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GENERALIZED LAGRANGIANS AND SPINNING PARTICLES

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We show that the spin structure of elementary particles can be naturally described by the generalized Ostrogradskii Lagrangians depending on higher-order derivatives. One component of a spin is related to the rotation of a particle and the other one, caused by the dependence of a Lagrangian on the acceleration, is known as a zitterbewegung component of spin.

Introduction

As early as 1736, Leonard Euler presented the book *Mechanica,* in which he established for the first time Newtonian dynamics in terms of rational mechanics. At that time, Euler's law of mechanics took the form:

"The increase dv of the velocity is proportional to $p\,dt$, where p is the power acting on the body during the time dt" [1].

Later, in 1740, in his work *Methodus Inveniedi Linears Curvas,* he introduced the calculus of variations, fundamental in the subsequent development of analytical dynamics. He stated the variational principle in the following form:

"Since all the effects of Nature obey some law of maximum or minimum, it cannot be denied that the curves described by projectiles under the influence of some forces will enjoy the same property of maximum or minimum. It seems easy to define, a priori, using metaphysical principles, what this property is. But since, with the necessary application, it is possible to determine these curves by the direct method, it may be decided which is a maximum or a minimum."

The magnitude which he considered to be stationary was $mds\sqrt{h}$, where m is the mass of the body, ds is the element of distance travelled, and h is the height of fall. It was in 1749 at the Academy of Sciences of Berlin when he presented Newton's law in the standard form $\boldsymbol{f} = m\boldsymbol{a}$. Explicitly,

$$\text{(i)}\ \ \frac{2d\,dx}{dt^2} = \frac{X}{M}, \qquad \text{(ii)}\ \ \frac{2d\,dy}{dt^2} = \frac{Y}{M}, \qquad \text{(iii)}\ \ \frac{2d\,dz}{dt^2} = \frac{Z}{M},$$

where X, Y, and Z are the Cartesian components of the external force, and the left-hand sides are a peculiar form of writing the second derivative of the position variables. Euler left Berlin and moved to St. Petersburg in 1766, where he wrote as much as half of his extensive work and where he died in 1783.

Joseph Louis Lagrange succeeded Euler in 1766 as the director of mathematics at the Academy of Sciences of Berlin. In 1787, he became a member of the Paris Academy of Sciences, where he published in 1788 the book *Mécanique Analytique,* in which the methods of Lagrangian dynamics were introduced. He said:

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"In the general motion of any system of bodies actuated by mutual forces of attraction or by attraction towards fixed centers which are proportional to any function of the distance, the curves described by the different bodies, and their velocities, are necessarily such that the sum of the products of each mass by the integral of the product of the velocity and the element of the curve is necessarily a maximum or a minimum, provided that the first and last points of each curve are regarded as fixed, so that the velocities of the corresponding coordinates at those points are zero" [2].

In contrast with Newton's *Principia,* in which many geometrical diagrams are used to produce the corresponding proofs, Lagrange enhanced the role of analysis and made the following declaration:

"No diagrams will be found in this work. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended this field."

As a consequence of Lagrange's equations, mechanics rested on the principle of least action, or the *principle of the greatest or least living force.*

Mikhail Vasilevich Ostrogradskii, whose bicentennial we are celebrating, left Ukraine at the age of 21 to study in Paris. In 1822–1827, he attended lectures by Laplace, Fourier, Legendre, Poisson, and Cauchy and published several papers at the Paris Academy. He went to St. Petersburg in 1828. Since then Ostrogradskii lectured at the Naval Academy, from 1830 at the Institute of Communication, and from 1832 also at the Pedagogical Institute. In 1847, he became the chief inspector for the teaching of mathematical sciences in military schools. He established conditions that allowed Chebyshev's school to flourish in St. Petersburg. He should also be considered as the founder of the Russian school of theoretical mechanics. I am not an expert in the History of Physics, but needless to say that Ostrogradskii was probably impregnated of Euler and Lagrange ideas during his stay in Paris and St. Petersburg. He was aware of the importance of variational methods in mechanics, and the Lagrangian formalism in particular. We are concerned in this contribution with his suggestion in 1850 of using Lagrangians depending on higher-order derivatives [3] and their usefulness for the description of classical spinning particles.

What is Classical Spin?

Because Newton's equations for point particles are second-order ordinary differential equations for the position variables of particles, the action principle can be written in terms of the Lagrangian function, which is, therefore, an explicit function of the position variables and their first-order time derivatives. In rational mechanics, this has been extended to arbitrary systems in the sense that Lagrangians are postulated as functions of the independent degrees of freedom q_i and also of only their first-order derivatives.

