

# The Spinning Electron

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A classical model for a spinning electron is described. It has been obtained within a kinematical formalism proposed by the author to describe spinning particles. The model satisfies Dirac's equation when quantized. It shows that the charge of the electron is concentrated at a single point but is never at rest. The charge moves in circles at the speed of light around the centre of mass. The centre of mass does not coincide with the position of the charge for any classical elementary spinning particle. It is this separation and the motion of the charge that gives rise to the dipole structure of the electron. The spin of the electron contains two contributions. One comes from the motion of the charge, which produces a magnetic moment. It is quantized with integer values. The other is related to the angular velocity and is quantized with half integer values. It is exactly half the first one and points in the opposite direction. When the magnetic moment is written in terms of the total observable spin, one obtains the  $g = 2$  gyromagnetic ratio. A short range interaction between two classical spinning electrons is analysed. It predicts the formation of spin 1 bound states provided some conditions on their relative velocity and spin orientation are fulfilled, thus suggesting a plausible mechanism for the formation of a Bose-Einstein condensate.

## 1. Introduction

The spin of the electron has for many years been considered a relativistic and quantum mechanical property, mainly due to the success of Dirac's equation describing a spinning relativistic particle in a quantum context. Nevertheless, in textbooks and research works one often reads that the spin is neither a relativistic nor a quantum mechanical property of the electron, and that a classical interpretation is also possible. The work by Levy-Leblond [1] and subsequent papers by Fushchich *et al.* [2], which show that it is possible to describe spin  $\frac{1}{2}$  particles in a pure Galilean framework, with the same  $g = 2$  gyromagnetic ratio, spin-orbit coupling and Darwin terms as in Dirac's equation, lead to the idea that spin is not strictly a relativistic property of the electron.

The spin is the angular momentum of the electron, and the classical and quantum mechanical description of spin is the main subject of the kinematical formalism of elementary spinning particles published by the author [3]. This work presents the main results of this formalism and, in particular, an analysis of a model of a classical spinning particle whose states are described by Dirac's spinors when quantized. Other contributions are also discussed.

## 2. Classical elementary particles

To understand what a classical elementary particle is from the mathematical point of view, we consider first the example of a point particle. It is the simplest geometrical object with which we can build any other geometrical body of any size and shape. The point particle is the classical elementary particle of Newtonian mechanics and has no spin. Yet we know today that spin is one of the intrinsic properties of all known elementary particles. The description of spin is related to the representation of the generators of the rotation group, and we know it is an intrinsic property since it is related to one of the Casimir operators of the Galilei and Poincaré groups.

From the Lagrangian point of view, the initial (and final) state of the point particle is a point on the continuous space-time manifold. In fact what we fix as boundary conditions for the variational problem are the position  $r_1$  at time  $t_1$  and the position  $r_2$  at the final time  $t_2$ . We call *kinematical variables* of any mechanical system the variables which define the initial (and final) configuration of the system in this Lagrangian description, and *kinematical space* the manifold covered by these variables. The point particle is a system of three degrees of freedom with a four-dimensional kinematical space.

In group theory, a homogeneous space of any Lie group is the quotient structure between the group and any of its continuous subgroups. The important property of the kinematical space of a point particle, from the mathematical viewpoint, is that it is a homogeneous space of the Galilei and Poincaré groups.

In the example of the point particle, the kinematical space manifold is the quotient structure between the Poincaré group and the Lorentz group in the relativistic case, and also the quotient between the Galilei group and the homogeneous Galilei group in the non-relativistic one.

We use this idea to arrive at the following definition.

**Definition:** *A classical elementary particle is a mechanical system whose kinematical space is a homogeneous space of the kinematical group.*

The spinless point particle fulfils this definition, but it is not the most general elementary particle that can be described, because we have larger homogeneous spaces with a more complex structure. The largest structured particle is the one for which the kinematical space is either the Galilei or Poincaré group or any of its maximal homogeneous spaces.

With this definition we have a new formalism, based upon group theory, to describe elementary particles from a classical point of view. It will be quantized by means of Feynman's path integral method, where the kinematical variables are precisely the common end points of all integration paths. The wave function of any mechanical system will be a complex function defined on the kinematical space. In this way, the structure of an elementary particle is basically related to the kinematical group of space-time transformations that implements the Special Relativity Principle. It is within the kinematical group of

symmetries that we must look for the independent and essential classical variables to describe an elementary object.

When we consider a larger homogeneous space than the space-time manifold, for both Galilei and Poincaré groups, we have variables additional to time and space to describe the states of a classical elementary particle. These additional variables will produce a classical description of spin.

### 3. Main features of the formalism

When we write the Lagrangian of any mechanical system in terms of the introduced kinematical variables, and the dynamics is expressed in terms of some arbitrary evolution parameter  $\tau$  (not necessarily the time parameter), we get the following properties:

- The Lagrangian is independent of the evolution parameter  $\tau$ . The time evolution of the system is obtained by choosing  $t(\tau) = \tau$ .
- The Lagrangian is only a function of the kinematical variables  $x_i$  and their first  $\tau$  derivatives  $\dot{x}_i$ .
- The Lagrangian is a homogeneous function of first degree in terms of the derivatives of the kinematical variables  $\dot{x}_i$  and therefore Euler's theorem implies that it can be written as  $L(x, \dot{x}) = F_i(x, \dot{x})\dot{x}_i$ , where  $F_i = \partial L / \partial \dot{x}_i$ .
- If some kinematical variables are time derivatives of any other kinematical variables, then the Lagrangian is necessarily a generalised Lagrangian depending on higher order derivatives when expressed in terms of the essential or independent degrees of freedom. Therefore, the dynamical equations corresponding to these variables are no longer of second order, but, in general, of fourth or higher order. This will be the case for the charge position of a spinning particle.
- The transformation of the Lagrangian under a Lie group that leaves the dynamical equations invariant is  $L(gx, g\dot{x}) = L(x, \dot{x}) + d\alpha(g; x)/d\tau$ , where  $\alpha(g; x)$  is a gauge function for the group  $G$  and the kinematical space  $X$ . It only depends on the parameters of the group element and on the kinematical variables. It is related to the exponents of the group [4].
- When the kinematical space  $X$  is a homogeneous space of  $G$ , then  $\alpha(g; x) = \xi(g, g_x)$ , where  $\xi(g_1, g_2)$  is an exponent of  $G$ .
- When quantizing the system, Feynman's kernel is the probability amplitude for the mechanical process between the initial and final state. It will be a function, or more precisely a distribution, over the  $X \times X$  manifold. Feynman's quantization establishes the link between the description of the classical states in terms of the kinematical variables and its corresponding quantum mechanical description in terms of the wave function.
- The wave function of an elementary particle is thus a complex square integrable function defined on the kinematical space.
- The Hilbert space structure of this set of functions is achieved by a suitable choice of a group invariant measure defined over the kinematical space.

- The Hilbert space of a classical system, whose kinematical space is a homogeneous space of the kinematical group, carries a projective, unitary, irreducible representation of the group. In this way, the classical definition of an elementary particle has a correspondence with Wigner's definition of an elementary particle in the quantum case.

#### 4. The classical electron model

The latest LEP experiments at CERN suggest that the electron charge is confined within a region of radius  $R_e < 10^{-19}$  m. Nevertheless, the quantum mechanical effects of the electron appear at distances of the order of its Compton's wavelength  $\lambda_c = \hbar/mc \cong 10^{-13}$  m, which are six orders of magnitude larger.

One possibility to reconcile these features is the assumption, from the classical viewpoint, that the charge of the electron is a point, but at the same time this point is never at rest and it is affected by an oscillating motion in a confined region of size  $\lambda_c$ . This motion is known in the literature as *zitterbewegung*. This is the basic structure of spinning particle models that will be obtained within the proposed kinematical formalism, and also suggested by Dirac's analysis of the internal motion of the electron [5]. It is shown that the charge of the particle is at a single point  $r$ , but this point is not the centre of mass of the particle. Furthermore, the charge of the particle is moving at the speed of light, as shown by Dirac's analysis of the electron velocity operator. Here, the velocity corresponds to the velocity of the point  $r$ , which represents the position of the charge. In general, the point charge satisfies a fourth-order differential equation, which is the most general differential equation satisfied by any three-dimensional curve.

We shall see that the charge moves around the centre of mass in a kind of harmonic or central motion. It is this motion of the charge that gives rise to the spin and dipole structure of the particle. In particular, the classical relativistic model that when quantized satisfies Dirac's equation shows, for the centre of mass observer, a charge moving at the speed of light in circles of radius  $R_0 = \hbar/2mc$  and contained in a plane orthogonal to the spin direction [6,7]. This classical model of electron is what we will obtain when analysing the relativistic spinning particles.

