Based on JM Aguirregabiria's textbook.

1. Find the differential equation that all the circles in the plane satisfy.

2. $\dfrac{d^5y}{dx^5} - \dfrac{1}{x}\dfrac{d^4y}{dx^4} = 0.$

3. $y'' - xy''' + (y''')^3 = 0.$

4. Write the following equation

$$y'' + f(y) = 0$$

as quadratures. Even without solving the integral, we can get interesting qualitative information using physics. Why?

5. Are the functions $y_1 = x$ and $y_2 = |x|$ linearly independent in the domain $(-1, 1)$? Calculate the Wrosnkian and explain the result.

6. Show that the functions $\left\{x^{p_i}e^{k_i x} : p_i = 0, \ldots, n_i;\ i = 1, \ldots, n\right\}$ are linearly independent in any domain. *Note:* when $i \neq j$, then $k_i \neq k_j$

7. Find the second order linear homogeneous equation that these pair of functions satisfy (a) x, x^2, (b) $x + 1$, $x^2 + 1$, (c) x, e^{2x}.

8. $(x + 1)y'' + xy' - y = (x + 1)^2.$

9. Derivation method. Sometimes it is helpful to differentiate a high-order equation, even if the equation is linear and solved in the highest derivative. For example, solve the equation in the previous exercise by taking derivatives.

10. Discuss the fundamental problem of the equation $y' + A(x)y = B(x)$

11. Two-point Green's function of the harmonic oscillator: Find and simplify the solution of the following boundary-problem:

$$G''(x, s) + \omega^2 G(x, s) = \delta(x - s), \qquad G(0, s) = G(\ell, s) = 0.$$

Discuss the solution for different values of the parameter $0 \leq s \leq \ell$

12. $y'' - y = xe^x.$

13. $y'' + y = xe^{-x}\cos x.$

14. $\left(\mathrm{D}^3 + \mathrm{D}\right) y = 1 + e^{2x} + \cos x.$

15. $(\mathrm{D} + 1)^3 y = e^{-x} + x^2.$

16. $x^2y'' - xy' + y = x\ln^3 x.$

17. In a smooth table, there is a 6 m chain, with one 1 m hanging from the edge. When will the last link of the chain fall?

18. $yy'' + (y')^2 = \dfrac{yy'}{\sqrt{1+x^2}}$.

19. $\left(x^2-1\right)y'' - 6y = 1$.
Suggestion: The homogeneous equation has a polynomial solution.

20. $y'' + 10y' + 25y = 2^x + xe^{-5x}$.

21. $xy'' = y' \ln \dfrac{y'}{x}$.

22. Show that the function $x^{-1/2}\sin x$, is a solution of Bessel's equation

$$x^2y'' + xy' + \left(x^2 - 1/4\right)y = 0$$

23. Write the following equation in quadratures $y'' - xf(x)y' + f(x)y = 0$

24. $(2x-3)^2y'' - 6(2x-3)y' + 12y = 0$.

25. Three solutions of the complete equation: Suppose that the three functions y_1, y_2 and y_3 are particular solutions of

$$y'' + a_1(x)y' + a_2(x)y = b(x)$$

satisfying the following condition

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ 1 & 1 & 1 \end{vmatrix} \neq 0.$$

Show that the general solution to the equation is given by $y = C_1(y_1 - y_3) + C_2(y_2 - y_3) + y_3$ and that the equation itself can be written as $W\left[y_1 - y_3, y_2 - y_3, y - y_3\right] = 0$.

26. Show that if the functions y_1 and y_2 are zero in the same point $x \in I$, they cannot form a system of solutions to the following equation in the domain I

$$y'' + a_1(x)y' + a_2(x)y = 0.$$

27. Riccati's equation and second order homogeneous linear equation Show that with the change of variables $u = y'/y$, the following equation $y'' + a_1(x)y' + a_2(x)y = 0$ becomes Riccati's equation. Find the transformation that can transform any Ricatti's equation into a second order linear homogeneous equation.