But this, which is worldwide accepted as the basis of point-particle dynamics, will no longer be considered as such for the description of elementary spinning particles. In general, classical spin is considered as some kind of a vector of constant magnitude attached to the point particle, constant in time for free motion, and when some interaction is present, it experiences some precession due to the torques of external forces. But this idea of classical spin is difficult to agree with the notion of electron's spin when considered under the analysis of Dirac's equation [4, 5]. According to Dirac, the spin for a free electron satisfies the dynamical equation

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{P} \times \boldsymbol{u}, \tag{1}$$

where

$$\boldsymbol{S} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix},$$

is the spin operator expressed in terms of Pauli's $\boldsymbol{\sigma}$ matrices, $\boldsymbol{P}$ is the total constant linear momentum of the electron, and $\boldsymbol{u} = c\boldsymbol{\alpha}$ is Dirac's velocity operator written in terms of Dirac's α matrices. If we take the scalar product with $\boldsymbol{S}$ of Eq. (1), we get

$$\boldsymbol{S} \cdot \frac{d\boldsymbol{S}}{dt} = \frac{1}{2}\frac{dS^2}{dt} = \boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{u}) \neq 0$$

and, therefore, neither the spin nor its absolute value are constants of the motion for a free electron.

It is clear that the idea of classical spin as a kind of pin stuck to the point particle which remains constant whenever the particle is free and satisfies some plausible dynamical equation when interacting must be abandoned or at least revised. Therefore, if we do not know what are the necessary variables to describe spin at the classical level, we do not know whether they must satisfy or not second-order differential equations.

This has been one of the leading arguments to a thorough revision of kinematics and dynamics, to consider the possibility of a classical description of spin. This has been done in previous works [6, 7], and a more detailed and comprehensive analysis is collected in [8]. The second important argument has been the strength given in Lagrangian dynamics to the endpoint variables of the variational formalism, as was already considered by Lagrange in the mentioned statement of the previous section. We shall call the variables that define these endpoints *kinematical variables* and the manifold they span the *kinematical space.*

Classical Elementary Particles

For the revision of the classical description of matter, we can take a look at the successful way quantum mechanics describes both kinematics and dynamics. By kinematics, we understand the basic statements that define the physical objects we go to work with and their analytical description. In quantum mechanics, a state of an elementary particle is a vector state of an irreducible representation of the kinematical group of space–time transformations that describes the relativity principle [9]. This is a group-theoretical definition of an elementary particle. Intrinsic attributes of the particles are then interpreted in terms of group invariants and are therefore related to the Casimir operators of the kinematical group or, properly speaking, to the Casimir operators of their projective unitary irreducible representations.

Quantum dynamics describes the probability amplitudes for a whole process in terms of the endpoint kinematical variables that characterize the initial and final states of the system. The details concerning the intermediate flight of the particles involved are not explicit in the final form of the result. They are all removed in the calculation process, enhancing the role, as far as the theoretical analysis is concerned, of the initial and final data. This looks similar, at least in a formal way, to the variational formalism. But when one quantizes a Lagrangian system by means of Feynman's path integral approach [10], this probability amplitude (there called Feynman's kernel) becomes a function of only the initial and final kinematical variables of the classical system.

Therefore, in the classical approach, we define a classical elementary particle by giving a group-theoretical characterization to the kinematical variables of the action integral of the Lagrangian. We postulate the following:

Definition. *A classical elementary particle is a Lagrangian system whose kinematical space is a homogeneous space of the kinematical group.*

From the mathematical viewpoint, the largest homogeneous space of a Lie group is the group itself and, therefore, this definition restricts the maximum number of kinematical variables to the number of group parameters. Any homogeneous space of a group inherits not only a part of the structure of the group it comes from, but at the same time the physical dimensions of the corresponding parameters. It is the group and the variables we use to parametrize it that determine the basic variables that define a classical elementary particle, and also their physical or geometrical interpretation.

In this way, we do not state that Lagrangians depend only on first-order derivatives. This method depends only on the kind of variables we fix as endpoints of the variational process. Once the endpoints are fixed, the variational problem requires for the Lagrangian to depend on the next order of derivation of all the kinematical variables. It is this construction that will produce, or not, the dependence on higher-order derivatives, so that the restriction of using first-order Lagrangians is a consequence of the particular selection of the kinematical variables.

Mathematical Properties of the Lagrangian

When we have a generalized Lagrangian $L\left(t, q_i, \ldots, q_i^{(k)}\right)$, which is an explicit function of time t, of n degrees of freedom q_i, $i = 1, \ldots, n$, and their derivatives up to some finite order k, $q_i^{(k)} \equiv d^k q_i / dt^k$, the kinematical variables are therefore the time t, the n degrees of freedom q_i, $i = 1, \ldots, n$, and their derivatives up to order $k-1$. These are the variables we have to leave fixed at the initial time t_1 and final time t_2 of the action functional. We denote them generically by x_l, $l = 1, \ldots, nk$, where we take $x_0 = t$, $x_i = q_i$, $x_{n+i} = q_i^{(1)}$, and so on. We clearly see that L depends also on the next-order derivative of the kinematical variables.

