1. Consider the following ordinary and homogeneous second order differential equation,

$$P(x)y'' + Q(x)y' + R(x)y = 0\,. \qquad (i)$$

Let us find an integrating factor $\mu(x)$, such that multiplying the equation by $\mu(x)$ we obtain

$$\left[\mu(x)P(x)y'\right]' + \mu(x)R(x)y = 0\,. \qquad (ii)$$

a) Show that function μ has to be a solution of the equation $P\mu' = (Q - P')\mu$ and therefore that

$$\mu(x) = C\frac{1}{P(x)}\exp\int \mathrm{d}x\,\frac{Q(x)}{P(x)}$$

holds, with some parameter C.

b) Transform the following equations to form (ii):

$$\begin{aligned} y'' - 2xy' + \lambda y &= 0 && \text{Hermite's eqn.}\\ x^2y'' + xy' + (x^2 - \nu^2)y &= 0 && \text{Bessel's eqn.}\\ xy'' + (1-x)y' + \lambda y &= 0 && \text{Laguerre's eqn.}\\ (1-x^2)y'' - xy' + \alpha^2 y &= 0 && \text{Tchebyschev's eqn.} \end{aligned}$$

What is the point of this transformation?

2. Compute the scalar product in the space $L_2(0,\infty)_w$ of the functions $f(x) = 2$ and $g(x) = 3x$, with weight $w(x) = e^{-x^2}$. Compute the sum $\alpha f + \beta g$, such that it is orthogonal to the function f, and such that it is of unit norm. Follow the same process in the space $L_2(-\infty,\infty)_w$.

$$\text{Remember :} \qquad \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}\,.$$

3. By the orthogonalization method of Gram-Schmidt one transforms a set of linearly independent vectors $\{v_1, v_2 ... v_n\}$ into an orthogonal set $\{e_1, e_2 ... e_n\}$, by induction:

$$\begin{aligned} e_1 &= v_1\\ e_2 &= v_2 - \frac{< e_1, v_2 >}{< e_1, e_1 >}\, e_1\\ \vdots\ & \quad \vdots\\ e_n &= v_n - \sum_{i=1}^{n-1} \frac{< e_i, v_n >}{< e_i, e_i >}\, e_i \end{aligned}$$

a) Check (**graphically**) the method for the case $n = 3$.

b) The functions $f_k(x) = x^k$, $k = 0, 1, 2...$ form a non-orthogonal basis of the space $L_2[-1,1]$. Compute the first four orthogonal vectors given the Gram-Schmidt procedure. c) We can obtain the Legendre polinomioals, $\{P_0(x), P_1(x), \ ...\ \}$, from that orthogonal basis, by imposing the normalization condition $P_i(1) = 1$. Compute $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$. (In which physical problem do these functions appear?)

4. Compute **graphically** the eigenfunctions of the Sturm-Liouville problems in which the equation is $y''+\lambda y = 0$ and the boundary conditions $i)$ $y(0) = y'(1) = 0$, $ii)$ $y'(3) = y'(7) = 0$, $iii)$ $y(-\frac{\pi}{2}) = y(\frac{\pi}{2}) = 0$ and $iv)$ $y'(-5.2) = y(-3.4) = 0$.

5.* Show that the eigenfunctions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) - 2y'(1) = 0$$

are $\{\sin\sqrt{\lambda_n}x\}$. Except for the eigenvalue $\lambda = 0$, all the eigenvalues are solutions of the equation $\tan\sqrt{\lambda} = 2\sqrt{\lambda}$. By analysing that trascendental equation, check that in the limit $n \to \infty$ the eigenvalues tend to $\lambda_n \approx (2n-1)^2\pi^2/4$. Expand the function $f(x) = x$ in the eigenbasis (notice that this expansion is not harmonic, that is to say, the frequencies that do turn up are not simply multiples of a basic frequency; therefore, the usual Fourier expansion and this one are rather different).

6. Consider the equation $y'' + 2y' + y + \lambda y = 0$ in the interval $(0, \pi)$, under the condition that $y(0) = y(\pi)$ and $y'(0) = e^{2\pi}y'(\pi)$. Is there degeneracy?

7. Show that the eigenvalues associated to the problem given by the equation

$$y'' + a\delta(x)y + \lambda y = 0\,,$$

with boundary conditions $y(\pm\pi) = 0$, and a being real, are also real, and defined by the equation

$$\tan(\pi\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{a}\,.$$

[Actually, on top of this family there is another one, given by $\lambda_n^{(e)} = n^2$] What would be needed for negative eigenvalues to appear? By comparing to Schrödinger's equation, give a physical description.