To describe the dynamics of a classical charged spinning particle, we must therefore follow just the charge trajectory or, alternatively, the centre of mass motion and the motion of the charge around the centre of mass. In general the centre of mass satisfies second-order, Newton-like dynamical equations, in terms of the total external force. But this force has to be evaluated not at the centre of mass position, but rather at the position of the charge. We will demonstrate all these features by considering different examples.

## 5. Non-relativistic elementary particles

Let us first consider the non-relativistic formalism because the mathematics involved is simpler. In the relativistic case the method is exactly the same, [3,6,7] and we limit ourselves here to giving only the main results. We start with the description of the Galilei group to show how we obtain the variables that determine a useful group parameterization. These variables associated with the group will later be transformed into the kinematical variables of the elementary particles. We end this section with an analysis of some different kinds of classical elementary particles.

### 5.1 Galilei group

The Galilei group is a group of space-time transformations characterised by ten parameters  $g \equiv (b, \vec{a}, \vec{v}, \vec{\alpha})$ . The action of a group element  $g$  on a space-time point  $x \equiv (t, \vec{r})$ , represented by  $x' = gx$ , is considered in the following form

$$x' = \exp(bH) \exp(\vec{a} \cdot \vec{P}) \exp(\vec{v} \cdot \vec{K}) \exp(\vec{\alpha} \cdot \vec{J})x$$

It is a rotation of the point, followed by a pure Galilei transformation, and finally a space and time translation. Explicitly, the above transformation becomes

$$t' = t + b, \quad (1)$$

$$\vec{r}' = R(\vec{\alpha})\vec{r} + \vec{v}t + \vec{a}. \quad (2)$$

The group action (1)-(2) represents the relationship between the coordinates  $(t, \vec{r})$  of a space-time event, as measured by the inertial observer O, and the corresponding coordinates  $(t', \vec{r}')$  of the same space-time event as measured by another inertial observer O'. Parameter  $b$  is a time parameter,  $\vec{a}$  has dimensions of space,  $\vec{v}$  of velocity and  $\vec{\alpha}$  is dimensionless, and these dimensions will be shared by the corresponding variables of the different homogeneous spaces of the group.

The variables  $b$  and  $\vec{a}$  are the time and position of the origin of frame O at time  $t = 0$  as measured by observer O'. The variables  $\vec{v}$  and  $\vec{\alpha}$  are respectively the velocity and orientation of frame O as measured by O'.

The composition law of the group  $g'' = g'g$  is:

$$b'' = b + b', \quad (3)$$

$$\vec{a}'' = R(\vec{\alpha}')\vec{a} + \vec{v}'b + \vec{a}', \quad (4)$$

$$\vec{v}'' = R(\vec{\alpha}')\vec{v} + \vec{v}', \quad (5)$$

$$R(\vec{\alpha}'') = R(\vec{\alpha}')R(\vec{\alpha}). \quad (6)$$

The generators of the group in the realization (1, 2) are the differential operators

$$H = \partial/\partial t, \quad P_i = \partial/\partial x^i, \quad K_i = t\partial/\partial x^i, \quad J_k = \epsilon_{kli} x^l \partial/\partial x^i \quad (7)$$

and the commutation relations of the Galilei Lie algebra are

$$[\vec{J}, \vec{J}] = -\vec{J}, \quad [\vec{J}, \vec{P}] = -\vec{P}, \quad [\vec{J}, \vec{K}] = -\vec{K}, \quad [\vec{J}, H] = 0, \quad (8)$$

$$[H, \vec{P}] = 0, \quad [H, \vec{K}] = \vec{P}, \quad [\vec{P}, \vec{P}] = 0, \quad [\vec{K}, \vec{P}] = 0. \quad (9)$$

The Galilei group has the non-trivial exponents [4]

$$\xi(g, g') = m \left( \frac{1}{2} \vec{v}^2 b' + \vec{v} \cdot R(\vec{\alpha}) \vec{a}' \right). \quad (10)$$

They are characterised by the non-vanishing parameter  $m$ . The gauge functions for the Lagrangians defined on the different homogeneous spaces of the Galilei group are of the form

$$\alpha(g; x) = m \left( \frac{1}{2} \vec{v}^2 t + \vec{v} \cdot R(\vec{\alpha}) \vec{r} \right)$$

They all vanish if the boost parameter  $\vec{v}$  vanishes. This implies that a Galilei Lagrangian for an elementary particle is invariant under rotations and translations, but not under Galilei boosts. In the quantum case this means that the Hilbert space for this system carries a unitary representation of a central extension of the Galilei group. In the classical case, the generating functions of the canonical Galilei transformations, with the Poisson bracket as the Lie operation, satisfy the commutation relations of the Lie algebra of the central extension of the Galilei group [4].

The central extension of the Galilei group [8] is an 11-parameter group with an additional generator  $I$  which commutes with the other ten,

$$[I, H] = [I, \vec{P}] = [I, \vec{K}] = [I, \vec{J}] = 0, \quad (11)$$

while the remaining commutation relations are the same as above (8, 9), the only exception being the last, which now appears as

$$[K_i, P_j] = -m \delta_{ij} I. \quad (12)$$

If the following polynomial operators are defined on the group algebra

$$\vec{W} = I\vec{J} - \frac{1}{m} \vec{K} \times \vec{P}, \quad U = IH - \frac{1}{2m} \vec{P}^2, \quad (13)$$

we see that  $U$  commutes with all generators of the extended Galilei group and that  $\vec{W}$  satisfies the commutation relations

$$[\vec{W}, \vec{W}] = -I\vec{W}, \quad [\vec{J}, \vec{W}] = -\vec{W}, \quad [\vec{W}, \vec{P}] = [\vec{W}, \vec{K}] = [\vec{W}, H] = 0.$$

We find that  $\vec{W}^2$  also commutes with all generators. It turns out that the extended Galilei group has three functionally independent Casimir operators. In those representations in which the operator  $I$  becomes the unit operator, for instance, in the irreducible representations they are, respectively, interpreted as the mass,  $M = mI$ , the internal energy  $H_0 = H - P^2/2m$ , and the absolute value of the spin

$$\vec{S} = \vec{J} - \frac{1}{m} \vec{K} \times \vec{P}, \quad \Rightarrow \quad S^2 = \left( \vec{J} - \frac{1}{m} \vec{K} \times \vec{P} \right)^2. \quad (14)$$

In what follows we take the above definition (14) as the definition of the spin of a nonrelativistic particle. In those representations in which  $I$  is the unit operator, the spin operator  $\vec{S}$  satisfies the commutation relations:

$$[\vec{S}, \vec{S}] = -\vec{S}, \quad [\vec{J}, \vec{S}] = -\vec{S}, \quad [\vec{S}, \vec{P}] = [\vec{S}, \vec{K}] = [\vec{S}, H] = 0,$$

*i.e.*, it is an angular momentum operator, transforms like a vector under rotations and is invariant under space and time translations and under Galilei boosts, respectively.

Furthermore, it reduces to the total angular momentum operator  $\vec{J}$  in those frames in which  $\vec{P} = \vec{K} = 0$ .

## 5.2 The spinless point particle

The kinematical variables of the point particle are  $\{t, \vec{r}\}$ , time and position, respectively. The nonrelativistic Lagrangian written in terms of the  $\tau$  derivatives of the kinematical variables is the first order homogeneous function

$$L_{NR} = \frac{m \dot{\vec{r}}^2}{2} = T\dot{t} + \vec{R} \cdot \dot{\vec{r}},$$

where we define  $T = \partial L / \partial \dot{t}$  and  $R_i = \partial L / \partial \dot{r}^i$ . The constants of motion obtained through the application of Noether's theorem to the different subgroups of the Galilei group are

$$\text{energy} \quad H = -T = \frac{m}{2} \left( \frac{d\vec{r}}{dt} \right)^2,$$

$$\text{linear momentum} \quad \vec{P} = \vec{R} = m \frac{d\vec{r}}{dt},$$

$$\text{kinematical momentum} \quad \vec{K} = m\vec{r} - \vec{P}t,$$

$$\text{angular momentum} \quad \vec{J} = \vec{r} \times \vec{P}.$$

The spin for this particle is  $\vec{S} = \vec{J} - \vec{K} \times \vec{P} / m = 0$ .