Now, to produce a coherent relativistic formalism, we need first to withdraw time as an evolution parameter and assume that evolution is described in terms of some arbitrary parameter τ. Therefore, the time derivative $q_i^{(1)} = dq_i / dt$ should be replaced by $q_i^{(1)} = \dot{q}_i / \dot{t}$, $q_i^{(2)} = \dot{q}_i^{(1)} / \dot{t}$, etc., where the dot over a variable means its τ-derivative, so that the action functional can be rewritten as

$$\int_{t_1}^{t_2} L\left(t, q_i^{(k)}\right) dt = \int_{\tau_1}^{\tau_2} L\left(t, q_i, \frac{\dot{q}_i}{\dot{t}}, \ldots, \frac{\dot{q}_i^{(k-1)}}{\dot{t}}\right) \dot{t}\, d\tau \equiv \int_{\tau_1}^{\tau_2} \hat{L}(x_l, \dot{x}_l)\, d\tau .$$

We see from the above change of variables that the new Lagrangian $\hat{L}(x_l, \dot{x}_l) \equiv L\dot{t}$, written in terms of the kinematical variables, has the following properties:

1. It is independent of the evolution parameter τ.

2. It is a homogeneous function of first degree in the derivatives $\dot{x}_l$ of the kinematical variables and, according to Euler's theorem on homogeneous functions, it satisfies the relation

$$\sum_j \left(\frac{\partial \hat{L}}{\partial \dot{x}_j}\right) \dot{x}_j = \hat{L}.$$

3. It therefore admits the general form

$$\hat{L} = \sum_j F_j(x_l, \dot{x}_l)\dot{x}_j, \tag{2}$$

where the functions $F_j = \partial\hat{L} / \partial\dot{x}_j$ are homogeneous functions of degree zero in the variables $\dot{x}_l$.

4. If G is a Lie group of transformations of the kinematical variables x such that the dynamical equations remain invariant under the transformation $x' = gx$ and the corresponding induced transformation $\dot{x}' = g\dot{x}$ for any $g \in G$, then the Lagrangian transforms under G as follows:

$$\hat{L}(gx, g\dot{x}) = \hat{L}(x, \dot{x}) + \frac{d\alpha(g; x)}{d\tau}, \tag{3}$$

where the function $d\alpha(g; x) / d\tau$ is not arbitrary. It depends only on the kinematical variables and the group parameters and is analytically related to the exponents of the group G [11].

5. Noether's theorem. The invariance of the action functional under an r-parameter Lie group G defines r constants of motion N_α, $\alpha = 1, \dots, r$, which can be written in terms of only the functions $F_l(x, \dot{x})$ and their first time derivatives and of the first-order functions of the infinitesimal transformations of x_l.

Simple Examples

Let us start first with the Newtonian point particle. By definition, its kinematical variables for its Lagrangian formalism are time t and position $\boldsymbol{r}$. Let us assume first that the set of inertial observers are all at rest with their Cartesian frames parallel with respect to each other so that the kinematical group is just the space–time translation group. Then the kinematical relation between observers is given by the group action

$$t'(\tau) = t(\tau) + b, \qquad \boldsymbol{r}'(\tau) = \boldsymbol{r}(\tau) + \boldsymbol{a}, \tag{4}$$

at any instant of the evolution parameter τ. This four-parameter group has four generators H and $\boldsymbol{P}$. In action (4), the generators are the differential operators

$$H = \frac{\partial}{\partial t}, \qquad \boldsymbol{P} = \nabla.$$

The group law $g'' = g'g$ is

$$b'' = b' + b, \qquad \boldsymbol{a}'' = \boldsymbol{a}' + \boldsymbol{a}.$$

We see that the kinematical space of our point particle is, in fact, isomorphic to the whole kinematical group so that our kinematical variables $x \equiv (t, \boldsymbol{r})$ have the same domains and dimensions as the group parameters $(b, \boldsymbol{a})$, respectively. The kinematical space X is clearly the largest homogeneous space of the kinematical group.

According to this restricted relativity principle, the Lagrangian for a point particle will be a function of the variables $(t, \boldsymbol{r}, \dot{t}, \dot{\boldsymbol{r}})$, and a homogeneous function of first degree in terms of the derivatives $(\dot{t}, \dot{\boldsymbol{r}})$. Then, according to (2), it can be written as

$$L = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}}, \tag{5}$$

where $T = \partial L / \partial \dot{t}$ and $\boldsymbol{R} = \partial L / \partial \dot{\boldsymbol{r}}$. Since the space–time translation group has no central extensions and thus no nontrivial exponents, Lagrangians can be taken strictly invariant under this group. Therefore, dynamical equations can be any autonomous second-order differential equations for the functions $\boldsymbol{r}(t)$ that do not depend explicitly on the variables $\boldsymbol{r}$ and t.

When applying Noether's theorem to this kinematical group, we obtain, as constants of motion, the energy $H = -T$ and linear momentum $\boldsymbol{P} = \boldsymbol{R}$. Possible Lagrangians for this kind of systems are very general and might be any function of the components of the velocity dr_i / dt. The homogeneity condition in terms of kinematical variables implies that, e.g., expressions of the form

$$a_{ij}\frac{\dot{r}_i\dot{r}_j}{\dot{t}} + b_{ijk}\frac{\dot{r}_i\dot{r}_j\dot{r}_k}{\dot{t}^2} + \dots ,$$

with arbitrary constants a_{ij}, b_{ijk}, etc., or expressions like

$$\sqrt{a_0\dot{t}^2 + a_i\dot{t}\dot{r}_i + b_{ij}\dot{r}_i\dot{r}_j + \frac{c_{ijk}\dot{r}_i\dot{r}_j\dot{r}_k}{\dot{t}} + \dots},$$

homogeneous of first degree in the derivatives, can be taken as possible Lagrangians.

