Ordinary differential equations
4th topic

Systems of equations

4.1 Definition and general properties, 4.2 Solutions methods, 4.3 First order linear systems, 4.4 Homogeneous linear systems, 4.5 Complete linear systems

4.1 Definition and general properties

- In three dimensions, the intersection of the surfaces $\varphi_1(x,y,z)=0$ and $\varphi_2(x,y,z)=0$ defines a curve.
  - Let us consider the two-parameter families of curves $\varphi_1(x,y,z,C_1,C_2)=0 \quad \text{and} \quad \varphi_2(x,y,z,C_1,C_2)=0$ defined in a domain
  - this will be a congruency if and only if there is only one single curve of the family going through every point (x,y,z)
  - It is always possible then to write the equations of a congruency as
$$\psi_1(x,y,z)=C_1,$$
$$\psi_2(x,y,z)=C_2.$$
  - Deriving with respect to the independent variable x we get the differential equations for the congruency:
$$\frac{\partial\psi_i}{\partial x}+\frac{\partial\psi_i}{\partial y}y'+\frac{\partial\psi_i}{\partial z}z'=0,\ i=1,2.$$

- There are two main ways of expressing the equations of congruences
  - Solving for the the derivative, we get the normal form:

$$y' = f_1(x, y, z)$$

$$z' = f_2(x, y, z)$$

  - Isolating the differentials we get the canonical form:

$$\frac{dx}{g_1(x, y, z)} = \frac{dy}{g_2(x, y, z)} = \frac{dz}{g_3(x, y, z)}.$$

Exercise 4.2

- find the differential equation of the circles

$$x^2 + y^2 + z^2 = A^2$$

$$x + y + z = B$$

both in normal and canonical ways.

- Taking derivatives and simplifying:

$$x + yy' + zz' = 0,$$

$$1 + y' + z' = 0.$$

Solving for z' and substituting in the first equation:

$$x + yy' + z(-1 - y') = x + (y - z)y' - z = 0.$$

Taking the same steps with y':

$$x + y(-1 - z') + zz' = x - y - z'(y - z) = 0.$$

- From the last two equations we can get the normal form:
$$(y-z)y' = z-x \Rightarrow \frac{dy}{dx} = \frac{z-x}{y-z},$$
$$(y-z)z' = x-y \Rightarrow \frac{dz}{dx} = \frac{x-y}{y-z}.$$
And now the canonical form:
$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}.$$

- For systems, we will use a more convenient notation:
  - the coordinates in a space of dimension $n+1$ will be
$$(t, x_1, x_2, \dots, x_n)$$
  - The equations for the congruences:
$$\psi_i(t, x_1, \dots, x_n) = C_i, \quad i = 1, \dots, n.$$
  - Normal form:
$$\dot{x}_i = f_i(t, x_1, \dots, x_n), \ i = 1, \dots, n.$$
  - Canonical form:
$$\frac{dt}{g_0} = \frac{dx_1}{g_1} = \frac{dx_2}{g_2} = \dots \frac{dx_n}{g_n}$$

Uniqueness and existence theorem

- In this context, the theorem of existence and uniqueness is also valid.
- For a system written in the normal form

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n), \; i = 1, \ldots, n,$$

, if the functions f_i and $\partial f_i / \partial x_j$ are continuous, there is only one solution for the system with n initial conditions given by

$$x_i(t_0) = x_{i0}, \; i = 1, \ldots, n,$$

4.2 Solution methods

- There is no general way of solving systems.
- We will study two methods:
  - Reduction to one equation
  - First integrals

Reduction to one equation

- We saw in the 3rd topic that any equation of order n can be reduced to a system of first order equations
- This works the other way too: a system of n first order equations can be re-expressed as a differential equation of order n.

