



ELSEVIER

21 June 1999

PHYSICS LETTERS A

Physics Letters A 257 (1999) 21–25

A pure kinematical explanation of the gyromagnetic ratio $g = 2$ of leptons and charged bosons

M. Rivas¹, J.M. Aguirregabiria², A. Hernández³*Dpto. de Física Teórica, Universidad del País Vasco-Euskal Herriko Unibertsitatea, Apdo. 644, 48080 Bilbao, Spain*

Received 10 March 1999; accepted 27 April 1999

Communicated by P.R. Holland

Abstract

By analysing the structure of the spin operator, we give a pure kinematical explanation of the origin of the gyromagnetic ratio of elementary particles. © 1999 Published by Elsevier Science B.V. All rights reserved.

PACS: 03.65.Sq; 11.10.Ef

Jackiw [1] has given recently another dynamical argument confirming that the gyromagnetic ratio of spin-1 fields is $g = 2$, provided a nonelectromagnetic gauge invariance is accepted. He also gives some ad hoc argument for $s = 2$ fields, consistent with the $g = 2$ prescription.

The $g = 2$ gyromagnetic ratio of the electron was considered for years a success of Dirac's electron theory [2]. Later, Levy-Leblond [3] obtained similarly $g = 2$ but from a $s = 1/2$ non-relativistic wave equation. Proca [4] found $g = 1$ for spin 1 particles and this lead Belinfante [5] to conjecture that the gyromagnetic ratio for elementary systems is $g = 1/s$, irrespective of the value s of its spin. He showed this to be true for quantum systems of spin $3/2$, and few years later the conjecture was analysed

and checked to be right for any half-integer spin by Moldauer and Case [6], and by Tumanov [7] for the value $s = 2$. In all these cases a minimal electromagnetic coupling was assumed.

Weinberg [8] made the prediction $g = 2$ for the intermediate boson of the weak interactions when analyzing the interaction of W bosons with the electromagnetic field by requiring a good high-energy behavior of the scattering amplitude. The discovery of the charged W^\pm bosons with $g = 2$, contradictory with Belinfante's conjecture, corroborated Weinberg's prediction and raised the question as to whether $g = 2$ for any elementary particle of arbitrary spin.

Ferrara et al. [9] in a Lagrangian approach for massive bosonic and fermionic strings, by the requirement of a smooth fixed-charge $M \rightarrow 0$ limit, get $g = 2$ as the most natural value for particles of arbitrary spin. However the only known particles which fulfill this condition are leptons and charged W bosons, i.e., charged fermions and bosons of the

¹ E-mail: wttripem@lg.ehu.es

² E-mail: wtpagagj@lg.ehu.es

³ E-mail: wtpheaaa@lg.ehu.es

lowest admissible values of spin. No other higher spin charged elementary particles have been found.

The aim of this work, instead of using dynamical arguments as in the previous attempts, is to give a kinematical description of the gyromagnetic ratio of elementary particles which is based upon the double content of their spin operator structure, as derived by quantizing a classical formalism of elementary spinning particles, developed by one of us [10,11].

This approach is based on the assumption that the *kinematical space* of an elementary classical particle is a *homogeneous space* of the kinematical group of space-time transformations. The kinematical space of a Lagrangian system is defined as the manifold spanned by the initial (or final) variables that are held fixed in the corresponding variational problem, and in terms of which the Feynman path integral approach is worked out. Feynman's quantisation of these classical particles is done in Ref. [12], where, among other things, Dirac's equation is obtained.

The main highlights of the mentioned approach are the following: The definition of classical elementary particle is kinematical. It depends on the structure of the kinematical group of space-time transformations that defines the relativity principle. When restricted to the Galilei \mathcal{G} (or Poincaré \mathcal{P}) group, the largest kinematical space of an elementary particle is the group itself, i.e., a 10-dimensional manifold spanned by the variables $x \equiv (t, \mathbf{r}, \mathbf{v}, \boldsymbol{\alpha})$ with the same domains and physical dimensions as the corresponding group parameters, and they are interpreted respectively as the time, position, velocity and orientation of the particle. In the relativistic approach and in a covariant notation, this amounts to characterise the end points of the action integral by the space-time point $x^\mu \equiv (t, \mathbf{r})$ and a tetrad or Lorentz matrix $\Lambda_\nu^\mu \equiv (\mathbf{v}, \boldsymbol{\alpha})$, as in the Hanson and Regge spinning top [13].