Let us go further and extend the kinematical group to include rotations. Then, the kinematical transformations are

$$t'(\tau) = t(\tau) + b, \qquad \boldsymbol{r}'(\tau) = R(\boldsymbol{\beta})\boldsymbol{r}(\tau) + \boldsymbol{a}, \tag{6}$$

where $R(\boldsymbol{\beta})$ represents a rotation matrix written in terms of three parameters β_i of a suitable parametrization of the rotation group. This group is called the Aristotle group G_A. In addition to H and $\boldsymbol{P}$, it has three new generators $\boldsymbol{J}$, which, in the above action (6), are given by the operators $\boldsymbol{J} = \boldsymbol{r} \times \nabla$. This group also does not have central extensions, and, thus, it does not have nontrivial exponents. Lagrangians in this case will also be invariant. This additional rotation invariance leads to the conclusion that L, which still has the general form (5), will be an arbitrary function of $\dot{\boldsymbol{r}}^2$.

When applying Noether's theorem, we have, in addition to the energy $H = -\partial L / \partial \dot{t} = -T$ and linear momentum $P = \partial L / \partial \dot{\boldsymbol{r}} = \boldsymbol{R}$, a new observable related to the invariance under rotations, namely, the angular momentum $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P}$.

The group elements are parametrized in terms of the seven parameters $g \equiv (b, \boldsymbol{a}, \boldsymbol{\beta})$ and the group G_A has the composition law $g'' = g'g$ given by

$$b'' = b' + b, \qquad \boldsymbol{a}'' = R(\boldsymbol{\beta}')\boldsymbol{a} + \boldsymbol{a}', \qquad R(\boldsymbol{\beta}'') = R(\boldsymbol{\beta}')R(\boldsymbol{\beta}). \tag{7}$$

We clearly see, by comparing (7) with (6), that the kinematical space X of our point particle is isomorphic to the homogeneous space of the group, $X \simeq G_A / SO(3)$. It corresponds to the coset space of elements of the form $(t, \boldsymbol{r}, \boldsymbol{0})$ when acting on the subgroup $SO(3)$ of elements $(0, \boldsymbol{0}, \boldsymbol{\beta})$. The kinematical variables $(t, \boldsymbol{r})$ span the same manifold and have the same dimensions as the set of group elements of the form $(b, \boldsymbol{a}, \boldsymbol{0})$.

But once we have a larger symmetry group, we can extend our definition of elementary particle to the whole group G_A. The physical system might have three new kinematical variables $\boldsymbol{\alpha}$ (the angular variables). In a τ-evolution description of dynamics, with the identification in $g'' = g'g$ of $g'' \equiv x'(\tau)$, $g \equiv x(\tau)$ and g' playing the role of the group element g acting on the left on x, we get $x' = gx$. In view of (7), they explicitly transform as

$$t'(\tau) = t(\tau) + b, \qquad \boldsymbol{r}'(\tau) = R(\boldsymbol{\beta}')\boldsymbol{r}(\tau) + \boldsymbol{a}$$

as in (6), and also for the new degrees of freedom

$$R(\boldsymbol{\alpha}'(\tau)) = R(\boldsymbol{\beta})R(\boldsymbol{\alpha}(\tau)).$$

The seven kinematical variables of our elementary particle are now time t, position $\boldsymbol{r}$, and orientation $\boldsymbol{\alpha}$. Our system can be interpreted as a point with a local Cartesian frame attached to it. This local frame can rotate, and the rotation of this frame is described by the evolution of the new variables $\boldsymbol{\alpha}$. Then the Lagrangian for this system will also be a function of $\boldsymbol{\alpha}$ and $\dot{\boldsymbol{\alpha}}$, or, equivalently, of the angular velocity $\boldsymbol{\omega}$ of the moving frame. The homogeneity condition allows us to write $\hat{L}$ as

$$\hat{L} = T\dot{t} + \boldsymbol{R} \cdot \dot{\boldsymbol{r}} + \boldsymbol{W} \cdot \boldsymbol{\omega},$$

where T and $\boldsymbol{R}$ are defined as before in (5), and $\boldsymbol{W} = \partial\hat{L} / \partial\boldsymbol{\omega}$.

Total energy is now $H = -T$, the linear momentum is $\boldsymbol{P} = \boldsymbol{H}$, but the angular momentum takes the form

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{W}.$$

The particle, in addition to the angular momentum $\boldsymbol{r} \times \boldsymbol{P}$, now has a translation-invariant angular momentum. The particle, a point and a rotating frame like the usual description of a rigid body, has spin $\boldsymbol{W}$.

Nevertheless, we have seen that, while restricting ourselves to the Aristotle kinematical group, we do not obtain generalized Lagrangians. All above Lagrangians depend only on the first-order derivative of the variables $\boldsymbol{r}$ and $\boldsymbol{\alpha}$.