Exercise 4.3

- Solve the following system $\dot{x} = 3x - 2y$, $\dot{y} = 2x - y$
- Taking derivatives and substituting,

$$\ddot{x} = 3\dot{x} - 2\dot{y} = 3\dot{x} - 2\dot{(2x - y)} =$$
$$3\dot{x} + 2y - 4x = 3\dot{x} + (3x - \dot{x}) - 4x = 2\dot{x} - x$$

Now we can solve this, by (for example) the method of characteristic polynomials

$$x = C_1 e^t + C_2 t e^t.$$

And y can be obtained easily from the first equation in the system

Exercise 4.4

- Solve $\dot{x} = y$, $\dot{y} = xy$
-
  - Taking derivatives and substituting:

$$\ddot{x} = \dot{y} = xy = x\dot{x}.$$

Integrating once we get:

$$\dot{x} = \frac{x^2}{2} + C_1.$$

Now, separate variables and integrate:

$$\frac{dx}{x^2 + C_1} = 2dt,$$

$$C_1 > 0, \quad 2t + C_2 = \frac{\arctan}{\sqrt{C_1}} \left(\frac{x}{\sqrt{C_1}} \right),$$

$$C_1 < 0, \quad 2t + C_2 = \frac{\operatorname{arctanh}}{\sqrt{C_1}} \left(\frac{x}{\sqrt{C_1}} \right),$$

$$C_1 = 0, \quad 2t + C_2 = -\frac{1}{x}.$$

First integrals

- Imagine that a function $\Phi(t, x_1, \ldots, x_n)$ is constant throughout the evolution of a system: $\dot{\Phi} = 0$.
  - In that case, the function $\Phi(t, x_1, \ldots, x_n)$ is a first integral for the system.
  - The equation $\Phi(t, x_1, \ldots, x_n) = C$ is a equation for different surfaces in $(t, x_1, \ldots, x_n)$ for every C.
- It is interesting to note that in practice, one does not need to find solution to get first integrals. And even more, knowing a first integral makes the solution finding easier.
  - To prove that a function is a first integral, one needs to prove that its derivative with respect to t is zero:

$$\frac{d\Phi}{dt} \equiv \frac{\partial \Phi}{\partial t} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} f_i = 0.$$

- For example, for the system $\dot{x} = y$, $\dot{y} = \dot{x}$, the function $\Phi = e^{-t}(x + y)$ is a first integral. Its first derivative is zero:

$$\dot{\Phi} = -e^{-t}(x + y) + e^{-t}(\dot{x} + \dot{y}) =$$

$$-e^{-t}(x + y) + e^{-t}(y + x) = 0.$$

- Now we can use this constant to solve for one of the unknowns:

$$x_i = \Psi(t, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

- For each first integral, we can solve for one unknown.
- In the example before, we can use $e^{-t}(x+y) = A$ to get $y = Ae^t - x$,
  and then, the only equation left to solve would be
  $\dot{x} = Ae^t - x$
- If it is possible to find n (functionally) independent first integrals, then we would be able to write a general solution, because in principle all the x_i can be expressed as functions of C and t
- In order for the n first integrals $\Phi(t, x_1, \ldots, x_n)$ to be independent, one needs

$$\frac{\partial(\phi_1, \ldots, \phi_n)}{\partial(x_1, \ldots, x_n)} \neq 0$$

Exercise 4.6

- Show that the first integrals $e^{-t}(x+y)$ and $e^{t}(x-y)$ are independent. Show also that $x^2 - y^2$ is not independent with respect to them.
- To prove that they are independent, let us calculate:

$$\begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{-t} \\ e^{-t} & -e^{-t} \end{vmatrix} = -1 - 1 = -2 \neq 0$$

- It is easily seen that $\phi_3 = \phi_1 \phi_2$, so they are dependent.

- How can first integrals be found?
  - In general, one needs to look for symmetries.
  - In physics, symmetries are usually linked to conservation-laws.
  - In any case, we will need to get some practice and learn to see things "by eye".

- For example, let us consider the following system:

$$\begin{aligned} \dot{x} &= y - z, \\ \dot{y} &= z - x, \\ \dot{z} &= x - y. \end{aligned}$$

- Adding the equations, we get $\dot{x} + \dot{y} + \dot{z} = 0$. Therefore, we get the first integral $x + y + z = A$
- On the other hand, if we multiply the equations by x, y and z respectively, we get $x\dot{x} + y\dot{y} + z\dot{z} = 0$, so another first integral would be $x^2 + y^2 + z^2 = A$

- Usually, the canonical form makes it easier to look for symmetries in the equation
- Consider the following system:

$$\dot{x} = \frac{2tx}{t^2 - x^2 - y^2}, \quad \dot{y} = \frac{2ty}{t^2 - x^2 - y^2}.$$

- In canonical form:

$$\frac{dt}{t^2 - x^2 - y^2} = \frac{dx}{2tx} = \frac{dy}{2ty}.$$

- By simplification, one gets $dx/(2x) = dy/(2y)$ and integrating $y = Ax$.

