Atala IV

Fourier-en transformatua

13 Zergaitia

Fourier-en analisitik Fourier-en Transformatura

Demagun f_l funtzio familia, periodikoak, $f_l(x+2l) = f_l(x)$, eta $f_l(x) \to f(x)$ limitean, $l \to \infty$:

$$f_l(x) = \frac{1}{l} \int_{-l}^{l} \mathrm{d}\xi \, f_l(\xi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi(\xi - x)}{l} \right) \right] .$$

Demagun

$$\int_{-l}^{l} \mathrm{d}\xi \, |f_l(\xi)| < +\infty .$$

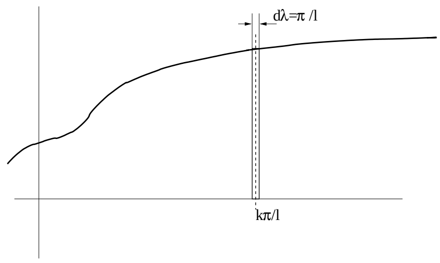
Fourier-en analisitik Fourier-en transformatura

Hots:

$$\lim_{l\to\infty} \frac{1}{l} \int_{-l}^{l} \mathrm{d}\xi \, f_l(\xi) = 0 .$$

Kontuan hartu

$$\int_{0}^{\infty} \mathrm{d}\lambda \, g(\lambda) \approx \sum_{k=0} \frac{\pi}{l} g\left(\frac{k\pi}{l} \right) .$$



Fourier-en analisitik Fourier-en transformatura

$$\begin{aligned} f(x) &= \lim_{l\to\infty} \frac{1}{l} \sum_{k=1}^{\infty} \int_{-l}^{l} \mathrm{d}\xi \, f_l(\xi) \cos \frac{k\pi(x-\xi)}{l} = \\ &= \frac{1}{\pi} \int_{0}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\xi \, f(\xi) \cos\left[\lambda(x-\xi)\right] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\xi \, f(\xi) \cos\left[\lambda(x-\xi)\right] \end{aligned}$$

Honezaz gain

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\xi\, f(\xi) \sin\left[\lambda(x-\xi)\right]$$

Fourier-en analisitik Fourier-en transformatura

Beraz

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\xi\, f(\xi) \cos\left[\lambda(x-\xi)\right] \\ &\quad + \frac{i}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\xi\, f(\xi) \sin\left[\lambda(x-\xi)\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\xi\, f(\xi) e^{i\lambda(\xi-x)} \end{aligned}$$

Fourier-en analisitik Fourier-en transformatura

Azkenez, defini dezagun

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\xi\, f(\xi) e^{ik\xi}\,.$$

Ondokoa frogatu dugu:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}k\, \hat{f}(k) e^{-ikx}\,.$$

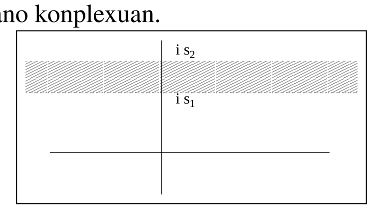
14 Fourier-en transformatua plano konplexuan

Definizioa

Demagun $f(x)$ aldagai *errealeko* funtzio konplexua,

$$\hat{f}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, f(x) e^{izx}\,.$$

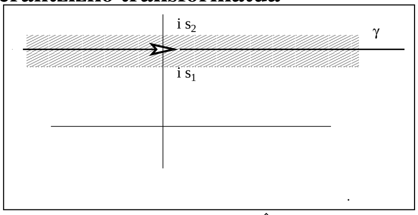
Definizio eremua: inplizitua, integralaren konbergentzia eremua. *Banda* horizontala plano konplexuan.



$\exists s_1, s_2$ / $\forall s \in (s_1, s_2)$ eta $x \in \Re\, \exists \hat{f}(x+is)$.

$$\left| \int_{-\infty}^{\infty} \mathrm{d}\xi f(\xi) e^{i(x+is)\xi} \right| \leq \int_{-\infty}^{\infty} \mathrm{d}\xi |f(\xi)| e^{-s\xi}$$

Alderantzizko transformatua

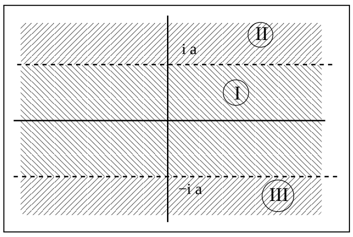


$$f(x) = \frac{1}{\sqrt{2\pi}} \int_\gamma \mathrm{d}z \, e^{-izx} \hat{f}(z) \, .$$

N.B.: plano konplexuko $\hat{f}$ funtzioa emanda, zein bandatan definitu dugun zehaztu behar dugu bere alderantzizko transformatua lortzeko.

