along $\boldsymbol{p}$; then we can write $\boldsymbol{S} = \alpha\boldsymbol{p} + \beta\dot{\boldsymbol{r}}$ with some constant α. Since $\boldsymbol{S}$ is only a function of $\boldsymbol{\rho}$ and not of $\dot{\boldsymbol{r}}$, then $\beta = 0$ and $\boldsymbol{S}$ and $\boldsymbol{p}$ lie along the same direction. Hence $\boldsymbol{S}$ is also another constant of the motion, and then $\dot{\boldsymbol{r}} \times \boldsymbol{p} = 0$ being consequently $\dot{\boldsymbol{r}}$ and $\boldsymbol{p}$ parallel.

We have no dynamical equation to determine $\boldsymbol{r}(\tau)$ but we know that $\dot{\boldsymbol{r}}(\tau)$ has a constant direction and evolves between the points $\boldsymbol{r}(\tau_1)$ and $\boldsymbol{r}(\tau_2)$ and thus a possible solution is:

$$\boldsymbol{r}(\tau) = f(\tau) \frac{\boldsymbol{r}(\tau_2) - \boldsymbol{r}(\tau_1)}{\|\boldsymbol{r}(\tau_2) - \boldsymbol{r}(\tau_1)\|} \tag{56}$$

i.e. a straight line joining the end points at constant time, and being $f(\tau)$ unknown.

Since the physical significance of τ is unknown, we associate the motion of the particle with the direction of $\boldsymbol{p}$. If $\boldsymbol{p}$ is pointing from $\boldsymbol{r}(\tau_1)$ to $\boldsymbol{r}(\tau_2)$ we say that the particle travels from $\boldsymbol{r}(\tau_1)$ to $\boldsymbol{r}(\tau_2)$ at constant time, i.e. at infinite speed, carries no mass and energy, linear momentum $\boldsymbol{p}$ and spin $\boldsymbol{S}$ which has no transversal components to the direction of the motion, and such that p and S are the same for every Galilei observer. We call such a particle a Galilei photon.

We see that the kinematical space for the Galilei photon is the Euclidean group $\mathscr{E} = \mathscr{T} \square \mathscr{R}$, which has no central extensions, and the gauge functions for it are zero, so that the most general Euclidean particle is this Galilei photon.

3.4. General Galilei particle

Let us consider that mechanical system for which $X = \mathscr{G}$. An element of X will be given by the ten real numbers $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho})$ which under $\mathscr{G}$ transform as follows:

$$t' = t + b \tag{57}$$

$$\boldsymbol{r}' = R(\boldsymbol{\mu})\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a} \tag{58}$$

$$\boldsymbol{u}' = R(\boldsymbol{\mu})\boldsymbol{u} + \boldsymbol{v} \tag{59}$$

$$\boldsymbol{\rho}' = \frac{\boldsymbol{\mu} + \boldsymbol{\rho} + \boldsymbol{\mu} \times \boldsymbol{\rho}}{1 - \boldsymbol{\mu} \cdot \boldsymbol{\rho}} \tag{60}$$

and we interpret $t(\tau)$ as the *time*, $\boldsymbol{r}(\tau)$ the *position*, $\boldsymbol{u}(\tau)$ the *velocity* and $\boldsymbol{\rho}(\tau)$ the *orientation* of the system.

Since we assume $\boldsymbol{u}$ to be definite, we interpret it as $\boldsymbol{u} = \dot{\boldsymbol{r}}/\dot{t} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, so we are faced in a $\dot{t} \neq 0$ situation.

The possible gauge functions are still the same as in (24) and we define the momenta $H = -\partial L/\partial \dot{t}$, $p_i = \partial L/\partial \dot{r}^i$, $mk_i = \partial L/\partial \dot{u}^i$, and $Z_i = \partial L/\partial \omega^i$, where $\boldsymbol{\omega}$ is again given by (20). We also define the *angular velocity* $\boldsymbol{\Omega} = \boldsymbol{\omega}/\dot{t}$, and the *acceleration* $\boldsymbol{\gamma} = \dot{\boldsymbol{u}}/\dot{t} = \mathrm{d}\boldsymbol{u}/\mathrm{d}t$ which transform as

$$\boldsymbol{\Omega}(\tau)' = R(\boldsymbol{\mu})\boldsymbol{\Omega}(\tau) \tag{61}$$