The principle of inertia by Galilei enlarges the Aristotle kinematical group G_A to the whole Galilei group $\mathcal{G}$. The physical laws of dynamics must be independent of the uniform relative motion between inertial observers, and this sets up a new kinematical group with a more complex structure. The action of the Galilei group is defined as

$$t'(\tau) = t(\tau) + b, \qquad \boldsymbol{r}'(\tau) = R(\boldsymbol{\beta})\boldsymbol{r}(\tau) + \boldsymbol{v}t(\tau) + \boldsymbol{a},$$

which contains three new parameters $\boldsymbol{v}$ (the relative velocity between observers).

In addition to the generators H, $\boldsymbol{P}$, and $\boldsymbol{J}$, the Galilei group has three new generators $\boldsymbol{K}$, which, in the above group action, are given by $\boldsymbol{K} = t\nabla$.

We see that, once we have a larger group, we can also enlarge, in an appropriate way, the kinematical variables of our point particle. The largest homogeneous space will contain as kinematical variables time t, position $\boldsymbol{r}$, and orientation $\boldsymbol{\alpha}$, as the corresponding parameters of the Aristotle group but also the velocity of the particle $\boldsymbol{u} = d\boldsymbol{r}/dt$, which comes from the corresponding group parameter $\boldsymbol{v}$. The Lagrangian is now a function of these kinematical variables and their next-order derivatives, i.e., it must necessarily depend on the acceleration $d\boldsymbol{u}/dt$ of the particle. It is a Lagrangian that depends on the second derivative of the position variables $\boldsymbol{r}$. We thus get a generalized Lagrangian for describing an elementary particle that has a spin structure such that, in addition to the rotational spin as in the previous model, it has spin related to the zitterbewegung, as we shall describe in the next nonrelativistic example.

A Nonrelativistic Spinning Particle

We thus see that the most general nonrelativistic particle will be described by a Lagrangian which is a function of the variables t, $\boldsymbol{r}$, $\boldsymbol{u}$, $\boldsymbol{\alpha}$ and $\dot{t}$, $\dot{\boldsymbol{r}}$, $\dot{\boldsymbol{u}}$, $\dot{\boldsymbol{\alpha}}$, being homogeneous of first degree in terms of these last ones. It therefore can be written as

$$\hat{L} = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} + \boldsymbol{U}\cdot\dot{\boldsymbol{u}} + \boldsymbol{W}\cdot\boldsymbol{\omega}, \tag{8}$$

where we have replaced $\dot{\boldsymbol{\alpha}}$ by the angular velocity $\boldsymbol{\omega}$, which is a linear function of it, and where we have used the new functions $U_i = \partial\hat{L}/\partial\dot{u}_i$.

The Galilei group has nontrivial exponents [12, 13] and, thus, according to (3), the Lagrangian is not invariant under the whole Galilei group but it transforms with the gauge function

$$\alpha(g, x) = \frac{m}{2}\left(\boldsymbol{v}^2 t + 2\boldsymbol{v}\cdot R(\boldsymbol{\beta})\boldsymbol{r}\right). \tag{9}$$

We see that if the group parameter $\boldsymbol{v} = 0$, this gauge function vanishes so that the noninvariance of the Lagrangian is coming only from its change under pure Galilei transformations.

Let us consider as a simpler example an elementary particle whose kinematical space is $X = \mathcal{G}/SO(3)$. Any point $x \in X$ can be characterized by the seven variables $x = (t, \boldsymbol{r}, \boldsymbol{u})$, $\boldsymbol{u} = d\boldsymbol{r}/dt$, which are interpreted as time, position, and velocity of the particle respectively.

The Lagrangian will also depend on the next-order derivatives, i.e., on the velocity, which is already considered as a kinematical variable, and on the acceleration of the particle. Rotation and translation invariance implies that $\hat{L}$ will be a function of only $\boldsymbol{u}^2$, $(d\boldsymbol{u}/dt)^2$, and $\boldsymbol{u}\cdot d\boldsymbol{u}/dt = d(u^2/2)/dt$, but this last term is the total time derivative and it will not be considered here.

Let us assume that our elementary system is represented by the following Lagrangian in terms of the kinematical variables and their derivatives and in terms of some group invariant evolution parameter τ:

$$\hat{L} = \frac{m}{2}\frac{\dot{\boldsymbol{r}}^2}{\dot{t}} - \frac{m}{2\omega^2}\frac{\dot{\boldsymbol{u}}^2}{\dot{t}},$$

where the dot means the τ-derivative. The parameter m is the mass of the particle because the first term is gauge variant in terms of the gauge function (9) defined by this constant m, while the parameter ω of dimensions of time^{-1} represents an internal frequency. It is the frequency of the internal zitterbewegung. If we assume that the evolution parameter is dimensionless, all terms in the Lagrangian have dimensions of action. Because the Lagrangian is a homogeneous function of first degree in terms of the derivatives of the kinematical variables, $\hat{L}$ can also be written as

$$\hat{L} = T\dot{t} + \boldsymbol{R}\cdot\dot{\boldsymbol{r}} + \boldsymbol{U}\cdot\dot{\boldsymbol{u}},$$

where the functions accompanying the derivatives of the kinematical variables are defined and explicitly given by

$$T = \frac{\partial\hat{L}}{\partial\dot{t}} = -\frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2,$$

$$\boldsymbol{R} = \frac{\partial\hat{L}}{\partial\dot{\boldsymbol{r}}} = m\frac{d\boldsymbol{r}}{dt}, \tag{10}$$

$$\boldsymbol{U} = \frac{\partial\hat{L}}{\partial\dot{\boldsymbol{u}}} = -\frac{m}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \tag{11}$$

