Based on JM Aguirregabiria's textbook.

1. Prove **without solving the integral** that when $s > 0$ these relations are satisfied

$$\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}, \qquad \mathcal{L}[\cos at] = \frac{s}{s^2 + a^2}.$$

2. Calculate $\mathcal{L}[\sin^2 at]$ **without solving the integral**

3. Find the Laplace transform of the following function

$$f(t) = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ \sin t + \cos t, & t \ge 2\pi. \end{cases}$$

4. Find the inverse Laplace transform of $F(s) = \dfrac{1 - e^{-2s}}{s^2}$.

5. Find the inverse Laplace transform of $F(s) = \dfrac{1}{s^2 - 4s + 5}$.

6. Let us suppose that $F(s) = \mathcal{L}[f(t)]$. Prove that

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(u)\, du$$

is satisfied if there exists $\displaystyle\lim_{t \to 0+} \frac{f(t)}{t}$. Why is this last condition necessary?

7. Prove that when $F(s) = \mathcal{L}[f(t)]$ we can obtain:

$$\mathcal{L}\left[\int_a^t f(u)\, du\right] = \frac{1}{s}\left[F(s) - \int_0^a f(u)\, du\right].$$

8. Using the definition and properties of Euler's gamma function, calculate $\mathcal{L}\left[t^b\right]$ and $\mathcal{L}\left[t^b e^{at}\right]$, for $b > -1$. Specifically, what are $\mathcal{L}\left[t^{-1/2}\right]$ and $\mathcal{L}\left[t^{1/2}\right]$? What happens when b is integer and non-negative? Check what is the value of $sF(s)$ in the limti $s \to \infty$ for $-1 < b < 0$. Comment the result.

9. Using the properties of the error-function, calculate $\mathcal{L}\left[\operatorname{erf}\left(a\sqrt{t}\right)\right]$. What happens when $a < 0$?

10. Calculate the Laplace transform of $\operatorname{Si}(t)$, $\operatorname{Ci}(t)$ and $-\operatorname{Ei}(-t)$. These functions are defined as follows for $t > 0$:

$$\begin{aligned} \operatorname{Si}(t) &= \int_0^t \frac{\sin u}{u}\, du, \\ \operatorname{Ci}(t) &= -\int_t^\infty \frac{\cos u}{u}\, du, \\ -\operatorname{Ei}(-t) &= \int_t^\infty \frac{e^{-u}}{u}\, du. \end{aligned}$$

Suggestion: Make an appropriate change of variables to turn the lower limit of the last two integrals into constant.

11. Find the inverse Laplace transform of $F(s) = \dfrac{1}{(s+1)^2\,(s^2+1)}$.

12. Volterra's integral equation Explain how to solve the integral equation of the following type

$$x(t) = g(t) + \int_0^t k(t-u)\,x(u)\,du.$$

using Laplace transforms.

13. Solve $x(t) = \cos t + \displaystyle\int_0^t e^{-(t-u)}\,x(u)\,du.$

14. Solve the following problem:

$$\ddot{x} + 2\dot{x} + 2x = \delta(t-\pi), \qquad x(0) = \dot{x}(0) = 0$$

15. Let $f(t)$ be a periodic function with period T: $f(t+T) = f(t)$, with $t > 0$. Prove

$$\mathcal{L}\left[f(t)\right] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st}\,f(t)\,dt.$$

16. Prove that Volterra's integral equation

$$x(t) + \int_0^t (t-u)\,x(u)\,du = \sin 2t$$

is equivalent to the initial value problem

$$\ddot{y}(t) + y(t) = \sin 2t, \qquad y(0) = \dot{y}(0) = 0$$

if $\ddot{y}(t) \equiv x(t)$. Solve both problems.

17. By definition, the Bessel function $J_0(x)$ is a solution of the zeroth order Bessel's equation

$$xy'' + y' + xy = 0,$$

with initial conditions $J_0(0) = 1$, $J_0'(0) = 0$.

(a) Calculate the Laplace transform of J_0

(b) By integrating the series expansion of the function in $1/s$ term by term, obtain the expansion of J_0

Suggestion: Use the binomial series

$$(1+\alpha)^2 = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n\,, \quad (\alpha \in \mathbb{R},\ |x| < 1)$$

and simplify the solution using the subfactorial function

$$(2n-1)!! = \frac{(2n)!}{2^n n!}$$

18. The tautochrone A point particle is moving along a smooth curve without friction due to gravity. If the time needed to get to the minimum does not depend on the point from which the particle has started at rest, the curve is called the tautochrone. Let us suppose that to get the equation for this curve $y(x)$, we choose the axis y to be vertical and upwards, and s is the *curvilinear abscissa* measured from the origin.

