1. Does what follows hold for all $n = 0, \pm 1, \pm 2, \ldots$?

$$\begin{aligned} \int_0^{2\pi} \cos(x)\cos(nx)\; dx &= \frac{1}{2}\int_0^{2\pi} \left[\, \cos(n+1)x + \cos(n-1)x \,\right] dx \\ &= \frac{1}{2}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_0^{2\pi} = 0\,. \end{aligned}$$

2. Let us assume that $y_1(x)$ and $y_2(x)$ are both solutions of the same boundary-conditions problem, namely that given by the second order ordinary linear differential equation $\mathrm{L}y = 0$ and the boundary conditions $y(0) = 1$, $y(1) = 0$

True or false: The sum $y(x) = 3y_1(x) + 2y_2(x)$ is a solution of the problem.

3. If only the derivatives with respect to a single variable appear in a partial differential equation (PDE), it can be directly solved as if it were an ordinary differential equation (ODE), with the integration constants substituted by functions of the other variables.

a) Find the general solution of the equation

$$\frac{\partial^2 u}{\partial y^2} + 4x^4 u = xe^y\,.$$

b) What if the function u were dependent on the three variables x, y and z?

c) Compute the general solution of the equation $u_{xy} = 0$.

4. True or false:

$$\int_{-1}^{1} \frac{dt}{t^2} = -\frac{1}{t}\bigg|_{-1}^{1} = -2$$

5. Find a function ϕ such that its gradient is $\nabla\phi = y^2 z\;\hat{\imath} + (2xyz + 3)\;\hat{\jmath} + (xy^2 - 2z)\;\hat{k}$ Is there any relation between the surfaces defined by the equation $\phi(x, y, z) = \text{cons.}$ and the vector field $\nabla\phi(x, y, z)$?

6. Use the expression $\int_0^\infty \mathrm{d}x\, e^{-ax^2} = \frac{1}{2}\sqrt{\pi/a}$ to compute

$$\begin{aligned} I_{2n}(a) &:= \int_0^\infty \mathrm{d}x\, x^{2n} e^{-ax^2} = \frac{(2n-1)!!}{2^{n+1}}\sqrt{\frac{\pi}{a^{2n+1}}}\,, \\ I_{2n+1}(a) &:= \int_0^\infty \mathrm{d}x\, x^{2n+1} e^{-ax^2} = \frac{n!}{2a^{n+1}}\,. \end{aligned}$$

7. Even function are those such that $f(-x) = f(x)$; odd functions, on the other hand, fulfill $f(-x) = -f(x)$.

a) Show that any function defined on an interval $(-l, l)$ can be expressed as the sum of two functions, one even and the other one odd. (Therefore, the function e^x can be expressed as the sum of an even and an odd function. Which is the name of these functions?

b) Show that $\int_{-l}^{l} f(x)dx = 0$ holds if f is odd.

c) Which of the functions in the following relation are even and which ones odd?

$$i)\ f(x) = (2x - x^3)^4\,; \quad ii)\ f(x) = \ln|\sin x|\,; \quad iii)\ f(x) = x^3 - 2x + 1\,.$$

d) Show that the derivative of an odd function is even, and reciprocally. What can be said about the function $F(x) = \int_0^x f(t)dt$?

8. Investigate the periodicity of the following set of functions. If any of them is periodic, which is its period? [Except in case d), let $n = 0, \pm1, \pm2, ...$; in case d) let $n = 1, 2, ...$].

$$a) \qquad f(x) = \begin{cases} 0 & 2n-1 \le x < 2n \\ 1 & 2n \le x < 2n+1 \end{cases}$$

$$b) \qquad f(x) = \begin{cases} (-1)^n & 2n-1 \le x < 2n \\ 1 & 2n \le x < 2n+1 \end{cases}$$

$$c) \qquad f(x) = \tan(\pi x)$$

$$d) \qquad f(x) = \begin{cases} 1 & \frac{1}{2n+1} \le x < \frac{1}{2n} \\ 0 & \frac{1}{2n} \le x < \frac{1}{2n-1} \end{cases}$$

9. Compute the following integrals, using derivation with respect to a parameter:

$$\int_0^\infty \mathrm{d}x\, \frac{e^{-ax} - e^{-bx}}{x}\,, \qquad \int_0^\infty \mathrm{d}x\, \frac{1 - e^{-ax^2}}{x^2}\,.$$

Investigate the behaviour of the integrand at the origin and close to the point at infinity (that is, $x \approx 0$ and $x \to +\infty$). Could you compute the following integrals?

$$\int_0^\infty \mathrm{d}x\, \frac{e^{-ax}}{x}\,; \qquad \int_0^\infty \mathrm{d}x\, \frac{e^{-ax^2}}{x^2}\,.$$

10. True or false: Adding a further term to a Taylor expansion always improves the approximation.

Consider the first terms of the Taylor expansion of the cosine function around the point $x = 0$ and use them to compute the function at the point $x = 2\pi$. What is going on?

11. Sum the series, and discuss its convergence region:

$$\sum_{n=0}^{\infty} (n+1)x^{n+4}\,.$$

12. Consider the following system of ODEs:

$$\dot{x} = yz\,, \qquad \dot{y} = xz\,, \qquad \dot{z} = -2xy\,.$$

Which are the first integrals of this system? Can they be given a geometrical interpretation? Which is the geometrical interpretation of the system's solutions? Write down the general solution of the system.

13. Consider the ODE $y'' + \lambda^2 y = 0$. Among all solutions, select and write those that fulfill $y(0) = 0$. Is it possible for $y(\pi)$ to be zero as well for some solution of the equation?