## 5.3 A spinning elementary particle

According to the definition, the most general nonrelativistic elementary particle [9] is the mechanical system whose kinematical space  $X$  is the whole Galilei group  $\mathcal{G}$ . The kinematical variables are, therefore, the ten real variables  $x(\tau) \equiv \{t(\tau), \vec{r}(\tau), \vec{u}(\tau), \vec{\rho}(\tau)\}$ , with domains  $t \in \mathbb{R}$ ,  $\vec{r} \in \mathbb{R}^3$ ,  $\vec{u} \in \mathbb{R}^3$  and  $\vec{\rho} \in SO(3)$ . The latter, with  $\rho = \tan \alpha / 2$ , is a particular parameterization of the rotation group. In this parameterization the composition law of rotations is algebraically simple, as shown below. All these kinematical variables have the same geometrical dimensions as the corresponding group parameters. The relationship between the values  $x'(\tau)$  and  $x(\tau)$  take, at any instant  $\tau$ , for two arbitrary inertial observers

$$t'(\tau) = t(\tau) + b, \tag{15}$$

$$\vec{r}'(\tau) = R(\vec{\mu})\vec{r}(\tau) + \vec{v}t(\tau) + \vec{a}, \tag{16}$$

$$\vec{u}'(\tau) = R(\vec{\mu})\vec{u}(\tau) + \vec{v}, \tag{17}$$

$$\vec{\rho}'(\tau) = \frac{\vec{\mu} + \vec{\rho}(\tau) + \vec{\mu} \times \vec{\rho}(\tau)}{1 - \vec{\mu} \cdot \vec{\rho}(\tau)}. \tag{18}$$

The way the kinematical variables transform allows us to interpret them, respectively, as the time (15), position (16), velocity (17) and orientation (18) of the particle.

There exist three differential constraints among the kinematical variables:  $\vec{u}(\tau) = \dot{\vec{r}}(\tau)/\dot{t}(\tau)$ . These constraints, and the homogeneity condition on the Lagrangian  $L$  in terms of the derivatives of the kinematical variables, reduce from ten to six the essential degrees of freedom of the system. These degrees of freedom are the position  $\vec{r}(t)$  and the orientation  $\vec{\rho}(t)$ . Since the Lagrangian depends on the derivative of  $\vec{u}$  it thus depends on the second derivative of  $\vec{r}(t)$ . For the orientation variables the Lagrangian only depends on the first derivative of  $\vec{\rho}(t)$ . It can be written as

$$L = T\dot{t} + \vec{R} \cdot \dot{\vec{r}} + \vec{U} \cdot \dot{\vec{u}} + \vec{V} \cdot \dot{\vec{\rho}}, \quad (19)$$

where the functions written in capital letters are defined as before as  $T = \partial L / \partial \dot{t}$ ,  $R_i = \partial L / \partial \dot{r}^i$ ,  $U_i = \partial L / \partial \dot{u}^i$ ,  $V_i = \partial L / \partial \dot{\rho}^i$ . In general they will be functions of the ten kinematical variables  $(t, \vec{r}, \vec{u}, \vec{\rho})$  and homogeneous functions of zero degree of the derivatives  $(\dot{t}, \dot{\vec{r}}, \dot{\vec{u}}, \dot{\vec{\rho}})$ .

If we introduce the angular velocity  $\vec{\omega}$  as a linear function of  $\dot{\vec{\rho}}$ , then the last term of the expansion of the Lagrangian (19),  $\vec{V} \cdot \dot{\vec{\rho}}$ , can also be written as  $\vec{W} \cdot \vec{\omega}$ , where  $W_i = \partial L / \partial \omega^i$ .

The different Noether constants of motion are related to the invariance of the dynamical equations under the Galilei group, and are obtained by the usual Lagrangian methods. They are the following observables:

$$\text{energy} \quad H = -T - \vec{u} \cdot \frac{d\vec{U}}{dt}, \quad (20)$$

$$\text{linear momentum} \quad \vec{P} = \vec{R} - \frac{d\vec{U}}{dt}, \quad (21)$$

$$\text{kinematical momentum} \quad \vec{K} = m\vec{r} - \vec{P}t - \vec{U}, \quad (22)$$

$$\text{angular momentum} \quad \vec{J} = \vec{r} \times \vec{P} + \vec{u} \times \vec{U} + \vec{W}. \quad (23)$$

From  $\dot{\vec{K}} = 0$ , comparing with (21), we find  $\vec{R} = m\vec{u}$ , and the linear momentum has the form  $\vec{P} = m\vec{u} - d\vec{U}/dt$ . We see that the total linear momentum does not coincide with the direction of the velocity  $\vec{u}$ . The functions  $\vec{U}$  and  $\vec{W}$  are what distinguishes this system from the point particle case. The spin structure is thus directly related to the dependence of the Lagrangian on the acceleration and angular velocity.

We see that  $\vec{K}$  in (22) differs from the point particle case  $\vec{K} = m\vec{r} - \vec{P}t$ , in the term  $-\vec{U}$ . If we define the vector  $\vec{k} = \vec{U}/m$ , with dimensions of length, then  $\vec{K} = 0$  leads to the equation:

$$\vec{P} = m \frac{d(\vec{r} - \vec{k})}{dt}.$$



The vector  $\vec{q} = \vec{r} - \vec{k}$ , defines the position of the centre of mass of the particle. It is a different point from  $\vec{r}$ , whenever  $\vec{k}$  (and thus  $\vec{U}$ ) is different from zero. In terms of  $\vec{q}$  the kinematical momentum takes the form

$$\vec{K} = m\vec{q} - \vec{P}t,$$

which looks like the result in the case of the point particle, where the centre of mass and centre of charge are the same point.

The total angular momentum (23) has three terms. The first term  $\vec{r} \times \vec{P}$  resembles an orbital angular momentum, and the other two  $\vec{Z} = \vec{u} \times \vec{U} + \vec{W}$  can be taken to represent the spin of the system. In fact, the latter observable is an angular momentum. It is related to the new kinematical variables and satisfies the dynamical equation  $d\vec{Z}/dt = \vec{P} \times \vec{u}$ . Because  $\vec{P}$  and  $\vec{u}$  are not collinear vectors,  $\vec{Z}$  is not a conserved angular momentum. This is the dynamical equation satisfied by Dirac's spin operator in the quantum case. The observable  $\vec{Z}$  is the classical spin observable equivalent to Dirac's spin operator.

One important feature of the total angular momentum is that the point  $\vec{r}$  is not the centre of mass of the system, and therefore the  $\vec{r} \times \vec{P}$  part can no longer be interpreted as the orbital angular momentum of the particle. The angular momentum  $\vec{Z}$  is the angular momentum of the particle with respect to the point  $\vec{r}$ , but not with respect to the centre of mass.

The spin of the system is defined as the difference between the total angular momentum  $\vec{J}$  and the orbital angular momentum of the centre of mass motion  $\vec{L} = \vec{q} \times \vec{P}$ . It can assume the following different expressions:

$$\vec{S} = \vec{J} - \vec{q} \times \vec{P} = \vec{J} - \frac{1}{m} \vec{K} \times \vec{P} = \vec{Z} + \vec{k} \times \vec{P} = -m\vec{k} \times \frac{d\vec{k}}{dt} + \vec{W}. \quad (24)$$

The second form of the spin  $\vec{S}$  in (24) is exactly expression (14) which leads to one of the Casimir operators of the extended Galilei group. It is expressed in terms of the constants of the motion  $\vec{J}$ ,  $\vec{K}$  and  $\vec{P}$ , and it is therefore another constant of motion. Because the particle is free and there are no external torques acting on it, it is clear that the spin of the system is represented by this constant angular momentum and not by the other angular momentum observable  $\vec{Z}$ , which is related to Dirac's spin operator.

The third expression in (24) is the sum of two terms, one  $\vec{Z}$ , coming from the new kinematical variables, and another  $\vec{k} \times \vec{P}$ , which is the angular momentum, of the linear momentum located at point  $\vec{r}$ , with respect to the centre of mass. Alternatively we can describe the spin according to the last expression in (24) in which the term  $-\vec{k} \times m d\vec{k}/dt$  suggests a contribution of (anti) orbital type coming from the motion around the centre of mass. It is related to the *zitterbewegung*, or more precisely to the function  $\vec{U} = m\vec{k}$ , which comes from the dependence of the Lagrangian on acceleration. The term  $\vec{W}$  comes from the dependence on the other three degrees of freedom  $\rho_i$ , and thus on the angular velocity. This *zitterbewegung* is the motion of the centre of charge around the centre of mass, as we shall see in an example in section 5.6. That the point  $\vec{r}$

represents the position of the centre of charge has also been suggested in previous works for the relativistic electron [10].

To analyse the different contributions to the spin of the most general elementary particle we shall consider now two simpler examples. In the first one, the spin is related to the existence of orientation variables, and in the second, to the dependence of the Lagrangian on the acceleration.

#### 5.4 Spinning particle with orientation

The kinematical space is  $\mathcal{Q}/G_K$ , where  $G_K$  is the three-dimensional subgroup which consists of the commutative Galilei boosts, or pure Galilei transformations at a constant velocity. The kinematical variables are now  $\{t, \vec{r}, \vec{\alpha}\}$ , time, position and orientation, respectively. The possible Lagrangians are not unique in this case. They must be functions only of the velocity  $\vec{u} = d\vec{r}/dt$  and of the angular velocity  $\vec{\omega}$ . They have the general form

$$L = T\dot{t} + \vec{R} \cdot \dot{\vec{r}} + \vec{W} \cdot \vec{\omega},$$

where  $T = \partial L / \partial \dot{t}$ ,  $\vec{R} = \partial L / \partial \dot{\vec{r}}$ ,  $\vec{W} = \partial L / \partial \vec{\omega}$ .