Adibidea

$$\hat{f}(z) = \frac{1}{z^2 + a^2} \, .$$



$$\begin{aligned} f_{\mathrm{I}}(x) &= \sqrt{\frac{\pi}{2}} \frac{1}{a} e^{-a|x|} \\ f_{\mathrm{II}}(x) &= -\frac{\sqrt{2\pi}}{a} \theta(x) \sinh ax \\ f_{\mathrm{III}}(x) &= \frac{\sqrt{2\pi}}{a} \theta(-x) \sinh ax \end{aligned}$$

15 Erabilerak

15.1 EDAk

EDAk

$$\mathcal{F}[f'](\zeta) = -i\zeta \mathcal{F}[f](\zeta) \, .$$

Adibidea:

$$f' + \alpha f = g \, .$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_\gamma \mathrm{d}\zeta \, \frac{i\hat{g}(\zeta)}{\zeta + i\alpha} e^{-i\zeta x}$$

non γ (I) $-i\alpha$-ren gainean edo (II) $-i\alpha$-ren azpian dagoen.

$$f_{\mathrm{I}}(x) = \int_{-\infty}^{x} \mathrm{d}\xi \, e^{-\alpha(x-\xi)} g(\xi) \, .$$

$$f_{\mathrm{II}}(x) = -\int_{x}^{\infty} \mathrm{d}\xi \, e^{-\alpha(x-\xi)} g(\xi) \, .$$

15.2 DPEk

DPEk

$$u_t = \alpha^2 \nabla^2 u \, , u(\mathbf{x},0) = f(\mathbf{x}) \, , \int_{\mathbf{R}^3} \mathrm{d}^3\mathbf{x} \, |f(\mathbf{x})| \leq \infty \, .$$

Fourier-en transformatua bi alboetan, eta hastapen baldintza:

$$\int_{\mathbf{R}^3} \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^{3/2}} \, e^{i\mathbf{k}\cdot\mathbf{x}} u(\mathbf{x},t) = \hat{u}(\mathbf{k},t) \Rightarrow \hat{u}_t + \alpha^2 k^2 \hat{u} = 0 \, , \hat{u}(\mathbf{k},0) = \hat{f}(\mathbf{k})$$

$$\hat{u}(\mathbf{k},t) = \hat{f}(\mathbf{k}) e^{-\alpha^2 k^2 t} \Rightarrow u(\mathbf{x},t) = \int_{\mathbf{R}^3} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \, e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{u}(\mathbf{k},t) \, .$$

Hedatzailea

$$\begin{aligned}
u(x,t) &= \int_{\mathbf{R}^3} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \, e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{f}(\mathbf{k}) e^{-\alpha^2 k^2 t} \\
&= \int_{\mathbf{R}^6} \frac{\mathrm{d}^3\mathbf{k} \, \mathrm{d}^3\mathbf{r}}{(2\pi)} \, e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{r})} f(\mathbf{r}) e^{-\alpha^2 k^2 t} \\
&= \int_{\mathbf{R}^3} \mathrm{d}^3\mathbf{r} \left[\int_{\mathbf{R}^3} \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{r})} e^{-\alpha^2 k^2 t} \right] f(\mathbf{r}) \\
&= \int_{\mathbf{R}^3} \mathrm{d}^3\mathbf{r} \, G(\mathbf{x}-\mathbf{r},t) f(\mathbf{r})
\end{aligned}$$

non

$$G(\mathbf{x},t) = \frac{1}{(4\pi\alpha^2 t)^{3/2}} e^{-x^2/4\alpha^2 t} \, .$$

Alderantzizko espazioko hedatzailea eta hastapen datua

Honetaz konturatu (alderantzizko espazioa, Fourier-en espazioa, momentuaren espazioa, momentuaren irudia):

$$\hat{G}(\mathbf{k},t) = \frac{1}{(2\pi)^{3/2}} e^{-\alpha^2 k^2 t} \, .$$

Hastapen balioa:

$$\lim_{t\to 0^+} G(\mathbf{x},t) = \delta^{(3)}(\mathbf{x}) \, .$$

Uhin ekuazioaren Green-en funtzioa

$$u_{tt} = c^2 u_{xx} + f(x,t)\,.$$

Fourier-en transformatua idatzi,

$$\hat{u}(k,\omega) = \int_{\mathbf{R}^2} \frac{\mathrm{d}x\mathrm{d}t}{2\pi} e^{-i(kx-\omega t)} u(x,t)\,.$$

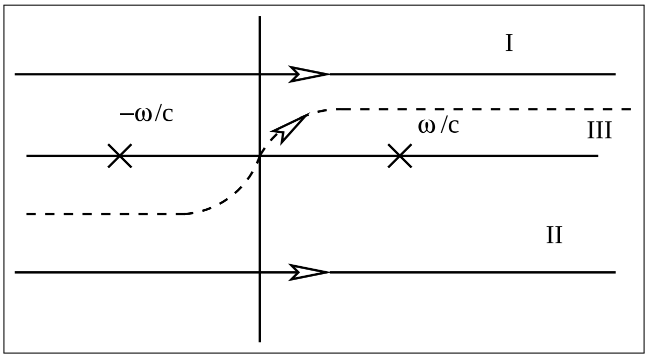
Ondorioz

$$\hat{u}(k,\omega) = \frac{\hat{f}(k,\omega)}{k^2c^2-\omega^2} = 2\pi \hat{G}(k,\omega)\hat{f}(k,\omega)\,.$$

$\hat{G}$-ren alderantzizko Fourier-en transformatua egiteko, *prozedura* zehaztu behar dugu, batzuen artean.