- Let us find another first integral by using the following property:

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a+c}{c+d}.$$

- Multiplying every fraction by t, x and y respectively,and adding them up:

$$\frac{tdt + xdx + yxy}{t(t^2 + x^2 + y^2)} = \frac{dx}{2tx}.$$

- Simplifying we get

$$\frac{tdt + xdx + yxy}{t^2 + x^2 + y^2} = \frac{dx}{2x},$$

It is clear that we have to exact differentials, so it is easy to integrate to give:

$$t^2 + x^2 + y^2 = Bx.$$

Exercise 4.9

- Solve:

$$\dot{x} = \frac{y}{x+y}, \quad \dot{y} = \frac{x}{x+y}.$$

- In canonical form, the system reads:

$$\frac{dt}{x+y} = \frac{dx}{y} = \frac{dy}{x}.$$

From the second equality one gets $xdx = ydy$, which can be easily integrated to give:

$$x^2 - y^2 = A.$$

- We also have these other two relations:

$$dx = \frac{ydt}{x+y},$$

$$dy = \frac{xdt}{x+y}.$$

Adding them up we get $dx + dy = ((x+y)/(x+y))dt = dt$, and by direct integration:

$$x + y - t = B$$

- The general solution to our system is thus this system of finite equations:

$$x^2 - y^2 = A,$$

$$x + y - t = B.$$

Exercise

- Solve the following system:

$$\dot{x} = \frac{ty}{y^2 - x^2}, \quad \dot{y} = -\frac{tx}{y^2 - x^2}.$$

- In canonical form this reads:

$$\frac{tdt}{y^2 - x^2} = \frac{dx}{y} = -\frac{dy}{x}.$$

From the second equality we get $xdx = -ydy$, and integrating:

$$x^2 + y^2 = A.$$

- The other two relations are:

$$dx = \frac{tydt}{y^2 - x^2},$$

$$dy = -\frac{txdt}{y^2 - x^2}.$$

Adding them up we get $dx + dy = ((y - x)/(y^2 - x^2))tdt = tdt/(y + x)$ and integrating

$$(x + y)^2 - t^2 = B$$

The two finite equations we obtained are the general solution.

4.3 Systems of first order linear equations

- We will now focus on systems of this form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t)$$

  ie, linear systems.
- Or course, we will demand the functions a_{ij} and b_i to be continuous in the domain I in order to have existence and uniqueness.

- We will use the following notation:

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

- This way, we can write the problem as $\dot{\vec{x}} = \mathbf{A}\vec{x} + \vec{b}$ or, using $L\vec{x} = \dot{\vec{x}} - \mathbf{A}\vec{x}$, we can write $L\vec{x} = \vec{b}$.
- It is easy to prove linearity:

$$L(a\vec{x} + a\vec{y}) = aL\vec{x} + bL\vec{y}.$$

Exercise 4.10

- Write the following system in matrix form:

$$\dot{x} = y,\ \dot{y} = -x.$$

- We clearly have $\vec{b} = \vec{0}$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

4.4 Linear homogeneous system

- Let us start with $L\vec{x} = 0$. Due to linearity, the superposition principle holds

$$L\vec{x}_i = 0 \Rightarrow L\sum c_i\vec{x}_i = \sum c_i L\vec{x}_i.$$

- Therefore, the group of solution of a linear homogeneous system is a vector space.
- In this space, the linear independence of the vectors $\vec{x}_i$ is defined as usual:

  The vectors $x_1, \ldots, x_n$ are linearly dependent if the system

$$\sum_{j=1}^{n} c_j\vec{x}_j = 0 \Leftrightarrow \sum_{j=1}^{n} x_{ij}c_j = 0 \quad \forall t \in I$$

  has non-zero solutions.

- If the system is dependent, its determinant (the Wronskian)

$$W[x_1,\ldots,x_n] \equiv |\vec{x}_1\vec{x}_2\ldots\vec{x}_n| = \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix},$$

  will be zero in all points in the domain I.
- In general, the inverse will not be true for some arbitrary set of functions.
- However, if the vectors $\vec{x}_1,\ldots,\vec{x}_n$

  are solutions of some known homogeneous linear system $L\vec{x}_i=0$,
  and their Wrosnkian is zero at some point $W(t_0)=0$,
  then it can be proved that it will be zero in all the interval I
  and the vectors will be linearly dependent.