In a time evolution description $\dot{t} = 1$ L can be written in terms of the three degrees of freedom and their time derivatives as

$$L = \frac{m}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - \frac{m}{2\omega^2}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)^2. \tag{12}$$

The dynamical equations obtained from Lagrangian (12) are as follows:

$$\frac{1}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4} + \frac{d^2\boldsymbol{r}}{dt^2} = 0; \tag{13}$$

their general solution is given by

$$\boldsymbol{r}(t) = \boldsymbol{A} + \boldsymbol{B}t + \boldsymbol{C}\cos\omega t + \boldsymbol{D}\sin\omega t \tag{14}$$

in terms of the 12 integration constants $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$.

When applying Noether's theorem to the invariance of dynamical equations under the Galilei group, the corresponding constants of motion can be written in terms of the above functions in the following form:

energy $$H = -T - \boldsymbol{u}\cdot\frac{d\boldsymbol{U}}{dt}, \tag{15}$$

linear momentum $$\boldsymbol{P} = \boldsymbol{R} - \frac{d\boldsymbol{U}}{dt} = m\boldsymbol{u} - \frac{d\boldsymbol{U}}{dt}, \tag{16}$$

kinematical momentum $$\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{U}, \tag{17}$$

angular momentum $$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{P} + \boldsymbol{u}\times\boldsymbol{U}. \tag{18}$$

It is the presence of the function $\boldsymbol{U}$ that distinguishes the features of this system with respect to the point particle case. We find that the total linear momentum is not lying along the direction of the velocity $\boldsymbol{u}$, and the spin structure is directly related to the dependence of the Lagrangian on the acceleration.

We can think that the observable $\boldsymbol{Z} = \boldsymbol{u}\times\boldsymbol{U}$ is the spin of the system. We shall define the spin properly after the definition of the center of mass of the particle. Nevertheless, the magnitude $\boldsymbol{Z}$ looks like Dirac's spin operator, since taking the time derivative of (18), which is a constant of motion, we get

$$\frac{d\boldsymbol{Z}}{dt} = \boldsymbol{P}\times\boldsymbol{u}$$

similarly as the dynamical equation for the spin of a free particle obtained from Dirac's equation.

If we substitute the general solution (14) in (15)–(18), we see in fact that the integration constants are related to the above conserved quantities:

$$H = \frac{m}{2}\boldsymbol{B}^2 - \frac{m\omega^2}{2}(\boldsymbol{C}^2 + \boldsymbol{D}^2),$$

$$\boldsymbol{P} = m\boldsymbol{B},$$

$$\boldsymbol{K} = m\boldsymbol{A},$$

$$\boldsymbol{J} = \boldsymbol{A}\times m\boldsymbol{B} - m\omega\boldsymbol{C}\times\boldsymbol{D}.$$

We see that the kinematical momentum $\boldsymbol{K}$ in (17) differs from the point particle case $\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t$ by the term $-\boldsymbol{U}$ such that if we define the vector $\boldsymbol{k} = \boldsymbol{U}/m$ with dimensions of length, then $\dot{\boldsymbol{K}} = 0$ leads from (17) to the equation

$$\boldsymbol{P} = m\frac{d(\boldsymbol{r} - \boldsymbol{k})}{dt},$$

and $\boldsymbol{q} = \boldsymbol{r} - \boldsymbol{k}$, defines the position of the center of mass of the particle. It is a point different from $\boldsymbol{r}$ and, using (11), it is given by

$$\boldsymbol{q} = \boldsymbol{r} - \frac{1}{m}\boldsymbol{U} = \boldsymbol{r} + \frac{1}{\omega^2}\frac{d^2\boldsymbol{r}}{dt^2}. \tag{19}$$

In terms of $\boldsymbol{q}$, the kinematical momentum takes the form

$$\boldsymbol{K} = m\boldsymbol{q} - \boldsymbol{P}t.$$

In terms of $\boldsymbol{q}$, the dynamical equations (13) can be separated as follows:

$$\frac{d^2\boldsymbol{q}}{dt^2} = 0, \tag{20}$$

$$\frac{d^2\boldsymbol{r}}{dt^2} + \omega^2(\boldsymbol{r}-\boldsymbol{q}) = 0, \tag{21}$$

where (20) is Eq. (13) after twice differentiating (19), and Eq. (21) is (19) after collecting all terms on the left-hand side.

From (20) we see that the point $\boldsymbol{q}$ moves along a straight trajectory at constant velocity, while the motion of the point $\boldsymbol{r}$, given in (21), is an isotropic harmonic motion with angular frequency ω around the point $\boldsymbol{q}$.