$$\boldsymbol{\gamma}(\tau)' = R(\boldsymbol{\mu})\boldsymbol{\gamma}(\tau) \tag{62}$$

and the momenta are functions of the variables $(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho}, \boldsymbol{\gamma}, \boldsymbol{\Omega})$ which under a general Galilei transformation transform as

$$H' = H + \tfrac{1}{2}mv^2 + \boldsymbol{v} \cdot R(\boldsymbol{\mu})\boldsymbol{p} \tag{63}$$

$$\boldsymbol{p}' = R(\boldsymbol{\mu})\boldsymbol{p} + m\boldsymbol{v} \tag{64}$$

$$\boldsymbol{k}' = R(\boldsymbol{\mu})\boldsymbol{k} \tag{65}$$

$$\boldsymbol{Z}' = R(\boldsymbol{\mu})\boldsymbol{Z}. \tag{66}$$

The observable $\boldsymbol{k}$ has dimensions of a length but it does not transform like the position of a point. It seems to be a relative position vector.

The Lagrangian takes the form

$$L = -H\dot{t} + \boldsymbol{p} \cdot \dot{\boldsymbol{r}} + m\boldsymbol{k} \cdot \dot{\boldsymbol{u}} + \boldsymbol{Z} \cdot \boldsymbol{\omega}. \tag{67}$$

By applying the generalised Noether's theorem (31) we get the following constants of the motion:

energy	H	(68)
linear momentum	$\boldsymbol{p}$	(69)
Galilei momentum	$\boldsymbol{g} = m\boldsymbol{r} - \boldsymbol{p}t - m\boldsymbol{k}$	(70)
angular momentum	$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{u} \times m\boldsymbol{k} + \boldsymbol{Z}.$	(71)

Since $\dot{\boldsymbol{g}} = 0$ we get that

$$\boldsymbol{p} = m\ (\dot{\boldsymbol{r}} - \dot{\boldsymbol{k}})/\dot{t} \tag{72}$$

and if we call $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$, it transforms as

$$\boldsymbol{q}' = R(\boldsymbol{\mu})\boldsymbol{q} + \boldsymbol{v}t + \boldsymbol{a} \tag{73}$$

so it is a position vector such that

$$\boldsymbol{p} = m\,\dot{\boldsymbol{q}}/\dot{t} = m\ \mathrm{d}\boldsymbol{q}/\mathrm{d}t \tag{74}$$

and we say that $\boldsymbol{q}$ represents the *centre-of-mass* position, and being $\boldsymbol{p}$ a constant of the motion, $\mathrm{d}\boldsymbol{q}/\mathrm{d}t$ is also a constant concluding that the centre of mass of the system moves with constant velocity.

Because $\boldsymbol{r} = \boldsymbol{q} + \boldsymbol{k}$, $\boldsymbol{k}$ is just the *relative position* of the system with respect to its centre of mass.

In terms of $\boldsymbol{q}$, the angular momentum takes the form:

$$\boldsymbol{J} = \boldsymbol{q} \times \boldsymbol{p} - \boldsymbol{k} \times m\ \mathrm{d}\boldsymbol{k}/\mathrm{d}t + \boldsymbol{Z} \tag{75}$$

and in the centre-of-mass frame, $\boldsymbol{p}=0$, $\boldsymbol{J}=\boldsymbol{S}$ is the *spin* of the system

$$\boldsymbol{S}=\boldsymbol{Z}-\boldsymbol{k}\times m\,\mathrm{d}\boldsymbol{k}/\mathrm{d}t \tag{76}$$

which is a constant of the motion and S a group invariant since $\boldsymbol{S}$ transforms as:

$$\boldsymbol{S}'=R(\boldsymbol{\mu})\boldsymbol{S}. \tag{77}$$

A non-vanishing spin is related to the existence of an orientation $\boldsymbol{\rho}$ and to the fact that the position of the system does not coincide with its centre of mass.

We see from (63) and (64) that $U=H-p^2/2m$ is again a group invariant, and in terms of centre-of-mass variables L becomes

$$L=\tfrac{1}{2}m\,\dot{\boldsymbol{q}}^2/\dot{t}-U\dot{t}+m(\dot{\boldsymbol{q}}\cdot\dot{\boldsymbol{k}})/\dot{t}+m\boldsymbol{k}\cdot[\dot{\boldsymbol{u}}+(\mathrm{d}\boldsymbol{k}/\mathrm{d}t)\times\boldsymbol{\omega}]+\boldsymbol{S}\cdot\boldsymbol{\omega}. \tag{78}$$

As U is a group invariant and $\boldsymbol{S}$ transforms as in (77), they are only functions of $(\boldsymbol{\rho},\boldsymbol{\gamma},\boldsymbol{\Omega})$ and we find anew that L is a function of the three invariants (m, U, S).

