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Ordinary differential equations
Topic 2

First order equations

2.1 Geometric meaning, 2.2 Existence-uniqueness theorem, 2.3 Exact equations, 2.4 Integrating factor, 2.5 Separable equations, 2.6 Special integrating factor, 2.7 Linear equations, 2.8 Transformation methods, 2.9 Homogeneous equations, 2.10 Equations type $y' = f(ax + by + c)$, 2.11 Equations of type $y' = f(\frac{ax+by+c}{\alpha x+\beta y+\gamma})$, 2.12 Bernouilli's equations, 2.13 Riccati's equations, 2.14 Envelopes and singular solutions, 2.15 Equations not soluble for the derivative

2.1 Geometric meaning

- We already know that the finite equation $\varphi(x, y) = 0$ defines a curve in the (x, y) plane
  - e.g. the equation $x^2 + y^2 = 1$ defines a unit circle centered at the origin
- But in order to describe a family of curves we need something like $\varphi(x, y, C) = 0$
  - e.g. the equation $x^2 + y^2 = C^2$ defines a family of circles of radious $C(\geq 0)$ centered at the origin
- But why wonder about curves in a class of differential equations?

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- Combining the equation for the family

$$\varphi(x, y, C) = 0$$

and its derivative

$$\frac{\partial \varphi}{\partial x}(x, y, C) + \frac{\partial \varphi}{\partial y}(x, y, C)y' = 0,$$

we can eliminate the parameter C
  - This shows the close relation between differential equations and families of curves
- The result of the previous combination gives us the differential equation for the family:

$$F(x, y, y') = 0.$$

  - This equation gives the relation between the slope at a point, and the curve that goes through that point

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Exercise 2.1

- Take the derivative of $x^2 + y^2 = C^2$ to show that the differential equation of circles centered at the origin is the following:

$$x + yy' = 0.$$

- From $(x^2 + y^2)' = (C^2)'$ it is easy to see that
  $x + yy' = 0$
  This is the result we are after
- This example is very easy, since the derivative is enough to get the equation. Generally, we would need both equations.

- By construction, the functions $\varphi(x, y, C) = 0$ are a uniparametric family of solutions for $F(x, y, y') = 0$
  - Besides, since the equation is first order it is the general solution
- Careful! By eliminating C, other general solutions and singular solutions can appear
- On the other hand, each particular case of $\varphi(x, y, C) = 0$ is the equation for an integral curve
  - integral curves are particular solutions obtained by integration of the differential equation of the family
  - e.g. the circles defined by $x^2 + y^2 = C^2$ are integral curves.

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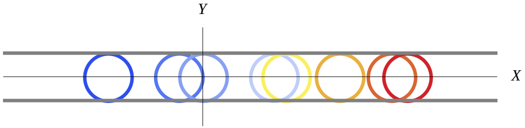
Exercise 2.2

- Find the differential equation for the unit circles whose center is in the x axis. Is there any singular solution?
- The finite equation for this family of curves is

$$(x - C)^2 + y^2 = 1,$$

  and by derivation plus some other simple operations we get $yy' = -(x - C)$
  This can be rewritten as $y^2(y')^2 = (x - C)^2$
  Using the finite equation we can eliminate C to get $y^2((y')^2 + 1) = 1$.
- By inspection, one can guess the singular solutions given by $y = \pm 1$



Particular (circles) and singular (straight lines) solutions.

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- If the curves of a family do not cross each other, we get a very intersting case: a congruence of curves
  - Are the families of circles from the previous exercises congruences?
    Those from exercise 2.1 yes, but not those of exercise 2.2
- In the case of congruences, there is a single curve that goes through a given point (and there is only corresponding value of the parameter)
  - In other words, there is only one single curve, and one single value of C that corresponds to a given point (x, y)
- This is why it is possible to use $\varphi(x, y, C) = 0$ to solve for C for a given point (x, y)
  - By doing that, we will be able to write the congruence as

$$u(x, y) = C$$

  - Remember the case $x^2 + y^2 = C^2$

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- In the case of congruencies, the differential equation is obtained by mere differentiation

$$(u(x,y) = C)'$$

$$u'(x,y) = \frac{\partial u}{\partial x}(x,y) + \frac{\partial u}{\partial y}(x,y)y' = C' = 0$$

  - Taking derivatives is enough to eliminate the parameter C
- We can obtain the symmetric form of the equation by multiplying by dx:

$$u'dx = \frac{du}{dx}dx = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0.$$

- From now on, we will use the following short-hands

$$P \equiv \frac{\partial u}{\partial x},\ Q \equiv \frac{\partial u}{\partial y},$$

$$du = Pdx + Qdy.$$

- e.g. for the equation $x + yy' = 0$ we get:

$$P \equiv \partial u/\partial x = x,\ Q \equiv \partial u/\partial y = y,\ xdx + ydy = 0.$$

- We can obtain the normal form by solving for the highest derivative:

$$y' = f(x, y)$$

- Sometimes the normal form is more convenient than the symmetric
- In any case, they are related by

$$f(x, y) \equiv -\frac{\partial u/\partial x}{\partial u/\partial y} = -\frac{P(x, y)}{Q(x, y)}$$

  - The normal form for the previous example is:

$$y' = -\frac{x}{y}.$$

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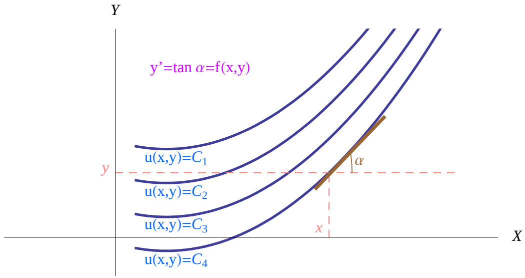

Figure 2.1 Congruence of curves, derivative and slope.