The basic conserved observables are:

$$\text{energy} \quad H = -T,$$

$$\text{linear momentum} \quad \vec{P} = m\vec{u},$$

$$\text{kinematical momentum} \quad \vec{K} = m\vec{r} - \vec{P}t,$$

$$\text{angular momentum} \quad \vec{J} = \vec{r} \times \vec{P} + \vec{W}.$$

For such a particle  $\vec{r} = \vec{q}$ , the centre of mass and centre of charge coincide and the spin  $\vec{S} = \vec{W} \neq 0$ . A particular Lagrangian which describes this system is the Lagrangian of a spherically symmetric body:

$$L = \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 + \frac{I}{2} \omega^2,$$

where the spin is  $\vec{S} = \vec{W} = I\vec{\omega}$ .

#### 5.5 Spinning particle with Zitterbewegung

The kinematical space is the manifold  $\mathcal{Q}/SO(3)$ , where  $SO(3)$ , is the three-dimensional subgroup of rotations. The kinematical variables are  $x(\tau) = \{t, \vec{r}, \vec{u}\}$ , time, position and velocity, respectively. The possible Lagrangians are not unique as in the previous case, and must be functions of the velocity  $\vec{u} = d\vec{r}/dt$  and the acceleration  $\vec{a} = d\vec{u}/dt$ .

The Lagrangians have the general form when expressed in terms of the kinematical variables and their  $\tau$ -derivatives

$$L = T\dot{t} + \vec{R} \cdot \dot{\vec{r}} + \vec{U} \cdot \dot{\vec{u}},$$

where  $T = \partial L / \partial \dot{t}$ ,  $\vec{R} = \partial L / \partial \dot{\vec{r}}$ ,  $\vec{U} = \partial L / \partial \dot{\vec{u}}$ . A particular Lagrangian could be, for example

$$L = \frac{m}{2} \frac{\dot{\vec{r}}^2}{\dot{t}} - \frac{m}{2\omega^2} \frac{\dot{\vec{u}}^2}{\dot{t}}, \quad (25)$$

If we consider that the evolution parameter is dimensionless, all terms in the Lagrangian have dimensions of action. The parameter  $m$  represents the mass of the particle while the parameter  $\omega$ , with dimension  $\text{time}^{-1}$ , represents an internal frequency: it is the frequency of the internal *zitterbewegung*. In terms of the essential degrees of freedom, which reduce to the three position variables  $\vec{r}$ , and using the time as the evolution parameter, the Lagrangian can also be written as

$$L = \frac{m}{2} \left( \frac{d\vec{r}}{dt} \right)^2 - \frac{m}{2\omega^2} \left( \frac{d^2\vec{r}}{dt^2} \right)^2. \quad (26)$$

The dynamical equations obtained from the Lagrangian (26) are:

$$\frac{1}{\omega^2} \frac{d^4\vec{r}}{dt^4} + \frac{d^2\vec{r}}{dt^2} = 0, \quad (27)$$

whose general solution is

$$\vec{r}(t) = \vec{A} + \vec{B}t + \vec{C} \cos \omega t + \vec{D} \sin \omega t, \quad (28)$$

in terms of the 12 integration constants  $\vec{A}$ ,  $\vec{B}$ ,  $\vec{C}$  and  $\vec{D}$ .

We see that the kinematical momentum  $\vec{K}$  in (22) differs from the point particle case in the term  $-\vec{U}$ . The definition of the vector  $\vec{k} = \vec{U}/m$ , implies that  $\vec{K} = 0$  leads to the equation  $\vec{P} = m d(\vec{r} - \vec{k})/dt$ , as before, and  $\vec{q} = \vec{r} - \vec{k}$  represents the position of the centre of mass of the particle. It is defined in this example as

$$\vec{q} = \vec{r} - \frac{1}{m} \vec{U} = \vec{r} + \frac{1}{\omega^2} \frac{d^2\vec{r}}{dt^2}. \quad (29)$$

In terms of the center of mass, the dynamical equations (27) can be separated into the form

$$\frac{d^2\vec{q}}{dt^2} = 0, \quad (30)$$

$$\frac{d^2\vec{r}}{dt^2} + \omega^2(\vec{r} - \vec{q}) = 0, \quad (31)$$

where (30) is just equation (27) after twice differentiation of (29), and equation (31) is (29) after all terms on the left hand side have been collected.

From (30) we see that the point  $\vec{q}$  moves in a straight trajectory at constant velocity while the motion of point  $\vec{r}$ , given in (31), is an isotropic harmonic motion of angular frequency  $\omega$  around the point  $\vec{q}$ .

The spin of the system  $\vec{S}$  is defined as

$$\vec{S} = \vec{J} - \vec{q} \times \vec{P} = \vec{J} - \frac{1}{m} \vec{K} \times \vec{P}, \quad (32)$$

and since it is written in terms of constants of motion it is clearly another constant of motion. Its magnitude  $S^2$  is also a Galilei invariant quantity which characterizes the system. From its definition we get

$$\vec{S} = \vec{u} \times \vec{U} + \vec{k} \times \vec{P} = -m(\vec{r} - \vec{q}) \times \frac{d}{dt}(\vec{r} - \vec{q}) = -\vec{k} \times m \frac{d\vec{k}}{dt}, \quad (33)$$

which appears as the (anti)orbital angular momentum of the relative motion of the point  $\vec{r}$  around the centre of mass position  $\vec{q}$  at rest, so that the total angular momentum can be written as

$$\vec{J} = \vec{q} \times \vec{P} + \vec{S} = \vec{L} + \vec{S}. \quad (34)$$

The total angular momentum is the sum of the orbital angular momentum  $\vec{L}$ , associated with the motion of the centre of mass, and the spin part  $\vec{S}$ . For a free particle both  $\vec{L}$  and  $\vec{S}$  are separate constants of motion. We use the term (anti)orbital to suggest that if the vector  $\vec{k}$  represents the position of a point of mass  $m$ , the angular momentum of its motion is in the opposite direction from what we obtain here for the spin observable. But, as we shall see in a moment, the vector  $\vec{k}$  represents not the position of the mass  $m$ , but the position of the charge of the particle.

## 5.6 Interaction with an external electromagnetic field

If the point  $\vec{q}$  represents the position of the centre of mass of the particle, then what position does point  $\vec{r}$  represent? The point  $\vec{r}$  represents the position of the charge of the particle. This can be seen by considering interaction with an external field. The homogeneity condition of the Lagrangian in terms of the derivatives of the kinematical variables suggests an interaction term of the form

$$L_I = -e\phi(t, \vec{r})\dot{t} + e\vec{A}(t, \vec{r}) \cdot \dot{\vec{r}}, \quad (35)$$

which is linear in the derivatives of the kinematical variables  $t$  and  $\vec{r}$ , and where the external potentials are only functions of  $t$  and  $\vec{r}$ .

The dynamical equations obtained from the Lagrangian  $L + L_I$  are

$$\frac{1}{\omega^2} \frac{d^4 \vec{r}}{dt^4} + \frac{d^2 \vec{r}}{dt^2} = \frac{e}{m} \left( \vec{E}(t, \vec{r}) + \vec{u} \times \vec{B}(t, \vec{r}) \right), \quad (36)$$

where the electric field  $\vec{E}$  and magnetic field  $\vec{B}$  are expressed in terms of the potentials in the usual form  $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$ ,  $\vec{B} = \nabla \times \vec{A}$ . Because the interaction term does not depend on  $\dot{\vec{u}}$ , the function  $\vec{U} = m\vec{k}$  has the same expression as in the free particle case. Therefore the spin and the centre of mass definitions, (33) and (29) respectively, remain the same as in the previous free case. Dynamical equations (36) can again be separated into the form

$$\frac{d^2 \vec{q}}{dt^2} = \frac{e}{m} \left( \vec{E}(t, \vec{r}) + \vec{u} \times \vec{B}(t, \vec{r}) \right), \quad (37)$$

$$\frac{d^2 \vec{r}}{dt^2} + \omega^2 (\vec{r} - \vec{q}) = 0. \quad (38)$$

The centre of mass  $\vec{q}$  satisfies Newton's equations under the action of the total external Lorentz force, while the point  $\vec{r}$  still satisfies the isotropic harmonic motion of angular frequency  $\omega$  around the point  $\vec{q}$ . But the external force and the fields are defined at the point  $\vec{r}$  and not at point  $\vec{q}$ . It is the velocity  $\vec{u}$  of

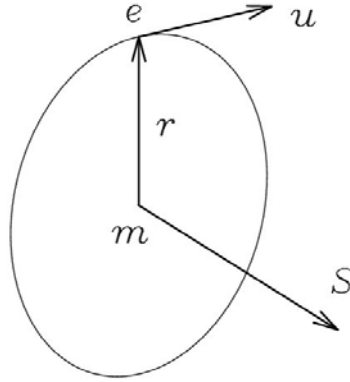


Figure 1: Charge motion in the C.M. frame.

the point  $\vec{r}$  which appears in the magnetic term of the Lorentz force. The point  $\vec{r}$  clearly represents the position of the charge. In fact, this minimal coupling we have considered is the coupling of the electromagnetic potentials with the particle current, which, in the relativistic case, can be written as  $j_\mu A^\mu$ . The current  $j_\mu$  is associated with the motion of a charge  $e$  at the point  $\vec{r}$ .