The term $m(\dot{\boldsymbol{q}}\cdot\dot{\boldsymbol{k}})/\dot{t}-U\dot{t}=(\mathrm{d}/\mathrm{d}\tau)(m(\dot{\boldsymbol{q}}/\dot{t})\cdot\boldsymbol{k}-Ut)$ is a total derivative and can be eliminated from L in order to obtain the dynamical equations. If we withdraw the centre-of-mass motion, we get in a time evolution description, for the centre-of-mass observer

$$L=\boldsymbol{S}\cdot\boldsymbol{\Omega}+m\boldsymbol{k}\cdot[\mathrm{d}^2\boldsymbol{k}/\mathrm{d}t^2+(\mathrm{d}\boldsymbol{k}/\mathrm{d}t)\times\boldsymbol{\Omega}] \tag{79}$$

which is a Lagrangian of a system with six degrees of freedom $\boldsymbol{\rho}(t)$ and $\boldsymbol{k}(t)$, where $\boldsymbol{S}$ is a vector function of $\boldsymbol{\rho}(t)$, $\boldsymbol{\gamma}(t)=\mathrm{d}^2\boldsymbol{k}/\mathrm{d}t^2$ and $\boldsymbol{\Omega}(t)$. The dynamical equations of this internal motion, usually called the zitterbewegung, are

$$\frac{\partial(\boldsymbol{S}\cdot\boldsymbol{\Omega})}{\partial\rho^j}-\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{2}{1+\rho^2}\right)\left(\frac{\partial(\boldsymbol{S}\cdot\boldsymbol{\Omega})}{\partial\Omega^j}+m\varepsilon_{jil}k^i\frac{\mathrm{d}k^l}{\mathrm{d}t}\right)\right.$$
$$\left.+\left(\frac{2}{1+\rho^2}\right)\varepsilon_{jil}\left(\frac{\partial(\boldsymbol{S}\cdot\boldsymbol{\Omega})}{\partial\Omega_i}+m\varepsilon^{i}{}_{rh}k^r\frac{\mathrm{d}k^h}{\mathrm{d}t}\right)\rho^l\right]=0 \tag{80}$$

and

$$2m\left(\frac{\mathrm{d}^2\boldsymbol{k}}{\mathrm{d}t^2}+\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t}\times\boldsymbol{\Omega}+\tfrac{1}{2}\boldsymbol{k}\times\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right)_j+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial(\boldsymbol{S}\cdot\boldsymbol{\Omega})}{\partial\gamma^j}\right)=0 \tag{81}$$

which depend on the specific model of particle, determined by the explicit form of its spin.

The solution of this system in the general case is a rather cumbersome task, and even in the almost trivial case of $\boldsymbol{k}=0$, $\boldsymbol{S}=I\boldsymbol{\Omega}$ with a constant moment of inertia I, equations (80) reduce to the spherically symmetric rigid body dynamical equations, which are solved for $\boldsymbol{\Omega}(t)$ in terms of elliptic functions and a further (numerical) integration will yield the orientation $\boldsymbol{\rho}(t)$.

However, we can analyse this general Galilei particle, looking to its structure when increasing its degrees of freedom.

If $X=\mathcal{G}/(\mathcal{V}\square\mathcal{R})$ the particle has three degrees of freedom, its position coincides with its centre of mass, it moves with constant velocity and we have a spinless massive point particle.

If $X=\mathcal{G}/\mathcal{R}$ which corresponds to (67) with $\boldsymbol{Z}=0$, the particle has six degrees of freedom, the three of the centre-of-mass position $\boldsymbol{q}$ and the relative position $\boldsymbol{k}$. From (76) we see that for the centre-of-mass observer $\boldsymbol{S}=-\boldsymbol{k}\times m(\mathrm{d}\boldsymbol{k}/\mathrm{d}t)$ and $\boldsymbol{S}$ being a constant of the motion $\boldsymbol{k}\times\mathrm{d}^2\boldsymbol{k}/\mathrm{d}t^2=0$. Thus we have a massive particle whose centre of mass travels at constant velocity and its relative motion around its centre of mass is a central motion. The particle has a spin of orbital nature, but changed in sign, and we can think of a certain size associated to some average value of k.