- The normal form gives the interpretation for the differential equation:
  - the differential equation gives the slope for the integral curve going through (x, y)
  - the value of the slope is $y' = \tan\alpha = f(x, y)$
  - there is only one curve per point, and the tangent defines a single direction

- Therefore, the equation $y' = f(x, y)$ assigns one direction to each point
  - Taking into account all points, it defines a direction field

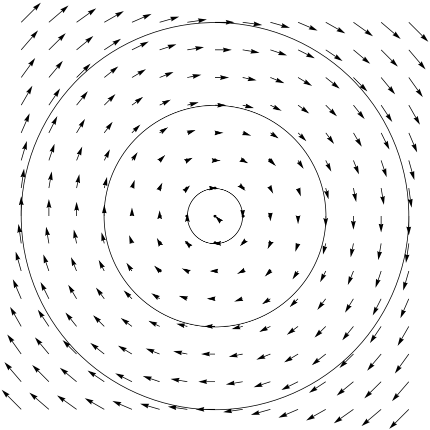

Figure 2.2 Congruence and direction field.

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- When is the relation between the equation and the congruence one-to-one?
- When the hypothesis for the theorem of existence and uniqueness are satisfied

Theorem (existence and uniqueness)

If the function f and its derivative $\partial f/\partial y$ are continuous in a domain, then the initial-value problem posed by

$$y' = f(x, y),\ y(x_0) = y_0$$

admits only one solution for each initial value (x_0, y_0)

- Due to this theorem, given some continuity properties, the curve obtained by integrating a differential equation is a congruence
- But, if there is a singular point, the theorem cannot be applied
- In those points, there can be more than one curve per point

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Exercise 2.4

- Show that the differential equation for the circles centered in the y axis that touch the x axis is given by

$$y' = \frac{2xy}{x^2 - y^2}.$$

- The equation for circles with center in the y is $x^2 + (y - y_0)^2 = R^2$
We also need the point $(x, y) = (0, 0)$ to be included in the circle
so $0^2 + (0 - y_0)^2 = R^2$, and thus $y_0 = \pm R$
This gives us the finite equation for the family

$$x^2 + (y \mp R)^2 = R^2.$$

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- It can be rewritten as $x^2 + y^2 \mp 2Ry = 0$, and therefore $\pm R = (x^2 + y^2)/(2y)$
By taking derivatives and simplifying

$$x + (y \mp R)y' = 0$$

Combining with the expression for $\pm R$ and multiplying by y
$xy + (y^2 - (y^2 + x^2)/2)y' = 0$
and finally we get

$$y' = \frac{2xy}{x^2 - y^2},$$

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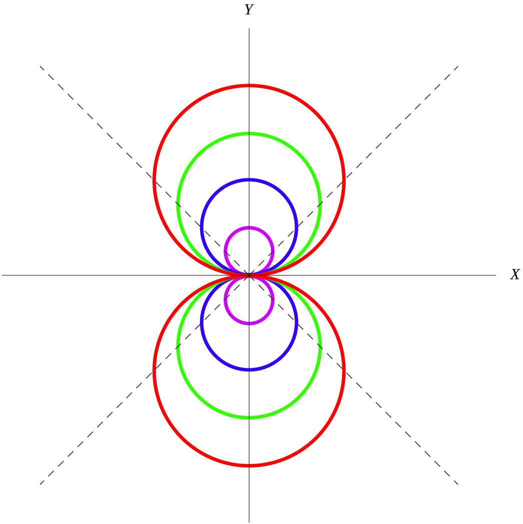

Figure 2.3 Circles that touch the x axis.

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- The diagonal lines $y = \pm x$ and the point $(x, y) = (0, 0)$ are singular
- At the origin there is no uniqueness, but it doesn't disagree the theorem (singular)
- At the diagonals there is no uniqueness problem

2.3 Exact equations

- The symmetric form for the equation $u(x, y) = C$ was given by

$$du = P(x, y)dx + Q(x, y)dy = (\partial u/\partial x)dx + (\partial u/\partial y)dy.$$

- Such differential equations are called exact

2.2 teorema

Following Schwarz's theorem, all exact solution satisfy the following property

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

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Example: integration of an exact equation

- Let us start with the following equation

$$xdx + ydy = 0$$

- Since $\partial P/\partial y = \partial Q/\partial x$ it is exact
- $\partial u/\partial x = x$, and integrating we obtain

$$u = x^2/2 + h(y)$$

  ($h(y)$ is not determined as of yet)
- Now, bear in mind that $\partial u/\partial y = y$, and on the other hand $\partial u/\partial y = h'(y)$.
- Comparing and integrating we get $h(y) = y^2/2 + D$, and therefore
- We can choose $D = 0$ by redefinitions