The charge has an oscillatory motion of very high frequency  $\omega$ , which in the case of the relativistic electron will be  $\omega = 2mc^2 / \hbar \approx 1,55 \times 10^{21} \text{ s}^{-1}$ , as shown later. The average position of the charge is the centre of mass, but it is this internal orbital motion which gives rise to the spin structure and also to the magnetic properties of the particle.

When analysed in the centre of mass frame (see Fig. 1),  $\vec{q} = 0$ ,  $\vec{r} = \vec{k}$ , and the system reduces to a point charge whose motion is in general an ellipse. If we choose  $C = D$ , and  $\vec{C} \cdot \vec{D} = 0$ , it reduces to a circle of radius  $r = C = D$ , orthogonal to the spin. Because the particle has a charge  $e$ , it produces a magnetic moment, which according to the usual classical definition is [11]

$$\vec{\mu} = \frac{1}{2} \int \vec{r} \times \vec{j} d^3\vec{r} = \frac{e}{2} \vec{k} \times \frac{d\vec{k}}{dt} = -\frac{e}{2m} \vec{S}, \quad (39)$$

where  $\vec{j} = e\delta^3(\vec{r} - \vec{k})d\vec{k}/dt$  is the vector current associated with the motion of a charge  $e$  located at the point  $\vec{k}$ . The magnetic moment is orthogonal to the *zitterbewegung* plane and opposite to the spin if  $e > 0$ . The particle also has a non-vanishing electric dipole moment with respect to the centre of mass  $\vec{d} = e\vec{k}$ . It oscillates and is orthogonal to  $\vec{\mu}$ , and therefore to  $\vec{S}$ , in the centre of mass frame. Its time average value vanishes for times larger than the natural period of this internal motion. Although this is a nonrelativistic example, it is interesting to compare this analysis with Dirac's relativistic analysis of the electron, [5] in which both momenta  $\vec{\mu}$  and  $\vec{d}$  appear, giving rise to two possible interacting terms in Dirac's Hamiltonian.

## 6. Relativistic elementary particles

The Poincaré group can be parameterised in terms of exactly the same ten parameters  $\{b, \vec{a}, \vec{v}, \vec{\alpha}\}$  as the Galilei group and with the same dimensions as before. We therefore maintain the interpretation of these variables respectively as the time, position, velocity and orientation of the particle. The homogeneous spaces of the Poincaré group can be classified in the same manner, but with some minor restrictions. For instance, the kinematical space of the example of the spinning particle with orientation as in section 5.4,  $X = \mathcal{G}/G_K$ , can no longer be defined in the Poincaré case, because the three dimensional set  $G_K$  of Lorentz boosts is not a subgroup of  $G$ ; but the most general structure of a spinning particle still holds.

The Poincaré group has three different maximal homogeneous spaces spanned by the variables  $\{b, \vec{a}, \vec{v}, \vec{\alpha}\}$ , which are classified according to the range of the velocity parameter  $\vec{v}$ . If  $v < c$  we have the Poincaré group itself. When  $v > c$ , this homogeneous space describes particles whose charge is moving faster than light. Finally, if  $v = c$ , we have a homogeneous space which describes particles whose position  $\vec{r}$  is always moving at the speed of light. This is the manifold which defines the kinematical space of photons and electrons [6,7]. The first manifold gives, in the low velocity limit, the same models as in the nonrelativistic case. It is the Poincaré group manifold, which is transformed into the Galilei group by the limiting process  $c \rightarrow \infty$ . But this limit cannot be applied to the other two manifolds. Accordingly, the Poincaré group describes a larger set of spinning objects.

### 6.1 Spinning relativistic elementary particles

We shall review the main points of the relativistic spinning particles whose kinematical space is the manifold spanned by the variables  $\{t, \vec{r}, \vec{u}, \vec{\alpha}\}$ , interpreted as the time, position, velocity and orientation of the particle, but with  $u = c$ . This is a homogeneous space homomorphic to the manifold  $G/V$ , where  $V$  is the one-dimensional subgroup of pure Lorentz transformations in a fixed arbitrary direction.

For these systems the most general form of the Lagrangian is

$$L = T\dot{t} + \vec{R} \cdot \dot{\vec{r}} + \vec{U} \cdot \dot{\vec{u}} + \vec{W} \cdot \dot{\vec{\alpha}},$$

where  $T = \partial L / \partial \dot{t}$ ,  $R_i = \partial L / \partial \dot{r}^i$ ,  $U_i = \partial L / \partial \dot{u}^i$  and  $W_i = \partial L / \partial \dot{\alpha}^i$  will be, in general, functions of the ten kinematical variables  $\{t, \vec{r}, \vec{u}, \vec{\alpha}\}$  and homogeneous functions of zero degree in terms of the derivatives  $\{\dot{t}, \dot{\vec{r}}, \dot{\vec{u}}, \dot{\vec{\alpha}}\}$ .

The Noether constants of motion are now the following conserved observables:

$$\text{energy} \quad H = -T - \vec{u} \cdot \frac{d\vec{U}}{dt}, \quad (40)$$

$$\text{linear momentum} \quad \vec{P} = \vec{R} - \frac{d\vec{U}}{dt}, \quad (41)$$

$$\text{kinematical momentum} \quad \vec{K} = H\vec{r}/c^2 - \vec{P}t - \vec{S} \times \vec{u}/c^2, \quad (42)$$

$$\text{angular momentum} \quad \vec{J} = \vec{r} \times \vec{P} + \vec{S}, \quad (43)$$

where

$$\vec{S} = \vec{u} \times \vec{U} + \vec{W}. \quad (44)$$

The difference from the Galilei case comes from the different behaviour of the Lagrangian under the Lorentz boosts when compared with the Galilei boosts. In the nonrelativistic case the Lagrangian is not invariant. However, the relativistic Lagrangian is invariant and the kinematical variables transform in a different way. This gives rise to the term  $\vec{S} \times \vec{u}/c^2$  instead of the term  $\vec{U}$  which appears in the kinematical momentum (42). The angular momentum observable (44) is not properly speaking the spin of the system, if we define spin as the difference between the total angular momentum and the orbital angular momentum associated with the centre of mass. It is the angular momentum of the particle with respect to the point  $\vec{r}$ , as in the nonrelativistic case. Nevertheless, the observable  $\vec{S}$  is the classical equivalent of Dirac's spin observable because in the free particle case it satisfies the same dynamical equation,

$$\frac{d\vec{S}}{dt} = \vec{P} \times \vec{u},$$

as Dirac's spin operator does in the quantum case. It is only a constant of motion for the centre of mass observer. This can be seen by taking the time derivative of the constant total angular momentum  $\vec{J}$  given in (43). We shall keep the notation  $\vec{S}$  for this angular momentum observable, because when the system is quantized it gives rise to the usual quantum mechanical spin operator in terms of the Pauli spin matrices.

## 6.2 Dirac's equation

Dirac's equation is the quantum mechanical expression of the Poincaré invariant linear relationship [6,7] between the energy  $H$  and the linear momentum  $\vec{P}$

$$H - \vec{P} \cdot \vec{u} - \vec{S} \cdot \left( \frac{d\vec{u}}{dt} \times \vec{u} \right) = 0,$$

where  $\vec{u}$  is the velocity of the charge ( $u = c$ ),  $d\vec{u}/dt$  the acceleration and  $\vec{S} = \vec{S}_u + \vec{S}_\alpha$  Dirac's spin observable (see Figure 2). This expression can be obtained from (42) by making the time derivative of that constant observable and a final scalar product with the velocity  $\vec{u}$ . The Dirac spin has two parts: one  $\vec{S}_u = \vec{u} \times \vec{U}$ , is related to the orbital motion of the charge, and  $\vec{S}_\alpha = \vec{W}$  is due to the rotation of the particle and is directly related to the angular velocity, as it corresponds to a spherically symmetric object.