Finally if $X = \mathcal{G}$ we have a particle which has a certain directional property, such as an intrinsic spin $\boldsymbol{\Sigma}$, whose change in orientation is described by the additional three degrees of freedom $\boldsymbol{\rho}$. The particle can have a moment of inertia, and thus a certain size, and we can think in some way of a spin which is the addition of the intrinsic $\boldsymbol{\Sigma}$, a rotating term $I\boldsymbol{\Omega}$ and the (anti)orbital part $-\boldsymbol{k}\times m\,\mathrm{d}\boldsymbol{k}/\mathrm{d}t$, and being this sum $\boldsymbol{S}$ a constant of the motion, the zitterbewegung will no longer be in general a central motion, but goes to it when $\boldsymbol{\Sigma}$ tends to zero.

4. Discrete symmetries

Since space and time inversions are automorphisms of $\mathcal{G}$

$$
\begin{aligned}
P&:(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu}) \rightarrow (b, -\boldsymbol{a}, -\boldsymbol{v}, \boldsymbol{\mu}) \\
T&:(b, \boldsymbol{a}, \boldsymbol{v}, \boldsymbol{\mu}) \rightarrow (-b, \boldsymbol{a}, -\boldsymbol{v}, \boldsymbol{\mu})
\end{aligned}
\tag{82}
$$

we assume they induce on $X = \mathcal{G}/\mathcal{V}$ the transformations:

$$
\begin{aligned}
P&:(t, \boldsymbol{r}, \boldsymbol{\rho}) \rightarrow (t, -\boldsymbol{r}, \boldsymbol{\rho}) \\
T&:(t, \boldsymbol{r}, \boldsymbol{\rho}) \rightarrow (-t, \boldsymbol{r}, \boldsymbol{\rho})
\end{aligned}
\tag{83}
$$

and on $X = \mathcal{G}$

$$
\begin{aligned}
P&:(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho}) \rightarrow (t, -\boldsymbol{r}, -\boldsymbol{u}, \boldsymbol{\rho}) \\
T&:(t, \boldsymbol{r}, \boldsymbol{u}, \boldsymbol{\rho}) \rightarrow (-t, \boldsymbol{r}, -\boldsymbol{u}, \boldsymbol{\rho})
\end{aligned}
\tag{84}
$$

and assuming that the τ parameter is invariant under the inversions, the derivatives for the homogeneous space $X = \mathcal{G}/\mathcal{V}$ transform as:

$$
\begin{aligned}
P&:(\dot{t}, \dot{\boldsymbol{r}}, \boldsymbol{\omega}) \rightarrow (\dot{t}, -\dot{\boldsymbol{r}}, \boldsymbol{\omega}) \\
T&:(\dot{t}, \dot{\boldsymbol{r}}, \boldsymbol{\omega}) \rightarrow (-\dot{t}, \dot{\boldsymbol{r}}, \boldsymbol{\omega})
\end{aligned}
\tag{85}
$$

and on $X = \mathcal{G}$ as:

$$
\begin{aligned}
P&:(\dot{t}, \dot{\boldsymbol{r}}, \dot{\boldsymbol{u}}, \boldsymbol{\omega}) \rightarrow (\dot{t}, -\dot{\boldsymbol{r}}, -\dot{\boldsymbol{u}}, \boldsymbol{\omega}) \\
T&:(\dot{t}, \dot{\boldsymbol{r}}, \dot{\boldsymbol{u}}, \boldsymbol{\omega}) \rightarrow (-\dot{t}, \dot{\boldsymbol{r}}, -\dot{\boldsymbol{u}}, \boldsymbol{\omega}).
\end{aligned}
\tag{86}
$$

The considered Lagrangians (38), (49), (51) and (78) are all invariant under space reversal but (49) and (78) are not longer time-reversal invariant because of the $\boldsymbol{\Sigma}$ term, in particular in the example (49) the term $\boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}$, $\boldsymbol{\Omega}$ changes into $-\boldsymbol{\Omega}$ while $\boldsymbol{\Sigma}$, being only a function of $\boldsymbol{\rho}$, does not change.

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