The spin of the system $\boldsymbol{S}$ is now defined as the angular momentum of the system with the orbital angular momentum of the motion of its center of mass subtracted, i.e.,

$$\boldsymbol{S} = \boldsymbol{J} - \boldsymbol{q}\times\boldsymbol{P} = \boldsymbol{J} - \frac{1}{m}\boldsymbol{K}\times\boldsymbol{P}.$$

Since it is finally written in terms of constants of motion, it is clearly a constant of the motion, and its magnitude S^2 is also a Galilei invariant quantity that characterizes the system. In terms of the integration constants, it is expressed as follows:

$$\boldsymbol{S} = -m\omega\boldsymbol{C}\times\boldsymbol{D}.$$

From its definition, we get

$$\boldsymbol{S} = \boldsymbol{u}\times\boldsymbol{U} + \boldsymbol{k}\times\boldsymbol{P} = -m(\boldsymbol{r}-\boldsymbol{q})\times\frac{d}{dt}(\boldsymbol{r}-\boldsymbol{q}) = -\boldsymbol{k}\times m\frac{d\boldsymbol{k}}{dt}, \tag{22}$$

which appears as the (anti)orbital angular momentum of the relative motion of the point $\boldsymbol{r}$ around the position of the center of mass $\boldsymbol{q}$, so that the total angular momentum can be written as

$$\boldsymbol{J} = \boldsymbol{q}\times\boldsymbol{P} + \boldsymbol{S} = \boldsymbol{L} + \boldsymbol{S}.$$

It is the sum of the orbital angular momentum $\boldsymbol{L}$ associated with the motion of the center of mass and the spin part $\boldsymbol{S}$. For a free particle, both $\boldsymbol{L}$ and $\boldsymbol{S}$ are separately constants of motion. We use the term (anti)orbital to suggest that if the vector $\boldsymbol{k}$ represents the position of a point mass m, the angular momentum of this motion is in the opposite direction as the obtained spin observable. But as we shall see in a moment, the vector $\boldsymbol{k}$ does not represent the position of the mass m but rather the position of the charge e of the particle.

One question now arises: If $\boldsymbol{q}$ represents the position of the center of mass, then what position does the point $\boldsymbol{r}$ represent? The point $\boldsymbol{r}$ represents the position of the charge of the particle. This can be seen by considering some interaction with an external field. The homogeneity condition of the Lagrangian in terms of the derivatives of the kinematical variables leads us to considering an interaction term of the form

$$\hat{L}_I = -e\phi(t,\boldsymbol{r})\dot{t} + e\boldsymbol{A}(t,\boldsymbol{r})\cdot\dot{\boldsymbol{r}},$$

which is linear in the derivatives of the kinematical variables t and $\boldsymbol{r}$ and where the external potentials are only functions of t and $\boldsymbol{r}$. More general interaction terms of the form $\boldsymbol{N}(t,\boldsymbol{r},\boldsymbol{u})\cdot\dot{\boldsymbol{u}}$ and also more general terms in which the functions ϕ and $\boldsymbol{A}$ also depend on $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ can be considered. But this will be something different than an interaction with an external electromagnetic field.

The dynamical equations obtained from $\hat{L}+\hat{L}_I$ are

$$\frac{1}{\omega^2}\frac{d^4\boldsymbol{r}}{dt^4}+\frac{d^2\boldsymbol{r}}{dt^2} = \frac{e}{m}(\boldsymbol{E}(t,\boldsymbol{r})+\boldsymbol{u}\times\boldsymbol{B}(t,\boldsymbol{r})), \tag{23}$$

where the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ are expressed in terms of the potentials in the usual form, $\boldsymbol{E} = -\nabla\phi-\partial\boldsymbol{A}/\partial t$, and $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. Because the interaction term does not modify the dependence of the Lagrangian on $\dot{\boldsymbol{u}}$, the function $\boldsymbol{U}=m\boldsymbol{k}$ has the same expression as in the free-particle case. Therefore, the definitions of spin and center of mass [(22) and (19), respectively] remain the same as in the previous case. The dynamical equations (23) can again be separated as follows:

$$\frac{d^2\boldsymbol{q}}{dt^2} = \frac{e}{m}(\boldsymbol{E}(t,\boldsymbol{r})+\boldsymbol{u}\times\boldsymbol{B}(t,\boldsymbol{r})),$$

$$\frac{d^2\boldsymbol{r}}{dt^2}+\omega^2(\boldsymbol{r}-\boldsymbol{q}) = 0,$$

where the center of mass $\boldsymbol{q}$ satisfies Newton's equations under the action of the total external Lorentz force, while the point $\boldsymbol{r}$ still satisfies the conditions of isotropic harmonic motion with angular frequency ω around the point $\boldsymbol{q}$. But the external force and the fields are defined at the point $\boldsymbol{r}$ and not at the point $\boldsymbol{q}$. It is the velocity $\boldsymbol{u}$ of the point $\boldsymbol{r}$ that appears in the magnetic term of the Lorentz force. The point $\boldsymbol{r}$ clearly represents the position of the charge. In fact, this minimal coupling we have considered is the coupling of the electromagnetic potentials with the particle current, but the current j_μ is associated with the motion of a charge e located at the point $\boldsymbol{r}$.