- Using the theorem of uniqueness and existence, it can be seen that the dimension of the space of solutions cannot be less than n.
- Due to the theorem, there are n linear independent solutions corresponding to the following initial conditions

$$\vec{x}_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \vec{x}_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \dots \quad \vec{x}_n(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- The same can be said for any initial condition satisfying

$$W[\vec{x}_1(t_0), \vec{x}_2(t_0), \dots, \vec{x}_n(t_0)] \neq 0$$

.

- The groups of n linearly independent solutions are known as fundamental systems of solutions
- Besides, each fundamental system $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ is a base of the space of solutions
- Any solution of $L\vec{x} = 0$ can be written as a linear combination of the fundamental system of solutions with coefficients C_j
- In order to calculate the value of the coefficients C_j, one has to calculate the unique solution at t_0,

$$\vec{x}(t_0) = \sum_{j=1}^{n} C_j \vec{x}_j(t_0) \Leftrightarrow \vec{x}_i(t_0) = \sum_{j=1}^{n} \vec{x}_{ij}(t_0) C_j$$

(This can be done with the determinant is not zero)

- Since the solution is unique, the solution corresponding to the initial conditions at the point t_0 can be written as

$$\vec{x}(t) = \sum_{j=1}^{n} C_j \vec{x}_j(t) \quad \forall t \in I$$

  with the coefficients that we have chosen at t_0.
- Therefore, the general solution of a linear homogeneous system is given by a linear combination of the vectors of the fundamental system with some arbitrary coefficients

$$\vec{x} = \sum_{j=1}^{n} C_j \vec{x}_j.$$

Exercise 4.11

- Prove that the following vectors

$$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

  form a fundamental system for the equations $\dot{x} = y, \ \dot{y} = -x,$. Write the general solution.

- Let us name the vectors as:

$$\vec{x}_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

  Let us write the system in matrix-form:

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Let us check that the proposed solutions are really solutions:

$$\dot{\vec{x}}_1 = \frac{d}{dt}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}_1 =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ -\cos t \end{pmatrix}.$$

- And the other one:

$$\dot{\vec{x}}_2 = \frac{d}{dt}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}_2 =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

- To check if they form a fundamental system we need to check the linear dependency, so we need to check the Wronskian:

$$W[\vec{x}_1, \vec{x}_2] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

  Since it is non-zero, the solutions form a fundamental system

- So the general solution is:

$$\vec{x}(t) = A \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Fundamental matrices

- Taken the n vectors of a fundamental system a columns, we can obtain a fundamental matrix:

$$\mathbf{F}(t) = (\vec{x}_1 \vec{x}_2 \dots \vec{x}_n = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}$$

- Fundamental matrices are not singular (by construction):

$$\det \mathbf{F}(t) = W|\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n| \neq 0,$$

- Besides, the fundamental matrix is a solution of a linear system:

$$L\mathbf{F} = \mathbf{0} \Leftrightarrow \dot{\mathbf{F}} = \mathbf{A} \cdot \mathbf{F}.$$

Exercise 4.12

- Find the fundamental matrix for $\dot{x} = y, \ \dot{y} = -x,.$
- Bearing in mind the result of exercise 4.11, the fundamental matrix is clearly

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Exam question (September-08)

- Let us study the matrix $\mathbf{F} = \begin{pmatrix} t & 1 \\ -1 & t \end{pmatrix}$. What system is this matrix a fundamental matrix of?
- The matrix $\mathbf{A}$ that corresponds to the linear system, will satisfy $\dot{\mathbf{F}} = \mathbf{A}\mathbf{F}$ therefore $\mathbf{A} = \dot{\mathbf{F}}\mathbf{F}^{-1}$.
  Then, since $\mathbf{F}^{-1} = (\mathrm{adj}(\mathbf{F}))^T/(\det \mathbf{F})$ and $(\det \mathbf{F}) = t^2 + 1 \neq 0$, we get

$$\mathbf{A} = \frac{1}{t^2+1}\begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix}.$$

- In general, for 2×2 matrices, we have:

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- Let us write a generic solution using fundamental matrices.
- The general solution is a linear combination of the fundamental system:

$$\vec{x} = \sum_{j=1}^{n} C_j \vec{x}_j \Rightarrow \vec{x}_i = \sum_{j=1}^{n} x_{ij} C_j = \sum_{j=1}^{n} F_{ij} C_j.$$