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Exercise 2.5

- Solve he following equation

$$(x + y + 1)dx + (x - y^2 + 3)dy = 0.$$

- In this case $\partial P/\partial y = 1$ and $\partial Q/\partial x = 1$, so it is exact
  Now, as $\partial u/\partial x = (x + y + 1)$, we obtain
  $u = x^2/2 + x(y + 1) + h(y)$
  This gives $\partial u/\partial y = x + h'(y)$
  Using $Q = \partial u/\partial y = x - y^2 + 3$ and comparing
  we get $h'(y) = 3 - y^2$
  This gives $h(y) = 3y - y^3/3$ and we thus obtain the solution

$$u = \frac{x^2}{2} + x(y + 1) + y(3 - \frac{y^2}{3}) = C$$

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1st special case: Equations without dependent variable

- These look as follows

$$y' = f(x).$$

- Both cross terms are zero, so it is exact
- Clearly

$$y = \int f(x)dx + C.$$

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- As an example, let us integrate $y^2((y')^2 + 1) = 1$
- It can be written as $(y')^2 = 1/y^2 - 1$,
  and also as $(y^2(y')^2)/1 - y^2 = 1$.
- Squaring and remembering $y' = dy/dx$, we can write
  $\frac{ydy}{\sqrt{1-y^2}} = \pm dx$ and by direct integration one obtains
  $\sqrt{1-y^2} = \pm(x - C)$

2nd special case: Equations with separated variables

- In this case, the independent and the dependent variables appear separated, in different terms:

$$P(x)dx + Q(y)dy = 0.$$

- It is exact since both cross derivatives are zero
- Clearly

$$y = \int P(x)dx + \int Q(y)dy + C.$$

Exercise 2.7

- Solve $(1+y)e^y y' = 2x$.
- In symmetric form, $(1+y)e^y dy - 2xdx = 0$, so it is an equation of the 2nd special case. Therefore $u = \int (1+y)e^y dy - \int 2xdx = ye^y - x^2 = C$

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General procedure If the equation

$$du = P(x,y)dx + Q(x,y)dy = 0$$

is exact, this is the way of obtaining the general solution:

- 1. Calculate $\int P(x,y)dx$ since $u(x,y) = \int P(x,y)dx + h(y)$
- 2. Write

$$\frac{d[\int P(x,y)dx]}{dy} + h'(y) = Q(x,y)$$

  and solve for $h'(y)$
- 3. Calculate $h(y)$ by direct integration

2.4 Integrating factor

- Let us start with an example

$$\frac{x}{y}dx + dy = 0.$$

  - This equation is not exact, but if we multiply it by y it becomes exact
- Sometimes it is possible to covert some equations into exact equations
- ZIf a non-exact equation $P(x,y)dx + Q(x,y)dy = 0$ becomes exact by multiplying with $\mu(x,y)$, then the function $\mu(x,y)$ is known as the integrating factor for that equation
- In general, the solutions for $\mu(x,y)(P(x,y)dx + Q(x,y)dy) = 0$ will also be solutions for $P(x,y)dx + Q(x,y)dy = 0$

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1st execption

- Let us consider the following equation $xydx + y^2dy = 0$

  This equation has one singular solution if the form $y = 0$

  On the other hand, it accepts an integrating factor $\mu = 1/y$

  But the euqation $\mu(P(x,y)dx + Q(x,y)dy) = xdx + ydy = 0$ does not have $y = 0$ as a solution: we have lost one solution!

  In this example, it cna be seen that the solution we have lost ($y = 0$) is the root of $1/\mu$

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- In general, one should check what happens with the roots of the inverse of the integrating factor, i.e., the solutions of $1/\mu(x, y) = 0$

  If there are solutions, and they are not part of the general solution for $\mu(P(x, y)dx + Q(x, y)dy) = 0$, then, they will be singular solutions for the original equation $P(x, y)dx + Q(x, y)dy = 0$

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2nd exception

- The integrating factor can bring "fake solutions"

  $\mu(x, y) = 0$ can describe solutions for the new equation
  $\mu(P(x, y)dx + Q(x, y)dy) = 0$
  But it can happen that those solutions are not solutions of the original equation
  It has to be checked

- In summary
  - If $\mu(x, y) = 0$, fake solutions can happen
  - If $1/\mu(x, y) = 0$ true solutions can disappear

Exercise 2.8

- Show that $\mu = 1/(xy^2)$ is an integrating factor for the following equation: $(xy + y^2)dx - x^2dy = 0$. Get the general solution. Is there any singular solution? Is there any root of μ that is not a solution for the differential equation?
- Let us start with:

$$\mu P = \frac{1}{xy^2}(xy + y^2) = \frac{1}{y} + \frac{1}{x}, \qquad \mu Q = -\frac{x^2}{xy^2} = -\frac{x}{y^2}$$

It can be seen that $\partial(\mu P)/\partial y = -1/y^2 = \partial(\mu Q)/\partial x$, so the new equation is exact
Using that $\partial u/\partial x = \mu P = 1/y + 1/x$, we get
$u = x/y + \log|x| + h(y)$
On the one hand, $\partial u/\partial y = \mu Q = -x/y^2$, but on the other $\partial u/\partial y = -x/y^2 + h'(y)$.
Therfore $h(y) = C$ and $u = x/y + \log|x| = C$.