The variational formalism when written in terms of the kinematical variables implies that the Lagrangian is necessarily a homogeneous function of first degree of the derivatives of these kinematical variables. Therefore, Lagrangians for describing elementary particles also depend on the acceleration and angular velocity of the particle. About the dependence of Lagrangians on the acceleration, Feynman and Hibbs [14] already quoted the possibility that the end points of the path integral could depend

on the velocity and, perhaps, higher-order derivatives. Therefore Lagrangians for free elementary spinning particles have the general form

$$L = T\dot{t} + \mathbf{R} \cdot \dot{\mathbf{r}} + \mathbf{V} \cdot \dot{\mathbf{v}} + \mathbf{Z} \cdot \boldsymbol{\omega}, \quad (1)$$

where $T = \partial L / \partial \dot{t}$, $R_i = \partial L / \partial \dot{r}_i$, $V_i = \partial L / \partial \dot{v}_i$, and $Z_i = \partial L / \partial \omega_i$, $\boldsymbol{\omega}$ is the angular velocity of the particle, and a dot means derivation with respect to some arbitrary evolution parameter.

It is the presence of the last two terms that distinguishes this kind of systems from spinless point particles and thus giving rise to the spin structure of the particle. It should be emphasised that in the quantised relativistic version of this formalism [12], the structure of the total spin observable, related to the analytical structure of the generator of rotations, has exactly the same form as in the non-relativistic case. Nevertheless, in the Galilei approach the discussion that follows to show the Zitterbewegung structure of the particle is simpler.

The total linear momentum does not lie along the velocity of point \mathbf{r} even for a free spinning particle, as it happens to Dirac's electron, but is expressed as

$$\mathbf{P} = m\mathbf{v} - \frac{d\mathbf{V}}{g} dt. \quad (2)$$

The total Galilei momentum, i.e., the constant of the motion associated to the invariance of dynamical equations under the Galilei boosts, has the form

$$\mathbf{K} = m\mathbf{r} - \mathbf{P}t - \mathbf{V}, \quad (3)$$

and the total angular momentum of the system is given by

$$\mathbf{J} = \mathbf{r} \times \mathbf{P} + \mathbf{v} \times \mathbf{V} + \mathbf{Z}. \quad (4)$$

The center-of-mass frame is defined as that reference system for which $\mathbf{P} = 0$ and $\mathbf{K} = 0$, and there the total angular momentum reduces to the spin of the system that is defined as

$$\mathbf{S} = \mathbf{v} \times \mathbf{V} + \mathbf{Z} = \mathbf{S}_v + \mathbf{S}_\alpha. \quad (5)$$

It is composed of two parts, one ($\mathbf{S}_v \equiv \mathbf{v} \times \mathbf{V}$) that depends on \mathbf{V} and, therefore, is a direct consequence of the dependence of the Lagrangian on the acceleration, and another ($\mathbf{S}_\alpha \equiv \mathbf{Z}$) that comes from the dependence on the angular velocity and related to the angular variables.

Feynman’s quantisation of this system requires the wave function to be a squared integrable complex function of the kinematical variables $\psi(t, \mathbf{r}, \mathbf{v}, \boldsymbol{\alpha})$ and the total angular momentum takes the form [12]

$$\mathbf{J} = \mathbf{r} \times \frac{\hbar}{i} \nabla + \mathbf{S} = \mathbf{r} \times \mathbf{P} + \mathbf{S}, \quad (6)$$

where the spin operator is

$$\mathbf{S} = \mathbf{v} \times \frac{\hbar}{i} \nabla_v + \mathbf{D}_\alpha = \mathbf{S}_v + \mathbf{S}_\alpha, \quad (7)$$

and ∇_v is the gradient operator with respect to the velocity variables and \mathbf{D}_α is a linear differential operator that operates only on the orientation variables $\boldsymbol{\alpha}$ and therefore commutes with the other. For instance, if we parameterise every rotation of angle θ by the three-vector $\boldsymbol{\alpha} = \mathbf{n} \tan \theta / 2$, where \mathbf{n} represents a unit vector along the rotation axis, \mathbf{D}_α is written as

$$\mathbf{D}_\alpha = \frac{\hbar}{2i} [\nabla_\alpha + \boldsymbol{\alpha} \times \nabla_\alpha + \boldsymbol{\alpha} (\boldsymbol{\alpha} \cdot \nabla_\alpha)]. \quad (8)$$

The first part in (7), \mathbf{S}_v , has integer eigenvalues because it has the form of an orbital angular momentum in terms of the \mathbf{v} variables. Half-integer eigenvalues come only from the operator (8). The first term is related to the Zitterbewegung while the second, \mathbf{S}_α , takes into account the change of orientation, i.e., the rotation of the particle.