- Thus, we have

$$\vec{x}(t) = \mathbf{F}(t) \cdot \vec{c},$$

where

$$\vec{c} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$$

Exercise 4.13

- Find the general solution of the system $\dot{x} = y,\ \dot{y} = -x$, using a fundamental matrix
- Bearing in mind the solution of exercise 4.12, we get

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

4.5 Complete linear systems

- As with the complete linear equations, the complete solution is obtained by adding up a particular solution with the general solution of the homogeneous equation

$$L\vec{x}_1 = 0,\ L\vec{x}_2 = b,\ \Rightarrow\ L(\vec{x}_1 + \vec{x}_2) = L\vec{x}_1 + L\vec{x}_2 = \vec{b}.$$

- And the difference of two complete solutions is the solution of the homogeneous:

$$L\vec{x}_1 = L\vec{x}_2 = b,\ \Rightarrow\ L(\vec{x}_1 - \vec{x}_2) = L\vec{x}_1 - L\vec{x}_2 = \vec{0}.$$

- For systems, the complete solution for the system $L\vec{x} = \vec{b}$ is obtained by adding two things:
  - the general solution of the homogeneous equation

$$L\vec{x} = \vec{0} \Leftrightarrow \vec{x} = \sum_{j=1}^{n} C_j \vec{x}_j$$

  - and any particular solution of the complete equation $L\vec{x}_p = \vec{b}$.
- The general solution of the complete equation is then

$$L\vec{x} = \vec{b} \Leftrightarrow \vec{x} = \sum_{j=1}^{n} C_j \vec{x}_j + \vec{x}_p.$$

Variation of parameters

- We can apply directly what we learned for systems.
- Let us suppose that for the homogeneous system $\dot{\vec{x}} = \mathbf{A} \cdot \vec{x}$, we have found a solution $\vec{x}(t) = \mathbf{F}(t) \cdot \vec{c}$.
- Then, in order to solve the whole system $\dot{\vec{x}} = \mathbf{A} \cdot \vec{x} + \vec{b}$, we will use a trial vector $\vec{x}(t) = \mathbf{F}(t) \cdot \vec{g}(t)$, where $\vec{g}(t)$ is arbitrary.
- Using Leibniz rule:

$$\dot{\vec{x}} = (\mathbf{F} \cdot \vec{g})^{\cdot} = \dot{\mathbf{F}} \cdot \vec{g} + \mathbf{F} \cdot \dot{\vec{g}} =$$

$$\mathbf{A} \cdot \mathbf{F} \cdot \vec{g} + \mathbf{F} \cdot \dot{\vec{g}}.$$

- From the initial hypothesis $\vec{x}(t) = \mathbf{F} \cdot \vec{g}$, so

$$\dot{\vec{x}} = \mathbf{A} \cdot \vec{x} + \mathbf{F} \cdot \dot{\vec{g}}.$$

- On the one hand we have

$$\dot{\vec{x}} = \mathbf{A} \cdot \vec{x} + \mathbf{F} \cdot \dot{\vec{g}},$$

but on the other, by definition

$$\dot{\vec{x}} = \mathbf{A} \cdot \vec{x} + \vec{b}.$$

- we conclude then:

$$\mathbf{F} \cdot \dot{\vec{g}} = \vec{b}, \quad \dot{\vec{g}} = \mathbf{F}^{-1} \cdot \vec{b}.$$

- The general solution of the complete is thus:

$$\vec{x} = \mathbf{F}(t) \cdot \vec{c} + \mathbf{F}(t) \cdot \int \mathbf{F}(t)^{-1} \cdot \vec{b}(t) dt.$$

Exercise 4.16

- Solve the system $\dot{x} = y, \ \dot{y} = -x + 1/\cos t,$.
- The fundamental system is:

$$\mathbf{F} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and its inverse:

$$\mathbf{F}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

- On the other hand

$$\mathbf{F}^{-1} \cdot \vec{b} = \begin{pmatrix} -\tan t \\ 1 \end{pmatrix},$$

and then

$$\int \mathbf{F}^{-1} \cdot \vec{b} = \begin{pmatrix} -\ln \cos t + C_1 \\ t + C_2 \end{pmatrix}.$$

- The general solution is then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t(-\ln\cos t + K_1) + \sin t(t + K_2) \\ -\sin t(-\ln\cos t + K_1) + \cos t(t + K_2) \end{pmatrix}$$