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- Let us now check $\mu = 0$ and $1/\mu$.
- The first one happens at $y = \infty$ so it is nothing to worry about
- the second one happens when $y = 0$
- From the general solution $x/y + \log|x| = C = 1/D$ one gets

$$y = \frac{Dx}{1 - D\log|x|}$$

- Setting $D = 0$ we recover $y = 0$ so we have not lost any solution

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- If an equation accepts μ as an integrating factor, $C\mu$ will also be an integrating factor, with C any constant
- All first order equations accept an integrating factor
  - The problem is...how can we calculate it in general?
- In some cases there is a method to obtain the integrating facto
  - If the new equation is exact, it should satisfy

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$$

  - We can rewrite it to obtain

$$Q\frac{\partial \log \mu}{\partial x} - P\frac{\partial \log \mu}{\partial y} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$$

which can be sometime useful

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2.5 Separable equations

- If in the symmetric form, the dependent and the independent variables can be written in different factors, the the equation is separable :

$$R(x)S(y)dx + U(x)V(y) = 0.$$

- This case accepts the following integrating factor $1/(S(y)U(x))$

$$\frac{1}{S(y)U(x)}(R(x)S(y)dx + U(x)V(y)) =$$

$$\frac{R(x)}{U(x)}dx + \frac{V(y)}{S(y)}dy = 0.$$

- Since the variables get separated, the solution is

$$u = \int \frac{R(x)}{U(x)}dx + \int \frac{V(y)}{S(y)}dy = C$$

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- In this case $1/\mu = S(y)U(x)$ so we have to check whether we are losing the $S(y) = 0$ solutions
- But we do not have to investigate the $U(x) = 0$ case:

  This expression does not give the values of the dependent variable, so it is not a solution for the differential equation

- Let us check that the equation $x(1+y)y' = y$ takes $\mu = 1/(xy)$ as an integrating factor and let us obtain the solution
- In the symmetric form we have $-ydx + x(1+y)dy = 0$, so

$$R(x) = -1, \quad S(y) = y, \quad U(x) = x, \quad V(y) = (1+y)$$
$$\mu = 1/(SU) = 1/(xy).$$

- Our new equation is

$$-\frac{1}{x}dx + \frac{1}{y}(1+y)dy = 0.$$

- By direct integration, the general solution is $\ln|y| + y = \ln|x| + \ln C$ or

$$|y|e^y = C|x|$$

- One should check whether the solution $y = 0$ is lost. That is not the case, since it corresponds to the case $C = 0$

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Exercise 2.9

- Solve the following equation
$$(x-4)y^4dx - x^3(y^2-3)dy = 0.$$
- For this case
$R(x) = x-4,\ S(y) = y^4,\ U(x) = -x^3,\ V(y) = y^3-3,$
$\mu = 1/(SU) = -1/(x^3y^4).$
- The new equation reads
$-((x-4)/x^3)dx + ((y^2-3)/y^4)dy$
- Therefore,
$$u = -\int (x-4)/x^3dx + \int (y^2-3)/y^4dy =$$
$$\frac{1}{x} + \frac{2}{x^2} - \frac{1}{y} + \frac{1}{y^3} = C = 1/D$$
- We shoudl check for $1/\mu = U(x)S(y) = 0$. One should worry if $y = 0$ is lost, but it corresponds to $D = 0$ in the general solution

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2.6 Special integrating factors

1st case: integrating factors of type $\mu(x)$

- In general, the integrating factor satisfies

$$Q\frac{\partial \log \mu}{\partial x} - P\frac{\partial \log \mu}{\partial y} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$$

- But for the case $\mu(x)$:

$$\frac{\partial \log \mu}{\partial x} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

- If that is satisfied, we get

$$\frac{d}{dy}\left[\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] = 0$$

and this is what one has to check

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- Then, integrating

$$\frac{\partial \log \mu}{\partial x} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

one gets

$$\mu(x) = C\exp \int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx$$

- We can choose the value C to our convinience.

- For example, let us consider $(2x^2 + y)dx + (x^2y - x)dy = 0$
- In this case,

$$\frac{d}{dy}\left[\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] =$$

$$\frac{d}{dy}\left[\frac{1}{x^2y - x}(1 - (2xy - 1))\right] = \frac{d}{dy}\left[\frac{2 - 2xy}{x^2y - x}\right] =$$

$$\frac{d}{dy}\left[-\frac{2}{x}\right] = 0$$

- therefore, it is possible to get an integrating factor of the form $\mu(x)$

$$\mu(x) = C\exp\int \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx =$$

$$C\exp\int -\frac{2}{x}dx = C\exp(-2\ln x) = \frac{1}{x^2}.$$

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- The new equation is

$$\left(2 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0$$

- Following the usual steps, we get

  on the one hand $u = 2x - y/x + h(y)$
  on the other

$$-\frac{1}{x} + h'(y) = y - \frac{1}{x},$$

  therfore $h(y) = y^2/2$.