This charge has an oscillatory motion of very high frequency ω, which, in the case of the relativistic electron, is $\omega=2mc^2/\hbar\simeq 1.55\times10^{21}\ \mathrm{s}^{-1}$. The average position of the charge is the center of mass, but it is this internal orbital motion, usually known as the *zitterbewegung,* that gives rise to the spin structure and also to the magnetic properties of the particle.

When analyzed in the center-of-mass frame $\boldsymbol{q}=0$, $\boldsymbol{r}=\boldsymbol{k}$, the system reduces to a point charge whose motion is, in general, an ellipse, but if we choose $C=D$ and $\boldsymbol{C}\cdot\boldsymbol{D}=0$, it reduces to a circle of radius $a=C-D$ orthogonal to the spin. Then if the particle has charge e, it has the following magnetic moment according to the usual classical definition [14]:

$$\boldsymbol{\mu} = \frac{1}{2}\int\boldsymbol{r}\times\boldsymbol{j}\,d^3\boldsymbol{r} = \frac{e}{2}\boldsymbol{k}\times\frac{d\boldsymbol{k}}{dt} = -\frac{e}{2m}\boldsymbol{S},$$

where $\boldsymbol{j}=e\delta^3(\boldsymbol{r}-\boldsymbol{k})d\boldsymbol{k}/dt$ is the current associated to the motion of a point charge e located at position $\boldsymbol{k}$. The magnetic moment is orthogonal to the zitterbewegung plane and opposite to the spin if $e>0$. It also has a nonvanishing oscillating electric dipole $\boldsymbol{d}=e\boldsymbol{k}$ orthogonal to $\boldsymbol{\mu}$ and, therefore, to $\boldsymbol{S}$ in the center-of-mass frame, such that its time average value vanishes for times larger than the natural period of this internal motion.

Although this is a nonrelativistic example, it is interesting to point out and compare with Dirac's relativistic analysis of the electron [5], in which both magnetic and electric momenta $\boldsymbol{\mu}$ and $\boldsymbol{d}$, respectively, appear, giving rise to two possible interacting terms in Dirac's Hamiltonian.

The Gyromagnetic Ratio

The most general spinning particle under the Galilei relativity principle is the one with the kinematical space $X = \mathcal{G}$. As mentioned before, the most general Lagrangian has the form (8). What is important of its analysis is the structure of the kinematical momentum $\boldsymbol{K}$ and angular momentum $\boldsymbol{J}$. As in the previous restricted example, they have the general form

$$\boldsymbol{K} = m\boldsymbol{r} - \boldsymbol{P}t - \boldsymbol{U},$$

where the observable $\boldsymbol{U}$, which comes from the dependence of the Lagrangian on the acceleration, is responsible for the separation between the center of mass and the center of charge. The zitterbewegung appears whenever we use generalized Lagrangians on the position variables and the point $\boldsymbol{r}$ represents the center of charge of the particle. If $\boldsymbol{U}$ does not vanish, the particle has magnetic moment. ,

For the total angular momentum, we get

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W}.$$

We obtain again an angular momentum observable

$$\boldsymbol{Z} = \boldsymbol{u} \times \boldsymbol{U} + \boldsymbol{W},$$

which satisfies Eq. (1) for a free particle. This is the classical equivalent of Dirac's spin operator, which consists of two parts. One part $\boldsymbol{u} \times \boldsymbol{U}$ is related to the zitterbewegung and, therefore, to the magnetic moment of the particle, and the other part $\boldsymbol{W}$ is related to the rotation of the particle as in the case of a rigid body, which plays no role in the dipole structure of the particle.

A constant spin can be defined for a free particle if we subtract from $\boldsymbol{J}$ the orbital angular momentum of the center of mass $\boldsymbol{q} \times \boldsymbol{P}$. In this case, the result is

$$\boldsymbol{S} = -m\boldsymbol{k} \times \frac{d\boldsymbol{k}}{dt} + \boldsymbol{W},$$

where, as before, $\boldsymbol{k} = \boldsymbol{r} - \boldsymbol{q} = \boldsymbol{U}/m$.

We see a clear kinematical feature: The magnetic moment is only related to the zitterbewegung part of the spin. Therefore, from the experimental point of view, we can measure mechanical and electromagnetic properties of the particle. When measuring the conserved spin of the particle, it is not possible to separate the measurement of both spin components. This implies that, when the magnetic moment is expressed in terms of total spin, their relationship is not the usual one and this produces the concept of gyromagnetic ratio g. The zitterbewegung part of the spin only quantizes with integer values because it has the structure of an orbital angular momentum. Half-integer values can come only from the rotation part of the spin. This salient feature has recently been shown to lead to the gyromagnetic ratio $g = 2$ for leptons and charged $W^{\pm}$ bosons whenever both components of spin contribute with their lowest admissible values [15]. Deviations of $g - 2$ are thus produced by radiation corrections as is shown in quantum electrodynamics.

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