The centre of mass observer is defined as the observer for whom  $\vec{K} = \vec{P} = 0$ , because this implies that  $\vec{q} = 0$  and  $d\vec{q}/dt = 0$ . By analysing the observable (42) in the centre of mass frame where  $H = mc^2$ , we get the dynamical equation of the point  $\vec{r}$ ,

$$\vec{r} = \vec{S} \times \vec{u} / mc^2$$

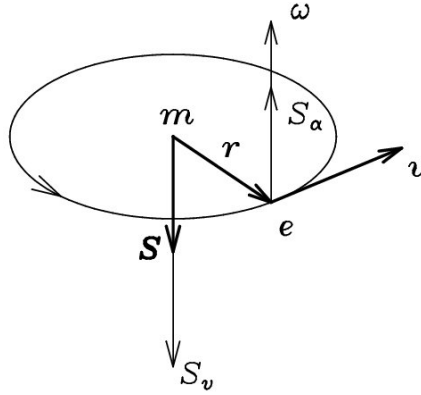


Figure 2. Motion of the centre of charge of the electron around its centre of mass in the C.M. frame.

where  $\vec{S}$  is a constant vector in this frame. The solution is the circular motion depicted in Figure 2.

The radius and angular velocity of the internal classical motion of the charge are, respectively,  $R = S/mc$ , and  $\omega = mc^2/S$ . The energy of this system is not definite positive. The particle of positive energy has the total spin  $\vec{S}$  oriented in the same direction as the  $\vec{S}_u$  part while the orientation is the opposite for the negative energy particle. This system corresponds to the time reversed motion of the other. When the system is quantized, the orbital component  $\vec{S}_u$ , which is directly related to the magnetic moment, quantizes with integer values, while the rotational part  $\vec{S}_\alpha$  requires half integer values. For these particles of spin  $1/2$ , the total spin is half the value of the  $\vec{S}_u$  part. When expressing the magnetic moment in terms of the total spin, we thus obtain a pure kinematical interpretation of the  $g = 2$  gyromagnetic ratio [12].

For the centre of mass observer this system appears as a system of three degrees of freedom. Two represent the  $x$  and  $y$  coordinates of the point charge, and the third is the phase of its rotational motion. However this phase is exactly the same as the phase of the orbital motion of the charge. Because the motion is at constant radius at constant speed  $c$ , only one independent degree of freedom is left—say the  $x$  variable. Therefore the system is reduced to a one-dimensional harmonic oscillator of angular frequency  $\omega$ . When the system is quantized, the stationary states of a one-dimensional harmonic oscillator have the energy

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega, \quad n = 0, 1, 2, \dots$$

But if the system is elementary, then it has no excited states, and in the C.M. frame it is reduced to the ground state of energy



$$E_0 = \frac{1}{2} \hbar \omega = mc^2.$$

If we compare this with the classical result  $\omega = mc^2/S$  we see that the constant classical parameter  $S$  takes the value  $S = \hbar/2$  when quantized. The radius of the internal motion is  $R = \lambda_C/2$ , half Compton's wavelength.

We see that all Lagrangian systems with the same kinematical space as the one considered in this model have exactly the same dynamics for the point  $r$ , describe spin  $1/2$  particles and satisfy Dirac's equation when quantized. The formalism describes an object whose charge is located at a single point  $\vec{r}$ , but it is nevertheless moving in a confined region of radius of order  $\lambda_C$ . It has a magnetic moment produced by the motion of the charge, and also an oscillating electric dipole moment, with respect to the centre of mass, of average value zero.

To end this section, and with the above model of the electron in mind, it is convenient to remember some of the features that Dirac obtained for the motion of a free electron [5]. Let the point  $\vec{r}$  be the position vector in terms of which Dirac's spinor  $\psi(t, \vec{r})$  is defined. When computing the velocity of the point  $\vec{r}$ , Dirac arrives at:

1. The velocity  $\vec{u} = i/\hbar[H, \vec{r}] = c\vec{\alpha}$ , is expressed in terms of the  $\vec{\alpha}$  matrices and he writes, ... *"a measurement of a component of the velocity of a free electron is certain to lead to the result  $\pm c$ ."*
2. The linear momentum does not have the direction of this velocity  $\vec{u}$ , but must be related to some average value of it: ... *"the  $x_1$  component of the velocity,  $c\alpha_1$ , consists of two parts, a constant part  $c^2 p_1 H^{-1}$ , connected with the momentum by the classical relativistic formula, and an oscillatory part, whose frequency is at least  $2mc^2/\hbar$ , ....."*
3. About the position  $\vec{r}$ : *"The oscillatory part of  $x_1$  is small... which is of order of magnitude  $\hbar/mc$  ...."*

And when analyzing the interaction of the electron with an external electromagnetic field in his original 1928 paper [13], after taking the square of Dirac's operator, he obtains two new interaction terms:

$$\frac{e\hbar}{2mc} \vec{\Sigma} \cdot \vec{B} + \frac{ie\hbar}{2mc} \vec{\alpha} \cdot \vec{E},$$

Here Dirac's spin operator is written as  $\vec{S} = \hbar\vec{\Sigma}/2$  where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},$$

in terms of  $\sigma$ -Pauli matrices.  $\vec{E}$  and  $\vec{B}$  are the external electric and magnetic fields, respectively. He says, *"The electron will therefore behave as though it has a magnetic moment  $(e\hbar/2mc)\Sigma$  and an electric moment  $(ie\hbar/2mc)\vec{\alpha}$ . The magnetic moment is just that assumed in the spinning electron model"* (Pauli model). *"The electric moment, being a pure imaginary, we should not expect to appear in the model."*

In the last sentence it is difficult to understand why Dirac, who did not reject the negative energy solutions, disliked the existence of this electric dipole, which was obtained from the formalism on an equal footing with the magnetic dipole term. Properly speaking this electric dipole does not represent the existence of a particular positive and negative charge distribution for the electron. The negative charge of the electron is at a single point but because this point is not the centre of mass, there exists a non-vanishing electric dipole moment with respect to the centre of mass even in the centre of mass frame. This is the observable Dirac disliked. It is oscillating at very high frequency and basically plays no role in low energy electron interactions because its average value vanishes, but it is important in high energy processes or in very close electron-electron interactions.

All real experiments to determine very accurately the gyromagnetic ratio are based on the determination of precession frequencies. But these precession frequencies are independent of the spin orientation. However, the difficulty separating electrons in a Stern-Gerlach type experiment suggests polarization experiments have to be done to determine in a direct way whether the spin and magnetic moment for elementary particles are either parallel or antiparallel to each other. One of the predictions of this formalism is that for both particle and the corresponding antiparticle the spin and magnetic moment have to have the same relative orientation, either parallel or antiparallel.

### 6.3 Dynamical equation of the relativistic spinning electron

We recall from elementary differential geometry some basic properties of any arbitrary three-dimensional curve  $\vec{r}(s)$ . If it is expressed in parametric form in terms of the arc length  $s$  as the parameter, it has associated the three orthogonal unit vectors  $\vec{v}_i$ ,  $i=1,2,3$  called respectively tangent, normal and binormal. These unit vectors satisfy the so called Frenet-Serret differential equations:

$$\begin{aligned}\dot{\vec{v}}_1(s) &= \kappa(s)\vec{v}_2(s) \\ \dot{\vec{v}}_2(s) &= -\kappa(s)\vec{v}_1(s) + \tau(s)\vec{v}_3(s), \\ \dot{\vec{v}}_3(s) &= -\tau(s)\vec{v}_2(s)\end{aligned}$$

where  $\kappa$  and  $\tau$  are respectively the curvature and torsion. Since the unit tangent vector is  $\vec{v}_1 = \dot{\vec{r}} \equiv \vec{r}^{(1)}$ , when successive derivatives are taken it yields

$$\begin{aligned}\vec{r}^{(1)} &= \vec{v}_1, \\ \vec{r}^{(2)} &= \kappa\vec{v}_2, \\ \vec{r}^{(3)} &= \dot{\kappa}\vec{v}_2 + \kappa\dot{\vec{v}}_2 = -\kappa^2\vec{v}_1 + \dot{\kappa}\vec{v}_2 + \kappa\tau\vec{v}_3, \\ \vec{r}^{(4)} &= -3\kappa\dot{\kappa}\vec{v}_1 + (\ddot{\kappa} - \kappa^3 - \kappa\tau^2)\vec{v}_2 + (2\dot{\kappa}\tau + \kappa\dot{\tau})\vec{v}_3.\end{aligned}$$

The elimination of the  $\vec{v}_i$  vectors between these equations implies that the most general curve in three-dimensional space satisfies the fourth-order ordinary differential equation:

$$\vec{r}^{(4)} - \left( \frac{2\dot{\kappa}}{\kappa} + \frac{\dot{\tau}}{\tau} \right) \vec{r}^{(3)} + \left( \kappa^2 + \tau^2 + \frac{\dot{\kappa}\dot{\tau}}{\kappa\tau} + \frac{2\dot{\kappa}^2 - \kappa\ddot{\kappa}}{\kappa^2} \right) \vec{r}^{(2)} + \kappa^2 \left( \frac{\dot{\kappa}}{\kappa} - \frac{\dot{\tau}}{\tau} \right) \vec{r}^{(1)} = 0.$$

All the coefficients in brackets, in front of the  $s$ -derivatives  $\vec{r}^{(i)}$ , can be expressed in terms of the scalar products  $\vec{r}^{(i)} \cdot \vec{r}^{(j)}$ ,  $i, j = 1, 2, 3$ . For helical motions there is a constant relationship  $\kappa/\tau = \text{constant}$ , and therefore the coefficient of  $\vec{r}^{(1)}$  vanishes.