- The final solution is

$$u = 2x - \frac{y}{x} + \frac{y^2}{2} = C.$$

Exercise 2.10

- Solve $(3xy + y^2) + (x^2 + xy)y' = 0$
- In this case

$$\frac{d}{dy}\left[\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right] =$$

$$\frac{d}{dy}\left[\frac{1}{x^2 + xy}(x + y)\right] = \frac{d}{dy}\left[\frac{1}{x}\right] = 0.$$

Therefore

$$\mu(x) = C\exp\int \frac{dx}{x} = Cx,$$

and the new equation reads
$xy(3x + y)dx + x^2(x + y)dy = 0$

- Following the usual procedure

$$u = x^3y + \frac{x^2y^2}{2} + h(y)$$

The other conditions give

$$x^3 + x^2y + h'(y) = x^2(x + y)$$

and therfore $h(y) = D$.
The solution is

$$u = x^3y + \frac{x^2y^2}{2} = C.$$

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2nd case: integrating factor of type $\mu(y)$

- The condition for the existence of this type of integrating factor is

$$\frac{d}{dx}\left[\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] = 0.$$

- and the integrating factor is obtained by

$$\mu(y) = C\exp\int \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy$$

Exercise 2.11

- Discuss whether the equation $(3xy + y^2) + (x^2 + xy)y' = 0$ accepts an integrating factor of type $\mu(y)$
- For this equation, we get

$$\frac{\partial P}{\partial y} = 3x + 2y, \qquad \frac{\partial Q}{\partial x} = 2x + y,$$

so it is not exact

Moreover,

$$\frac{d}{dx}\left[\frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] = \frac{d}{dx}\left[\frac{1}{3xy + y^2}(y - x)\right] \neq 0,$$

so it does nto accept an integrating factor of the form $\mu(y)$

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1.3 Integrating factor of the type $\mu(x, y) = g(h(x, y))$

- In order to be able to get an integrating factor that depends on the variables only via an intermidiate function, it should obey

$$\mu(h) = C\exp \int \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial h}{\partial x} - P\frac{\partial h}{\partial y}} dh,$$

only function of h

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Exercise 2.12

- Solve $(3xy + y^2)dx + (3xy + x^2)dy = 0$ using an integrating factor of the form $\mu(x + y)$
- For this equation

$$\frac{\partial P}{\partial y} = 3x + 2y, \quad \frac{\partial Q}{\partial x} = 3y + 2x, \quad \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = x - y.$$

We are told to use $h = x + y$, and then

$$Q\frac{\partial h}{\partial x} = 3xy + x^2 \quad P\frac{\partial h}{\partial y} = 3xy + y^2,$$

$$Q\frac{\partial h}{\partial x} - P\frac{\partial h}{\partial y} = x^2 - y^2 = (x + y)(x - y).$$

Therefore,

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q\frac{\partial h}{\partial x} - P\frac{\partial h}{\partial y}} = \frac{(x - y)}{(x - y)(x + y)} = \frac{1}{x + y}.$$

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- Then, bearing in mind $h = x + y$

$$\mu(h) = C\exp \int dh/h = C(x + y)$$

The new equation is $(x + y)\left[(3xy + y^2)dx + (3xy + x^2)dy\right] = 0$ and is of course exact

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 8xy + 3x^2 + 3y^2.$$

Now we can proceed as usual

$$u(x, y) = \int (x + y)(3xy + y^2)dx + h(y) =$$

$$x^3y + 2x^2y^2 + xy^3 + h(y)$$

which gives $\partial u/\partial y = x^3 + 4x^2y + 3xy^2 + h'(y)$ and also $\partial u/\partial y = Q = 4x^2y + x^3 + 3xy^2$, therefore $h(y) = D$. The solution reads $u = x^3y + 2x^2y^2 + xy^3 = C$

2.7 Linear equations

- First order linear equations have the following form

$$y' + A(x)y = B(x)$$

- There are two cases
  - If $B = 0$, the equation is homogeneous
  - If $B \neq 0$, the equation is inhomogeneous
- Linear equations accept the following integrating factor

$$\mu(x) = \exp \int A(x)dx.$$

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- The equation now reads

$$e^{\int A(x)}y' + A(x)e^{\int A(x)dx}y = B(x)e^{\int A(x)dx}.$$

- which can be rewritten as

$$\frac{d}{dx}\left[e^{\int A(x)dx}y\right] = B(x)e^{\int A(x)dx},$$

  and then, the general solution is

$$y = e^{-\int A(x)dx}\left[C + \int B(x)e^{\int A(x)dx}dx\right]$$

- The general solution for the liear equation is the sum of two terms
  the general solution for the homogeneous $(B = 0)$ +
  particular solution for the inhomogeneous $(C = 0)$

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Exercise 2.13

- Solve $xy' + (1+x)y = e^x$.
- In this case

$$A = (1+x)/x = (1/x) + 1, \qquad B = e^x/x.$$

Then $\mu = e^{\int A(x)dx} = e^{\int((1/x)+1)dx} = e^{\ln x + x} = xe^x$ and

$$y = e^{-\int A(x)dx}\left[C + \int B(x)e^{\int A(x)}dx\right] =$$

$$\frac{1}{x}e^{-x}\left[C + \int \frac{e^x}{x}xe^x dx\right] = \frac{1}{x}e^{-x}\left[C + \int e^{2x}dx\right] =$$

$$\frac{1}{x}e^{-x}\left[C + \frac{e^{2x}}{2}\right] = \frac{1}{x}\left[Ce^{-x} + \frac{e^x}{2}\right]$$

2.8 Transformation methods

- The difficulty in solving physical problems depends strongly on the coordinate choice
- Choosing an appropriate coordinate system helps
- In some cases it will be helpful to use transformation methods to transform either the dependent variable, the independent variable or both