If we define the vector $\mathbf{k} = \mathbf{V}/m$, then vector $\mathbf{q} = \mathbf{r} - \mathbf{k}$ represents the position of the center of mass, because in (3), $\dot{\mathbf{K}} = 0$ leads to $\mathbf{P} = m d\mathbf{q}/dt$. The nonvanishing function \mathbf{V} is therefore related to the separation between the center of mass and the position vector \mathbf{r} . It turns out that this function implies the existence of a relative motion between these two points. Vector \mathbf{r} may be interpreted as the position of the charge and its motion as the Zitterbewegung, as can be seen by considering next a particular Lagrangian system where the spin only contains the Zitterbewegung part $\mathbf{v} \times \mathbf{V}$. In our general kinematical formalism this corresponds to an elementary particle with a smaller kinematical space, $\mathcal{G}/SO(3)$, spanned by the variables $(t, \mathbf{r}, \mathbf{v})$.

In this model, translation invariance implies that the Lagrangian will be independent of t and \mathbf{r} , so

that it will depend only on the velocity \mathbf{v} and acceleration $d\mathbf{v}/dt$. Rotation invariance implies dependence on \mathbf{v}^2 , $(d\mathbf{v}/dt)^2$ and $\mathbf{v} \cdot d\mathbf{v}/dt$, but this last term is a total derivative and can be withdrawn. Then, let us consider the following Lagrangian for a free non-relativistic particle

$$L_0 = \frac{m}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 - \frac{m}{2\omega^2} \left(\frac{d^2\mathbf{r}}{dt^2} \right)^2. \quad (9)$$

This Lagrangian that arises naturally from the formalism was already used by Riewe [15] to describe a spinning particle. Recently, it has been used to compute a classical contribution to the tunnel effect and a plausible interpretation of Dirac’s electric dipole term and the so called spin polarised tunneling of magnetoresistive materials [16].

If we consider the usual interaction with an external electromagnetic field

$$L_I = -e\phi(t, \mathbf{r}) + e\mathbf{v} \cdot \mathbf{A}(t, \mathbf{r}), \quad (10)$$

then, taking into account the definition of the center of mass \mathbf{q} , dynamical equations can be written as

$$\begin{aligned} m \frac{d^2\mathbf{q}}{dt^2} &= e(\mathbf{E}(t, \mathbf{r}) + \mathbf{v} \times \mathbf{B}(t, \mathbf{r})), \\ \frac{d^2\mathbf{r}}{dt^2} + \omega^2(\mathbf{r} - \mathbf{q}) &= 0, \end{aligned} \quad (11)$$

and we see that the center of mass satisfies Newton’s equations with an external Lorentz force that is defined at point \mathbf{r} and not at point \mathbf{q} . In the same way, it is the velocity \mathbf{v} of point \mathbf{r} which enters into the magnetic force term. Point \mathbf{r} , that clearly represents the position of the charge, has an isotropic harmonic motion of frequency ω around the center of mass \mathbf{q} . The origin of the spin of the particle and its magnetic moment is related to this separation and its relative motion as we show in the sequel.

In the center-of-mass frame, the spin takes the form

$$\mathbf{S}_v = \mathbf{v} \times \mathbf{V} = -m\mathbf{k} \times \frac{d\mathbf{k}}{dt}, \quad (12)$$

as the (anti)-orbital angular momentum of the charge motion around the center of mass.

Let us consider the particular case in which the external magnetic field is uniform, as described by the vector potential $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$. In this case the variation of the angular momentum is found to be

$$\frac{d\mathbf{J}}{dt} = \frac{e}{2}(\mathbf{r} \times \mathbf{v}) \times \mathbf{B}. \quad (13)$$

Similarly, the interaction term of the Lagrangian can be written

$$e\mathbf{v} \cdot \mathbf{A} = \frac{e}{2}(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{B}. \quad (14)$$

When considered in the center-of-mass frame, $\mathbf{q} = 0$, $\mathbf{r} = \mathbf{k}$ and Eq. (13) becomes

$$\frac{dS_v}{dt} = \frac{e}{2} \left(\mathbf{k} \times \frac{d\mathbf{k}}{dt} \right) \times \mathbf{B} = \boldsymbol{\mu} \times \mathbf{B}, \quad (15)$$

the interaction term (14) can be written as $\boldsymbol{\mu} \cdot \mathbf{B}$ and the particle will therefore behave as though it has a magnetic moment

$$\boldsymbol{\mu} = \frac{e}{2} \left(\mathbf{k} \times \frac{d\mathbf{k}}{dt} \right) = -\frac{e}{2m} S_v. \quad (16)$$

This magnetic moment is the one produced by the particle current $\mathbf{j} = ev\delta^{(3)}(\mathbf{r}' - \mathbf{r})$, associated to the motion of a charge e at point \mathbf{r} , according to the usual definition.