Our example of the nonrelativistic spinning particle also satisfies the fourth order differential equation (27). Similarly, the point  $\vec{r}$  of the relativistic spinning electron also satisfies a fourth order ordinary differential equation which has been calculated from invariance principles [14]. It takes the following form for any arbitrary inertial observer:

$$\begin{aligned} \vec{r}^{(4)} - \frac{3(\vec{r}^{(2)} \cdot \vec{r}^{(3)})}{(\vec{r}^{(2)} \cdot \vec{r}^{(2)})} \vec{r}^{(3)} + \\ \left( \frac{2(\vec{r}^{(3)} \cdot \vec{r}^{(3)})}{(\vec{r}^{(2)} \cdot \vec{r}^{(2)})} - \frac{3(\vec{r}^{(2)} \cdot \vec{r}^{(3)})^2}{4(\vec{r}^{(2)} \cdot \vec{r}^{(2)})^2} - (\vec{r}^{(2)} \cdot \vec{r}^{(2)})^{1/2} \right) \vec{r}^{(2)} = 0. \end{aligned} \quad (46)$$

It corresponds to a helical motion since the term in the first derivative  $\vec{r}^{(1)}$  is lacking, and it reduces to circular central motion at constant velocity  $c$  in the centre of mass frame. Here we use space-time units such that the internal radius  $R = 1$  and the *zitterbewegung* frequency  $\omega = 1$ .

The centre of mass position is defined by

$$\vec{q} = \vec{r} + \frac{2(\vec{r}^{(2)} \cdot \vec{r}^{(2)})\vec{r}^{(2)}}{(\vec{r}^{(2)} \cdot \vec{r}^{(2)})^{3/2} + (\vec{r}^{(3)} \cdot \vec{r}^{(3)})} - \frac{3(\vec{r}^{(2)} \cdot \vec{r}^{(3)})^2}{4(\vec{r}^{(2)} \cdot \vec{r}^{(2)})}. \quad (47)$$

We can check that both  $\vec{q}$  and  $\vec{q}^{(1)}$  vanish for the centre of mass observer. The fourth order dynamical equation for the position of the charge (46) can also be rewritten as a system of two second order ordinary differential equations for the positions of the points  $\vec{q}$  and  $\vec{r}$

$$\vec{q}^{(2)} = 0, \quad \vec{r}^{(2)} = \frac{1 - \vec{q}^{(1)} \cdot \vec{r}^{(1)}}{(\vec{q} - \vec{r})^2} (\vec{q} - \vec{r}), \quad (48)$$

*i.e.*, a free motion for the centre of mass  $\vec{q}$  and a kind of central motion for the charge position  $\vec{r}$  around the centre of mass. Equation (46) emerges from (47) after differentiation twice with respect to time. The last equation of (48) is just (47) written in terms of  $\vec{q}$  and  $\vec{q}^{(1)}$ .

For the relativistic electron, when the centre of mass velocity is small,  $\vec{q}^{(1)} \rightarrow 0$ , and because  $|\vec{q} - \vec{r}| = 1$  in these units, we obtain the equations of the Galilei case

$$\vec{q}^{(2)} = 0, \quad \vec{r}^{(2)} = \vec{q} - \vec{r} \quad (49)$$

*i.e.*, a free motion for the centre of mass and a harmonic motion around  $\vec{q}$  of angular frequency  $\omega = 1$ , for the position of the charge, as happened in the nonrelativistic example analysed in (30) and (31).

### 6.4 Interaction with an external field

The free equation for the centre of mass motion  $\vec{q}^{(2)} = 0$  represents the conservation of linear momentum  $d\vec{P}/dt = 0$ . But the linear momentum is written in terms of centre of mass velocity as  $\vec{P} = m\gamma(q^{(1)})\vec{q}^{(1)}$ , so that the free dynamical equation (48) in the presence of an external field should be replaced by

$$\vec{P}^{(1)} = \vec{F}, \quad \vec{r}^{(2)} = \frac{1 - \vec{q}^{(1)} \cdot \vec{r}^{(1)}}{(\vec{q} - \vec{r})^2} (\vec{q} - \vec{r}), \quad (50)$$

where  $\vec{F}$  is the external force and the second equation is left unchanged. We consider the same definition of the centre of mass position (47) as in the free particle case, because it corresponds to the fact that the internal structure of an elementary particle is not modified by any external interaction, and the charge moves in the same way around the centre of mass as in the free case. Since

$$\frac{d\vec{P}}{dt} = m\gamma(q^{(1)})\vec{q}^{(2)} + m\gamma(q^{(1)})^3 (\vec{q}^{(1)} \cdot \vec{q}^{(2)})\vec{q}^{(1)}$$

it yields

$$m\gamma(q^{(1)})^3 (\vec{q}^{(1)} \cdot \vec{q}^{(2)}) = \vec{F} \cdot \vec{q}^{(1)}$$

and by leaving the highest derivative  $\vec{q}^{(2)}$  on the left hand side we finally obtain the differential equations that describe the evolution of a relativistic spinning electron in the presence of an external electromagnetic field:

$$m\vec{q}^{(2)} = \frac{e}{\gamma(q^{(1)})} \left[ \vec{E} + \vec{r}^{(1)} \times \vec{B} - \vec{q}^{(1)} \left( [\vec{E} + \vec{r}^{(1)} \times \vec{B}] \cdot \vec{q}^{(1)} \right) \right], \quad (51)$$

$$\vec{r}^{(2)} = \frac{1 - \vec{q}^{(1)} \cdot \vec{r}^{(1)}}{(\vec{q} - \vec{r})^2} (\vec{q} - \vec{r}). \quad (52)$$

## 7. Gyromagnetic ratio

The Hilbert space which describes the wave functions of the spinning electron is a complex vector space of squared integrable functions  $\psi(t, \vec{r}, \vec{u}, \vec{\alpha})$  of the kinematical variables. The general structure of the quantum mechanical angular momentum operator acting on this Hilbert space, in either the relativistic or nonrelativistic approach, is

$$\vec{J} = \vec{r} \times \frac{\hbar}{i} \nabla + \vec{S} = \vec{r} \times \vec{P} + \vec{S}, \quad (53)$$

where the spin operator takes the form  $S = Z + W$

$$\vec{S} = \vec{u} \times \frac{\hbar}{i} \nabla_u + \vec{W}. \quad (54)$$

The operator  $\nabla_u$  is the gradient operator with respect to the velocity variables and  $\vec{W}$  is a linear differential operator which depends only on the orientation variables  $\vec{\alpha}$ ; it therefore commutes with  $\nabla_u$ . For example, in the  $\vec{\rho} = \vec{n} \tan(\alpha/2)$  parameterization  $\vec{W}$  is written as

$$\vec{W} = \frac{\hbar}{2i} [\nabla_{\rho} + \vec{\rho} \times \nabla_{\rho} + \vec{\rho}(\vec{\rho} \cdot \nabla_{\rho})], \quad (55)$$

where  $\nabla_{\rho}$  is the gradient operator with respect to the  $\vec{\rho}$  variables.

The first part  $Z$  in (54) is related to the *zitterbewegung* spin and has only integer eigenvalues. This is because it has the form of an orbital angular momentum operator in terms of the  $\vec{u}$  variables. Half-integer eigenvalues come only from the operator (55). This operator takes into account the change of orientation, *i.e.*, the rotation of the particle.

We have seen, in both relativistic and non-relativistic examples, that if the only spin content of the particle  $\vec{S}$  is related to the *zitterbewegung* part  $\vec{Z} = \vec{u} \times \vec{U}$ , then the relationship between the magnetic moment and *zitterbewegung* spin is given by

$$\vec{\mu} = \frac{e}{2} \vec{k} \times \frac{d\vec{k}}{dt} = \frac{e}{2m} \vec{Z}, \quad (56)$$

*i.e.*, with a normal gyromagnetic ratio  $g = 1$ . If the electron has a gyromagnetic ratio  $g = 2$ , this necessarily implies that another part of the spin arises from the angular velocity of the body, but makes no contribution to the magnetic moment.