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2.9 Homogenenous equations

- A function obeying

$$f(ax, ay) = a^r f(x, y) \; \forall a$$

is an homogeneous fucntion of order r

- If P(x,y) and Q(x,y)are homogeneous functions of the same order

$$P(ax, ay) = a^r P(x, y), \quad Q(ax, ay) = a^r Q(x, y) \quad \forall a.$$

then $P(x, y)dx + Q(x, y)dy = 0$ is a homogeneous equation of order r

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- One can prove that

$$P(x,y)dx + Q(x,y)dy = 0 \text{ is homogeneous} \Leftrightarrow$$
$$-\frac{P(x,y)}{Q(x,y)} = f\left(\frac{y}{x}\right) \quad (1)$$

- This is why changing the variables to $u = y/x$ happens to be halpful for this type of equations

$$u = \frac{y}{x} \Rightarrow y = xu, \quad y' = u + xu'.$$

- Actually, this change of variables makes the equation separable

$$y' = f(u) = u + xu', \quad u' + \frac{1}{x}(u - f(u)) = 0,$$

- and the equation becomes anquadrature

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x} + C.$$

Exercise 2.16

- Solve $(\sqrt{x^2+y^2}+y)dx - xdy = 0.$
- In this case

$$f(x,y) = \frac{\sqrt{x^2+y^2}+y}{x}$$

and it can be seen that it is homogeneous

$$f(ay,ax) = \frac{\sqrt{a^2x^2+a^2y^2}+ay}{ax} = f(y,x),$$

therefore, $f(x,y) = f(y/x)$.
Let us make the change of variables $u = y/x$

$$f(u) = \frac{\sqrt{x^2+x^2u^2}+xu}{x} = u + \sqrt{1+u^2}.$$

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- The solution for our equation now reads

$$\int \frac{du}{f(u)-u} = \int \frac{du}{\sqrt{1+u^2}} = \frac{dx}{x} + \ln C.$$

Integrating

$$\text{arcsinh} u = \ln(u + \sqrt{1+u^2}) = \ln x + \ln C,$$

and finally

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = Cx.$$

2.10 Equations of type $y' = f(ax + by + c)$

- Using the change of variables $u = ax + by + c$ the equation becomes separable
- Let us prove that. First,

$$u = ax + by + c, \quad u' = a + by',$$

and for the type of equation we are dealing with

$$y' = f(ax + by + c) = f(u)$$

so

$$u' = a + bf(u).$$

The solution can be writen as a quadrature

$$\int \frac{du}{a + bf(u)} = \int dx + C.$$

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Exercise 2.17

- Solve $y' = (x + y + 1)^2$
- In this case $f(x, y) = f(ax + by + c) = (x + y + 1)^2$, so

$$u' = 1 + u^2$$

Therefore

$$\int \frac{du}{1 + u^2} = \int dx + C,$$

$$\arctan(x + y + 1) = x + C,$$

and finally

$$y = \tan(x + C) - (x + 1).$$

2.11 Equations of type $y' = f(\frac{ax+by+c}{\alpha x+\beta y+\gamma})$

- There are two cases depending on the geometrical relation between the two straight lines $ax + by + c = 0$ and $\alpha x + \beta y + \gamma = 0$

1st case $\alpha/a = \beta/b = k$ (parallel lines)

- In this case

$$y' = f\left(\frac{ax + by + c}{k(ax + by) + c}\right),$$

so this is the same case as the one seen in the previous section: $f(x, y) = f(ax + by + c)$.

- The change of variables $u = ax + by$ or $u = ax + by + c$ will help to solve the equation

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Exercise 2.18

- Solve $y' = (x-y)/(x-y-1)$
- In this eqution $a = 1$, $b = -1$ and $u = x - y$. Then $u' = 1 - u/(u-1) = -1/(u-1)$, and

$$\int (u-1)du = -\int dx + C,$$

$$\frac{u^2}{2} - u = -x + C,$$

$$\frac{(x-y)^2}{2} - (x-y) - C = -x,$$

$$(x-y)^2 + 2y = 2C.$$

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2.11 Equations of type $y' = f(\frac{ax+by+c}{\alpha x+\beta y+\gamma})$

2nd case $\alpha/a = \beta/b \neq k$ (not parallel lines)

- Let us suppose that the lines meet at (x_0, y_0)
  $ax_0 + by_0 + c = \alpha x_0 + \beta y_0 + \gamma = 0$.
- It is convenient to define $u = x - x_0$ and $v = y - y_0$:

$$ax + by + c = ax + by + c - (ax_0 + by_0 + c) = au + bv,$$
$$\alpha x + \beta y + \gamma = \alpha x + \beta y + \gamma - (\alpha x_0 + \beta y_0 + \gamma) = \alpha u + \beta v.$$

- For these variables $dy/dx = dv/dx = dv/du$and the equation becomes homogeneous

$$\frac{dv}{du} = f(\frac{ax + by + c}{\alpha x + \beta y + \gamma}) = f(\frac{au + bv}{\alpha u + \beta v}) = f(\frac{a + bv/u}{\alpha + \beta v/u}).$$