28. Forced oscillator This is the equation for a forced oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\cos\Omega t, \qquad (\gamma, \omega_0 > 0)$$

Show that the general solution to the homogeneous equation, the one corresponding to the **transient solution**, is the following

$$x = \begin{cases} e^{-\gamma t}\left(A\cos\omega t + B\sin\omega t\right), & \text{if } \gamma < \omega_0, \\ Ae^{-(\gamma+\lambda)t} + Be^{-(\gamma-\lambda)t}, & \text{if } \gamma > \omega_0, \\ (A+Bt)e^{-\omega_0 t}, & \text{if } \gamma = \omega_0, \end{cases}$$

where $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$ and $\lambda \equiv \sqrt{\gamma^2 - \omega_0^2} < \gamma$. Check that the particular solution we need to get the general solution to the complete equation can be given by :

$$\begin{aligned} x &= f\frac{\left(\omega_0^2 - \Omega^2\right)\cos\Omega t + 2\gamma\Omega\sin\Omega t}{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2} = A\cos(\Omega t + \alpha), \\ A &\equiv \frac{f}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}, \\ \alpha &\equiv \arctan\frac{\omega_0^2 - \Omega^2}{2\gamma\Omega} - \frac{\pi}{2}. \end{aligned}$$

This solution describes the **steady-state** solution

29. Damped oscillator Show that the oscillator

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \qquad \left(\gamma^2 - \omega_0^2 > 0\right)$$

goes through the equilibrium point the at most once. What happens in the case of critical damping?

30. Calculate the general derivative of the **sign function**:

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

31. Calculate the following generalized limit

$$\lim_{n\to\infty} ng\left[n(x-a)\right].$$

32. What will be the operation that the derivative $\delta'(x-a)$ will define inside an integral? What can it be used for in physics?

33. Using the defintion $\beta = \left(1-\gamma^{-2}\right)^{1/2}$, calculate the following limit

$$\lim_{\gamma\to\infty} \gamma g\left(\gamma(x-\beta t)\right)$$

where g is any summable function. This is useful in special and general relativity.

34. Consider the following function

$$\varphi_\varepsilon(x) = \frac{1}{\pi}\left[\arctan\left(\frac{x}{\varepsilon}\right) + \frac{\pi}{2}\right] \tag{1}$$

and calculate $\varphi_\varepsilon'(x)$. Plot the functions $\varphi_\varepsilon(x)$ and $\varphi_\varepsilon'(x)$, for the values $\varepsilon = 1, 10^{-1}, 10^{-2}$. Calculate $\varphi_\varepsilon(x)$ and $\varphi_\varepsilon'(x)$ in the limit $\varepsilon \longrightarrow 0$. Comment on the solution.

35. Exact second order linear differential equations The expression $a_0(x)y'' + a_1(x)y' + a_2(x)y$ is an exact second order linear equation, if it is the derivative of an appropriate first order expression $A_1(x)y' + A_2(x)y$

$$\left[A_1(x)y' + A_2(x)y\right]' = a_0(x)y'' + a_1(x)y' + a_2(x)y.$$

Find what is the necessary and sufficient condition that the second order expression has to satisfy to be exact. If it is not exact what is the condition that the integrating factor $\mu(x)$ has to satisfy so that the prodcut $\mu(x)\left[a_0(x)y'' + a_1(x)y' + a_2(x)y\right]$ is exact? Discuss the solving method for the equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x)$$

when then left hand side is exact. Use that method to solve the following equation

$$y'' + xy' + y = 0.$$

36. Equations in finite differences Euler's method for solving linear differential equations with constant coefficients can be directly extended to solve finite difference equations of the same type. For example, let us consider

$$x_n = x_{n-1} + x_{n-2}, \qquad n = 2, 3, \ldots$$

1. Trying solution of the right type, find the general solution for the equation
2. The solution corresponding to the initial values $x_0 = 0$ and $x_1 = 1$ is Fibonacci's series. Find the expression for F_n, the n-th number in Fibonacci's series.
3. Calculate the **golden number**, which can be defined in the following way

$$\phi = \lim_{n \to \infty} F_{n+1}/F_n.$$

37. If C_1 and C_2 are arbitrary constants, what is the **lowest order** differential equation that has $y = C_1 + \ln(C_2 x)$ as its solution? Discuss your answer.

38. Solve

$$xy'' = 2yy'.$$

39. Show that the function $y_1 = 1$ and $y_2 = 1/x$ are linear independent solutions to Burger's equation

$$y'' + 2yy' = 0$$

What is the general solution? (Careful!)

40.

$$xy'' + 2y' - xy = 0.$$

41. Find the general solution for

$$(2x^2 - 2x)y'' + (5x - 1)y' + y = 0.$$

42. Let us suppose that y_{h} is a particular solution to the first order linear homogeneous equation $y' + A(x)y = 0$. Use the method of variation of parameters to find the general solution for $y' + A(x)y = B(x)$ as a quadrature.