For the electron, therefore, both parts  $\vec{W}$  and  $\vec{Z}$  contribute to the total spin. But the  $\vec{W}$  part, which is related to the angular variables that describe its orientation in space, does not contribute to the separation  $\vec{k}$  between the centre of charge and the centre of mass. It turns out that the magnetic moment of a general particle is still related to the motion of the charge by the expression (56), *i.e.*, in terms of the  $\vec{Z}$  part, but not to the  $\vec{W}$  part. It is precisely when we express the magnetic moment in terms of the total spin  $\vec{S}$  that the concept of gyromagnetic ratio arises.

We now assume that both  $\vec{Z}$  and  $\vec{W}$  terms contribute to the total spin  $\vec{S}$  with their lowest admissible values. In the model of the spinning electron  $\vec{Z}$  and  $\vec{W}$  have opposite orientation.

For Dirac's particles, the classical *zitterbewegung* is a circular motion at the speed of light of radius  $R = \hbar/2mc$  and angular frequency  $\omega = 2mc^2/\hbar$ , on a plane orthogonal to the total spin. The total spin  $\vec{S}$  and the  $\vec{Z}$  part are both orthogonal to this plane, and parallel to each other. Let us define the gyromagnetic ratio by  $Z = gS$ . For the lowest admissible values of the quantized spins  $z = 1$  and  $w = 1/2$  in the opposite direction, this gives rise to a total  $s = 1/2$  perpendicular to the *zitterbewegung* plane, and therefore  $g = 2$ .

## 8. Bound motion of two electrons

If we have relativistic and nonrelativistic differential equations satisfied by the spinning electrons we can analyze the interaction between them by assuming, for example, a Coulomb interaction between their charges. This leads to a system of differential equations of the form (37-38) or (51-52) for each particle. For example, the external field acting on the charge  $e_1$  is replaced by the in-

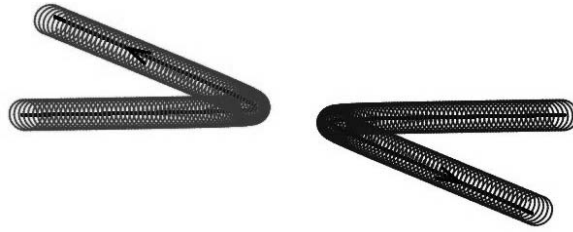


Figure 3. Scattering of two spinning electrons with parallel spins, in their centre of mass frame. It is also depicted the scattering of two spinless electrons with the same energy and linear momentum.

stantaneous Coulomb field created by the other charge  $e_2$  at the position of  $e_1$ , and similarly for the other particle. The integration is performed numerically by means of the numerical integration program *Dynamics Solver* [15].

Figure 3 represents the scattering of two spinning electrons analysed in their common centre of mass frame [14]. We send the particles with their spins parallel and with a non vanishing impact parameter. In addition to the helical motion of their charges, we can also depict the trajectories of their centre of mass. If we compare this motion with the Coulomb interaction of two spinless electrons coming from the same initial position and with the same velocity as the centre of mass of the spinning electrons, we obtain the solid trajectories marked with an arrow. Basically, this corresponds to the trajectory of the centre of mass of each spinning particle, provided the two particles do not approach each other below the Compton wavelength. This can be understood because the average position of the centre of charge of each particle approximately coincides with its centre of mass, and if they do not approach each other too closely the average Coulomb force is the same. The difference comes out when we consider a very deep interaction or very close initial positions.

Figure 4 represents the initial positions of a pair of particles with parallel spins. Recall that the radius of the internal motion is half the Compton wavelength. The initial separation of their centres of mass  $a$  is a distance smaller than the Compton wavelength. The centre of mass of each particle is considered to be moving with a velocity  $\vec{v}$ , as depicted.

That the spins of the two particles are parallel is reflected by the fact that the internal motions of the charges, represented by the oriented circles that surround the corresponding centre of mass, have the same orientation. It must be remarked that the internal motion of the charge around its centre of mass can always be characterised by a phase. The phases of the particles are chosen opposite to one another. We also depict the repulsive Coulomb force  $F$  computed in terms of the separation of charges. This interaction force  $F$  has also been drawn attached to the corresponding centre of mass, so that the net force acting on the point  $m_2$  is directed toward the point  $m_1$ , and conversely. This external force determines the motion of each centre of mass. We thus see that a repul-

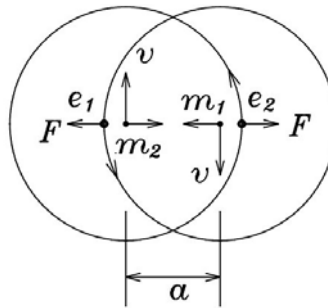


Figure 4: Initial position and velocity of the centre of mass and charges for a bound motion of a two-electron system with parallel spins. The circles would correspond to the trajectories of the charges if considered free. The interacting Coulomb force  $F$  is computed in terms of the separation distance between the charges.

sive force between the charges represents an attractive force between their centres of mass when located at such a short distance.

In Figure 5 we depict the evolution of the charges and masses of this two-electron system for  $a = 0,4\lambda_c$  and  $v = 0,004c$  during a short time interval. Figure 6 represents only the motions of the centres of mass of both particles for a longer time. It shows that the centre of mass of each particle remains in a bound region.

The evolution of the charges is not shown in this last figure because it blurs the picture, but it can be inferred from the previous figure. We have found bound motions at least for the range  $0 \leq a \leq 0,8\lambda_c$  and velocity  $0 \leq v \leq 0,01c$ . We can also obtain similar bound motions if the initial velocity  $v$  has a component along the  $OX$  axis. Bound motion can also be obtained for initial charge positions different from the ones depicted in Figure 4. This range for the relative phase depends on  $a$  and  $v$ , but in general bound motion is more likely if the initial phases of the charges are opposite to each other. If, instead of the instantaneous Coulomb interaction between the charges, we consider the retarded electromagnetic field of each charge, we obtain a similar behaviour for

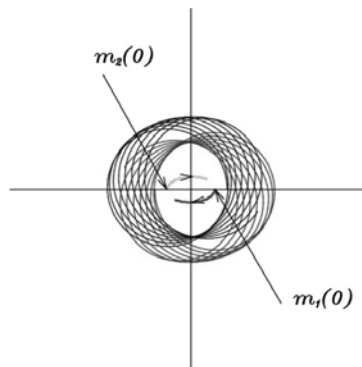


Figure 5: Bound motion of two electrons with parallel spins during a short period of time

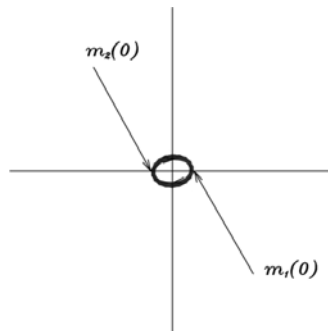


Figure 6. Evolution of the centres of mass of both particles for a longer time

the bound motion of this electron-electron interaction.

We thus see that if the separation between the centre of mass and centre of charge of a particle (*zitterbewegung*) is responsible of part of the spin structure, then this attractive effect can be easily interpreted.

A bound motion for classical spinless electrons is not possible. We can conclude that one of the salient features of the present formalism is the existence, from the classical viewpoint, of possible bound states for spinning electron-electron interaction. If the centres of mass of two electrons are separated by a distance greater than the Compton wavelength, they always repel each other as in the spinless case. But if the centres of mass of two electrons are separated by a distance less than the Compton wavelength, then from the classical viewpoint they can form bound states, provided certain initial conditions regarding their relative initial spin orientation, position of charges and centre of mass velocity are fulfilled. The difficulty may be to prepare a pair of electrons in the initial configuration depicted in Figure 4. A high-energy deep scattering can bring electrons to a very close approach. At low energy, if we consider the electrons in the conduction band of a solid, their interaction with the lattice could do this job. If we have a very thin layer under a huge external magnetic field perpendicular to the surface, as in the quantum Hall effect measurements, most of the electrons in this layer will have the spins parallel. If this happens to be true, we have a mechanism associated with the spin structure of the elementary particles for the plausible formation of a spin 1 Bose-Einstein condensate. This is just a classical prediction, not a quantum prediction, associated with a model which satisfies the Dirac equation when quantized. The possible quantum mechanical bound states must be obtained from the corresponding analysis of two interacting quantum Dirac particles, a problem which has not been solved yet. From the classical viewpoint, bound states for a hydrogen atom can exist for any negative energy and any arbitrary angular momentum. The quantum analysis of the atom gives the correct answer for the allowed stationary bound states.



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