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- The original equation becomes

$$\frac{dv}{du} = f(\frac{a + bv/u}{\alpha + \beta v/u}).$$

which is homogeneous

- Now, using earlier results, $z = v/u$ makes the equation separable
- In order to simplify notaion, we will denote the derivatives with respect to u with a $'$

$$\frac{dv}{du} = v' \text{ and } \frac{dz}{du} = z'$$

- We will then get

$$v' = z'u + z$$

and finally

$$z'u + z = f(\frac{a + bz}{\alpha + \beta z}).$$

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Eercise 2.19

- Solve
$$y' = \frac{x-y+1}{x+y-3}.$$

- The crossing point is obtained from $x_0 - y_0 + 1 = x_0 + y_0 - 3 = 0$ to give $(x_0, y_0) = (1, 2)$. We can use the new variables $u = x - 1$, $v = y - 2$ to transform the equation into
$$y' = \frac{u+1-(v+2)+1}{u+1+v+2-3} = \frac{u-v}{u+v}$$
Therefore
$$\frac{dv}{du} = \frac{1-v/u}{1+v/u}$$

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- Following the result in this section

$$v' = z + z'u = (1-z)/(1+z).$$

We then get

$$\frac{dz}{du} = \frac{1}{u}\left(\frac{1-z}{1+z} - z\right) = \frac{1}{u}\left(\frac{1-2z-z^2}{1+z}\right),$$

and

$$\int \frac{1+z}{1-2z-z^2}\,dz = \int \frac{du}{u} + \ln C,$$

$$-\frac{1}{2}\ln|1-2z-z^2| = \ln C|u|, \quad z^2+2z-1 = \frac{1}{C^2u^2}.$$

Reverting back to the original variables

$$\left(\frac{v}{u}\right)^2 + 2\left(\frac{v}{u}\right) - 1 = \frac{1}{C^2u^2},$$

$$\left(\frac{y-2}{x-1}\right)^2 + 2\left(\frac{y-2}{x-1}\right) - 1 = \frac{1}{C^2(x-1)^2},$$

$$y^2 + 2xy - 6y - x^2 - 2x = D^2.$$

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2.12 Bernouilli's equations

- These equations are of the form

$$y' + A(x)y = B(x)y^n \quad n \neq 0, 1.$$

  - If $n = 0$, it is linear inhomogeneous equation
  - If $n = 1$, it is a linear homogeneous equation
- The equation can be made linear by $u = y^{1-n}$

  First we have

$$u' = (1-n)y^{-n}y'.$$

  Substituting in the original equation

$$\frac{u'y^n}{(1-n)} + A(x)y = B(x)y^n, \quad \frac{u'}{(1-n)} + A(x)y^{1-n} = B(x),$$

  we obtain a linear equation

$$u' + (1-n)A(x)u = (1-n)B(x).$$

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Exercise 2.20

- Solve
$$y' - y\cos x = \frac{1}{2}\sin 2x.$$

- This is Bernouilli's equation with $n = 2$.
By doing $u = y^{-1}$ we get a linear equation:
$$u' + (1-2)(-\cos x)u = (1-2)\left(\frac{1}{2}\sin 2x\right).$$
The general solution is
$$u = e^{-\int \cos x dx}\left(C - \int \frac{1}{2}\sin 2x\, e^{\int \cos x dx}\right) =$$
$$u = e^{-\sin x}\left(C - \int \sin x\, \cos x\, e^{\sin x}\right) =$$
$$u = e^{-\sin x}\left(C - (\sin x - 1)e^{\sin x}\right) = Ce^{-\sin x} + 1 - \sin x.$$
Finally,
$$y = \left(Ce^{-\sin x} + 1 - \sin x\right)^{-1}.$$

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2.13 Riccati's equations

- These equations are of the form

$$y' + A(x)y + B(x)y^2 = C(x) \; B, C \neq 0.$$

If $B = 0$, it is a linear inhomogeneous equation
if $C = 0$, it is Bernouilli's equation

- There is no general method for solving it
- But if one particular solution $y_1(x)$ is known, the change of variables $u = y - y_1$ turns the equation into Bernouilli;s equation:

$$u' + (A(x) + 2B(x)y_1(x))u + B(x)u^2 = 0.$$

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- Let us show the relation between Bernouilli's and Ricatti's equations
- Let us start with $y = u + y_1$. Then

$$y' = u' + y_1'.$$

- Substituting in the equation

$$(u' + y_1') + A(u + y_1) + B(u + y_1)^2 = C.$$

- Expanding, we get

$$u' + Au + 2Buy_1 + Bu^2 + y_1' + Ay_1 + By_1^2 = C.$$

- But y_1 is a particular solution, so $y_1' + Ay_1 + By_1^2 = C$, and therefore

$$u' + Au + 2Buy_1 + Bu^2 = 0$$

Exercise 2.21

- Show that the function $y = 1/x$ is a solution for

$$y' = y^2 - \frac{2}{x^2}$$

and use it to find the general solution.