8. Consider the following differential operator,

$$Ly = \frac{1}{4}(1+x^2)^2y'' + \frac{1}{2}x(1+x^2)y' + ay\,,$$

in the interval $(-1, 1)$. a is a constant, and the domain of L is given by functions that are zero both at -1 and 1. What would be an adequate weight? By using the change of variables $x = \tan(\theta/2)$, compute the eigenvalues and eigenfunctions of the operator.

9.* (**Properties of Bessel functions**) The Bessel functions of the first kind $J_n(x)$ are solutions of the following ordinary differential equation:

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0\,.$$

For n an entire number, we have

$$J_n(0) = \delta_{n0}\,,$$

where δ_{nm} is Kronecker's delta, and, furthermore,

$$J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)\,.$$

Show that

$$g(x,t) = e^{x(t-1/t)/2}$$

is the generating function of Bessel functions, that is, that

$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

holds true. (*Hint:* Show the following:
1) both sides are solutions of the same partial derivative equation;
2) $g(0,t)$ and $\sum_{n=-\infty}^{\infty} J_n(0)t^n$ coincide;
3) $\partial_x g(0,t)$ and $\sum_{n=-\infty}^{\infty} J_n'(0)t^n$ are identical)

By derivation of the generating function, obtain the following set of equalities:

$$\begin{aligned} &\text{a)} \quad J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)\,; \\ &\text{b)} \quad J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)\,. \end{aligned}$$

Using these, obtain the next set:

$$\begin{aligned} &\text{c)} \quad \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)\,; \\ &\text{d)} \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)\,. \end{aligned}$$

Show that equality d) is a consequence of c) and $J_{-n}(x) = (-1)^n J_n(x)$.

10. Show directly (not using the Sturm-Liouville theorem) that $J_\nu(\lambda_n^{(\nu)}x)$ and $J_\nu(\lambda_m^{(\nu)}x)$ are orthogonal with respect to the weight x in the interval $(0,1)$, whenever $m \neq n$ and $J_\nu(\lambda_n^{(\nu)}) = J_\nu(\lambda_m^{(\nu)}) = 0$ holds. (Hint: use Bessel's equation itself and integration by parts to compute the Wronskian of the two functions)

11.* (**Properties of Legendre polynomials**) Using the generating function of Legendre,

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n\,, \quad |t| < 1\,,$$

derive the following expression for the distance between two points:

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\,, \quad \begin{cases} r_< = \min(r_1,r_2) & \text{and} \\ r_> = \max(r_1,r_2) & , \end{cases}$$

with $\cos\theta = \vec{r}_1 \cdot \vec{r}_2/(r_1 r_2)$.

By derivation of the generating function, obtain the following set of equalities:
a) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$;
b) $P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) = P_{n-1}(x)$;

and from those,
c) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$;
d) $xP_n'(x) - P_{n-1}'(x) = nP_n(x)$;

e) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$;
f) $(x^2-1)P'_n(x) = nxP_n(x) - nP_{n-1}(x)$.
Show the Legendre polynomials $P_n(x)$ are solutions of the equations

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0\,.$$

Check (it is advisable to use the generating function yet again) that $P_n(1) = 1$ holds true. Let us define the differential expression $Ly = -(1-x^2)y'' + 2xy'$. How would you classify it? How would you describe Legendre polynomials in regard to it? Will there be an inner product with respect to which they are orthogonal?

12.* **Tough one!** Consider the problem given by the equation $u'' + \omega^2\,(1 + a\delta(x))\,u = 0$ and the boundary conditions $u(l) = u(-l)$ and $u'(l) = u'(-l)$. For which values of ω does a non-trivial solution exist? (Warning: check also for the existence of complex values).

13. Consider the space of functions on the real line normalizable with unit weight function (notice that their continuous representative tends to zero at least as quickly as x^{-1} as $|x| \to \infty$). Determine whether any of the following operators is hermitian when defined over that space:

$$i)\qquad \frac{\mathrm{d}}{\mathrm{d}x} + x\,;\quad ii)\qquad -i\frac{\mathrm{d}}{\mathrm{d}x} + x^2\,;\quad iii)\qquad ix\frac{\mathrm{d}}{\mathrm{d}x}\,;\quad iv)\qquad i\frac{\mathrm{d}^3}{\mathrm{d}x^3}\,.$$