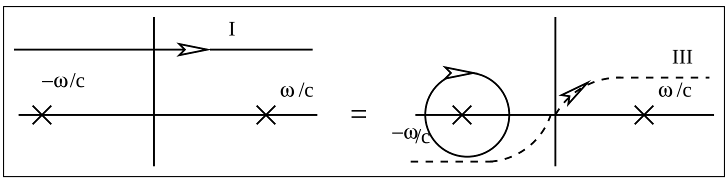
Prozedurak

ω integralaren ibilbidea erreal mantendu nahiez gero, k aldagaiari dagokionez aukera batzuk ditugu:



Prozedurak

Izatez, III. ibilbidea ez dago Fourier-en transformaturik lotuta. Hala ere, Green-en funtzioaren poloen hondarrak ekuazio *homogeneoaren* soluzioa da



$$
\begin{aligned}
G_{\rm I}(x,t) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi\omega c}\theta(-x)e^{-i\omega t}\sin\frac{\omega x}{c} \\
G_{\rm II}(x,t) &= \int_{-\infty}^{\infty} \frac{e^{-i\omega t}\mathrm{d}\omega}{4\pi\omega c}\left\{-i\cos\frac{\omega x}{c} + [\theta(-x)-\theta(x)]\sin\frac{\omega x}{c}\right\} \\
G_{\rm III}(x,t) &= -\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi\omega c}\theta(x)e^{-i\omega t}\sin\frac{\omega x}{c}
\end{aligned}
$$

Kasu guztietan $(\partial_t^2 - c^2\partial_x^2)G(x,t) = \delta(x)\delta(t)$ dugu. Adierazpen esplizituak ez dira oso adierazgarri...

$$
\begin{aligned}
G_{\rm I}(x,t) &= \frac{1}{2c}\theta(-x)\left[\theta\left(t+\frac{x}{c}\right) - \theta\left(t-\frac{x}{c}\right)\right] \\
G_{\rm II}(x,t) &= \frac{1}{4c}\left[\theta\left(\frac{x}{c}-t\right) - \theta\left(\frac{x}{c}+t\right)\right] + \\
&\quad \frac{1}{4c}[\theta(-x)-\theta(x)]\left[\theta\left(t+\frac{x}{c}\right) - \theta\left(t-\frac{x}{c}\right)\right] \\
G_{\rm III}(x,t) &= -\frac{1}{2c}\theta(x)\left[\theta\left(t+\frac{x}{c}\right) - \theta\left(t-\frac{x}{c}\right)\right]
\end{aligned}
$$

16 Laplace-ren Transformatua

Laplace-ren transformatua

$$
\mathcal{L}[f](s) = \int_0^{\infty} \mathrm{d}t\, e^{-st} f(t)\,.
$$

Izatez, Fourier-en transformatu bat baino ez da; demagun $\mathcal{F}[f](\zeta) = \int_{\mathbf{R}} \mathrm{d}x\, e^{izx} f(x)/\sqrt{2\pi}$; ondorioz

$$
\mathcal{L}[f](s) = \sqrt{2\pi}\mathcal{F}[P_+(f)](is),
$$

non $P_+(f)(x) = \theta(x)f(x)$ den.

Uhin ekuazioaren hedatzailea

$$
u_{tt} = c^2\nabla^2 u\,, \qquad u(\mathbf{r},0) = f(\mathbf{r})\,, \quad u_t(\mathbf{r},0) = 0\,.
$$

$$
\hat{U}(\mathbf{k},s) = \int_0^{\infty} \mathrm{d}t \int_{\mathbf{R}^d} \frac{\mathrm{d}\mathbf{r}}{(2\pi)^{d/2}} e^{-st} e^{-i\mathbf{k}\cdot\mathbf{r}} u(\mathbf{r},t)\,.
$$

Hau da,

$$
\hat{U}(\mathbf{k},s) = \frac{s}{s^2+c^2k^2}\hat{f}(\mathbf{k})\,.
$$

$$
\hat{u}(\mathbf{k},t) = \cos(ckt)\hat{f}(\mathbf{k})
$$

$$u(\mathbf{r},t) = \int_{\mathbf{R}^d} \mathrm{d}\mathbf{x}\, G_d(\mathbf{r}-\mathbf{x},t) f(\mathbf{x})\,.$$

non

$$G_d(\mathbf{r},t) = \int_{\mathbf{R}^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \theta(t)\cos(ckt) e^{i\mathbf{k}\cdot\mathbf{r}}\,.$$

$$G_1(r,t) = \frac{1}{2}\left[\delta(r+ct)+\delta(r-ct)\right]\;;$$

$$G_2(r,t) = \int_0^\infty \frac{\mathrm{d}k}{2\pi} k\cos(ckt) J_0(kr)\,;$$

$$G_3(r,t) = \frac{1}{2\pi^2 r}\int_0^\infty \mathrm{d}k\, k\cos(ckt)\sin(kr)\,;\ldots$$