- We have $y_1 = 1/x$, so $y_1' = -1/x^2$ and subsituting

$$y_1' = -1/x^2 = y_1^2 - 2/x^2 = 1/x^2 - 2/x^2$$

- As we have a particular solution, we can make the following change $u = y - 1/x$. As $A = 0$, $B = -1$ and $C = -2/x^2$ the new equation will be

$$u' - \frac{2}{x}u - u^2 = 0.$$

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- This is Bernoulli's equation da with $n = 2$, $A = -2/x$, and $B = 1$. It is convenient to define $v = u^{-1}$
  Now the equation takes a linear form:

$$v' - (-2/x)\, v = -1.$$

and the general solution is

$$v = e^{-\int \frac{2}{x} dx} \left[C - \int e^{\int \frac{2}{x} dx} dx \right] =$$

$$e^{-2 \log x} \left[C - \int x^2 dx \right] = \frac{1}{x^2} \left[C - \frac{x^3}{3} \right] = \frac{C}{x^2} - \frac{x}{3}.$$

- Let us undo the change of variables

$$1/u = (3C - x^3)/(3x^2),$$

and finally use the original variables to get

$$y = u + \frac{1}{x} = \frac{3x^2}{3C - x^3} + \frac{1}{x} = \frac{2x^3 + 3C}{x(3C - x^3)}.$$

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2.14 Envelopes and singular solutions

- There is and interesting geometrical concept that can be ilustrated with the follwoing figure

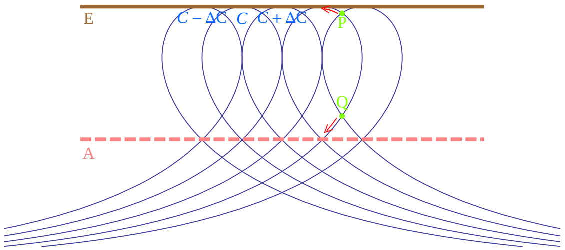

Figure 2.4.Envelope of a family of curves and multiple points.

  - The equations of the curves in the figure $\varphi(x, y, C) = 0$
    The curve E is not part of the family. However, the curve E touches one of the curves in the family at every single point. This is called an envelope
    The curve E obeys the same differential equation as the family of curves: $F(x, y, y') = 0$.

- In orfder to calculate the equation for the envelope, we can start by studying the point P

  This point obeys $\varphi(x, y, C) = 0$ and $\varphi(x, y, C + \Delta C) = 0$, and also the following combination:

$$\varphi(x, y, C) = 0 \quad \text{and} \quad \frac{\varphi(x, y, C + \Delta C) - \varphi(x, y, C)}{\Delta C} = 0.$$

  - In the limit $\Delta C \to 0$ the point P tends to be a point in the envelope, and the equations become

$$\varphi(x, y, C) = 0 \quad \text{and} \quad \frac{\partial \varphi(x, y, C)}{\partial C} = 0.$$

  - These two equations give the equation of the envelope

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Exercise 2.22

- Find the envelope of $(x-a)^2+y^2=1$
- The equations we have to solve are the following:

$$\varphi(x,y,a)=(x-a)^2+y^2-1=0$$

and

$$\frac{\partial\varphi(x,y,a)}{\partial a}=\frac{\partial((x-a)^2+y^2-1)}{\partial a}=0.$$

The second equation becomes

$$2(a-x)=0,$$

that is, $x=a$.
Using this result in the first equation

$$(x-a)^2+y^2=(a-a)^2+y^2=0^2+y^2=1,$$

we get the equation for the envelope

$$y=\pm 1.$$

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2.15 Equations not soluble for the derivative

- Sometimes, the easiest way of solving a differential equation is by differentiating the equation
- The new equation will be of a higher order, so it will have more solutions than the original one, but it will include those

Example: Kepler's problem

- Let us consider a particle moving in a newtonian potential of the form t $V = -k/r$. The equation that describes the dependence of the magnitude $u \equiv 1/r$ with respect to the angular position ϕ is $(' \equiv d/d\phi)$:

$$(u')^2 + u^2 - \frac{2\epsilon}{p}u = \frac{e^2 - 1}{p^2}.$$

- Bear in mind that $\epsilon = k/|k|$ and thus $\epsilon^2 = 1$.

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- By differentiating the equation, we get (forced harmonic oscillator)

$$2u'\left(u'' + u - 2\epsilon/p\right) = 0.$$

Its solution is $u = C\cos(\phi - \phi_0) + \epsilon/p$
As we have one parameter too many, we should substitute this in the original equation:

$$0 = (u')^2 + u^2 - \frac{2\epsilon}{p}u - \frac{e^2 - 1}{p^2} =$$

$$C^2(\sin(\phi - \phi_0))^2 + (C\cos(\phi - \phi_0) + \frac{\epsilon}{p})^2 -$$

$$\frac{2\epsilon}{p}(C\cos(\phi - \phi_0) + \frac{\epsilon}{p}) - \frac{e^2 - 1}{p^2} =$$

$$C^2 - \frac{\epsilon^2}{p^2} - \frac{e^2 - 1}{p^2}$$

.

- All in all, $C = e/p$

Exercise 2.52

- Solve $(y')^2 + 2y = 1$
- Differentiating, we get $2y'(y''+1) = 0$, and we have two equations: $y = D$ and $y'' = -1$.
  The second one solves to

$$y' = -x + C \ \text{ eta } y = -\frac{x^2}{2} + Cx + D.$$

The first solution is included in this one.
We still have to check with the original equation:

$$0 = (-x + C)^2 + 2(-\frac{x^2}{2} + Cx + D) - 1 =$$

$$x^2 - 2Cx + C^2 - x^2 + 2C^2x + 2D - 1.$$

This gives, $D = (1 - C^2)/2$ and this is the solution we are after