Coming back to the most general elementary spinning particle, one must remember that in the context of our formalism the spin also contains a S_α part. But this part is related to the angular variables that describe orientation and does not contribute to the separation \mathbf{k} between the center of charge and the center of mass. It turns out that the magnetic moment of a general particle is still related to the motion of the charge by the expression (16), i.e., in terms of the S_v part but not to the total spin S . It is precisely when we try to express the magnetic moment in terms of the total spin that the concept of gyromagnetic ratio arises.

Now, let us assume that both S_v and S_α terms contribute to the total spin S with their lowest admissible values.

The classical particles that when quantised have spin $s = 1/2$ and satisfy Dirac's equation have a classical Zitterbewegung that is a circular motion at the speed of light of radius $R = s/mc$ and angular frequency $\omega = mc^2/s$, in a plane orthogonal to the total spin [11,12]. The total spin S and the S_v part,

are both orthogonal to this plane. Then, let us define the gyromagnetic ratio by $S_v = gS$. For the lowest admissible values of the quantised spins $s_v = 1$ and $s_\alpha = 1/2$ in the opposite direction this gives rise to a total $s = 1/2$ along S_v and then $g = 2$.

For $s = 1$ particles the lowest possible values compatible with the above relative orientations are $s_v = 2$ and $s_\alpha = 1$ in the opposite direction, thus obtaining again $g = 2$. The possibility $s_v = 1$ and $s_\alpha = 0$ is forbidden in the relativistic case because necessarily $s_\alpha \neq 0$ to describe vector bosons with a multicomponent wave-function.

No higher spin charged elementary particles are known. The predictions of this formalism for hypothetical particles of $s = 3/2$ are $s_v = 1$ and $s_\alpha = 1/2$ in the same direction, and thus $g = 2/3$, or $s_v = 2$ and $s_\alpha = 1/2$ in the opposite direction, and therefore $g = 4/3$. Similarly, for $s = 2$ particles the lowest values are $s_v = 1$ and $s_\alpha = 1$ in the same direction, and thus $g = 1/2$, compatible with Belinfante's conjecture.

In summary, according to the proposed formalism of classical and quantal spinning particles, while restricted to the Galilei or Poincaré groups as kinematical groups, the spin consists of two parts: one S_α related to the rotational motion of the body and another S_v linked to the Zitterbewegung, or motion of the point charge around the center of mass. This interpretation is independent of whether the formalism is either relativistic or non-relativistic. The magnetic moment is produced by the current and therefore is only related to the Zitterbewegung part of spin with a normal, up to a sign, relation. It should be noticed that it is this double structure of the spin which, when expressing the magnetic moment in terms of the total spin, leads to a kinematical definition of the gyromagnetic ratio. The additional condition of minimum spin contribution of both components leads to $g = 2$ for $s = 1/2$ and $s = 1$ charged particles, and therefore it is only the difference $g - 2$ that should be justified on dynamical grounds, i.e., by means of electroweak corrections and thus by relativistic methods.

Acknowledgements

This work has been partially supported by the Universidad del País Vasco/Euskal Herriko Unibert-

sitatea under contract UPV/EHU 172.310 EB150/98 and by DGICYT project PB96-0250.

References

- [1] R. Jackiw, *Phys. Rev. D* 57 (1998) 2635.
- [2] P.A.M. Dirac, *Proc. Roy. Soc. (London) A* 117 (1928) 610.
- [3] J.M. Levy-Leblond, *Commun. Math. Phys.* 6 (1967) 286.
- [4] A. Proca, *Compt. Rend.* 202 (1936) 1420; *J. Phys. Radium* 49 (1938) 245.
- [5] F.J. Belinfante, *Phys. Rev.* 92 (1953) 997.
- [6] P.A. Moldauer, K.M. Case, *Phys. Rev.* 102 (1956) 279.
- [7] V.S. Tumanov, *Sov. Phys. JETP* 19 (1964) 1182.
- [8] S. Weinberg, in: S. Deser, M. Grisaru, H. Pendleton (Eds.), *Lectures on Elementary Particles and Quantum Field Theory*, MIT press, Cambridge, MA, 1970, p. 283.
- [9] S. Ferrara, M. Porrati, V.L. Telegdi, *Phys. Rev. D* 46 (1992) 3529.
- [10] M. Rivas, *J. Phys. A* 18 (1985) 1971.
- [11] M. Rivas, *J. Math. Phys.* 30 (1989) 318.
- [12] M. Rivas, *J. Math. Phys.* 35 (1994) 3380.
- [13] A.J. Hanson, T. Regge, *Ann. Phys.* 87 (1974) 498.
- [14] R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals*, MacGraw Hill, NY, 1965, p. 36.
- [15] F. Riewe, *Lett. Nuovo Cimento* 1 (1971) 807; *Nuovo Cimento B* 8 (1972) 271.
- [16] M. Rivas, *Phys. Lett. A* 248 (1998) 279.