(a) Integrate the conservation law of mechanical energy to obtain the time needed to get to the minimum

(b) Use Laplace transform in order to solve the integral equation obtained for ds/dy

(c) Find and integrate dx/dy, to find the equation of the tautochrone.

19. Calculate $\mathcal{L}\left[\frac{e^{-at}-e^{-bt}}{t}\right]$.

20. Solve this problem: $\dot{x}+2x+\int_0^t x(u)\,du = \sin t, \qquad x(0)=1.$

21. $\ddot{x}+x=\theta(t-\pi)-\theta(t-2\pi), \qquad x(0)=\dot{x}(0)=0.$

22. Solve the system

$$\begin{aligned}
&\ddot{x}-\dot{y}=t+1,\\
&\dot{x}+\dot{y}-3x+y=2t-1,\\
&x(0)=0, \quad y(0)=-11/9, \quad \dot{x}(0)=0.
\end{aligned}$$

23. $\frac{d^4x}{dt^4}+2\frac{d^2x}{dt^2}+x=0, \quad x(0)=0, \qquad \frac{dx}{dt}(0)=1, \quad \frac{d^2x}{dt^2}(0)=2, \quad \frac{d^3x}{dt^3}(0)=-3.$

24. $t\ddot{x}+(3t-1)\dot{x}-(4t+9)x=0, \qquad x(0)=0.$

25. $\ddot{x}+x=e^{-t}\cos t, \qquad x(\pi)=\dot{x}(\pi)=0.$

26. Prove that the rectified sinusoidal wave is

$$|\sin t| = \sin t + 2\sum_{k=1}^{\infty}\theta(t-k\pi)\sin(t-k\pi)$$

and its Laplace transform reads $\frac{1}{s^2+1}\coth\frac{\pi s}{2}$.

27. Dirac's delta Calculate the following integral using Laplace transforms:

$$\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}, \qquad \forall a>0.$$

Use this result and the one corresponding to the case $a<0$ to prove that the Fourier transform and inverse transform of the unit function are the generalized Dirac's delta, modulo some constant:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm ipx}\,dp = \delta(x).$$

Suggestion: Separate the real and imaginary part, and use the *sign* function in exercise sheet 3, and
$\lim_{a\to\infty}\int_{-a}^{a}\sin x dx = 0$

28. Fourier inversion formula. Use the result from the previous exercise to prove

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(p)e^{ipx}\,dp = f(x)$$

where

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)e^{-ipx}\,dx$$

is the Fourier transform of the function $f(x)$. This proves that Fourier inversion formula is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(p) e^{ipx} \, dp.$$

29. Parseval's theorem. Consider that the Fourier transforms of the functions $f(x)$ and $g(x)$ are given by $\hat{f}(p)$ and $\hat{g}(p)$ respectively. Use the results from the problem about *Dirac's delta* to prove:

$$\int_{-\infty}^{\infty} \overline{\hat{f}(p)} \, \hat{g}(p) \, dp = 2\pi \int_{-\infty}^{\infty} \overline{f(x)} \, g(x) \, dx.$$

30. Use Parseval's theorem to calculate the following integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} \, dt.$$

31. Retarded differential equations. Consider the solution to the retarded equation

$$\dot{x}(t) + x(t-1) = 0.$$

in the domain $-1 \leq t \leq 0$ with initial condition $x(t) = 1$. Calculate its Laplace transform and expand the solution using the series $s^{-1}e^{-s}$. Find the inverse transform to calculate the solution. Is there any other method to find the solution? Discuss the differentiability of the solution.

32. You may have heard that a group walking in military formation on a bridge could break it. Explain this case using the solution to this problem

$$\ddot{x} + x = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \qquad x(0) = \dot{x}(0) = 0.$$

Why doesn't this misfortune happen always?

33. The Laplace transform for the function $f(t) = \ln t$ is the following:

$$F(s) = -\frac{\ln s + \gamma}{s}.$$

Why is $sF(s)$ not bounded in the limit $s \to \infty$?

34. Give two different examples of functions that do not accept Laplace transforms

35. Duhamel's formula. Let us suppose that in the constant coefficient equation

$$ay'' + by' + cy = f(x),$$

the function f is piecewise continuous and of exponential order. Prove that the solution of the equation with null initial value problem $y(0) = y'(0) = 0$ can be written in the following way

$$y(x) = \int_0^x z'(u) f(x-u) \, du$$

given that the function z is a solution to the problem

$$az'' + bz' + cz = 1 \qquad \text{eta} \qquad z(0) = z'(0) = 0$$

Use this formula on the forced oscillator

$$y'' + \omega^2 y = f(x)$$

and compare it with the solution obtained using Cauchy's method.

36. How long does an external constant force need to be applied to a harmonic oscillator initially at rest , in order for the oscillator to be at rest for ever once the force disappears?

37. Find the solutions to the following equation

$$\int_0^t y(u)\,du = \int_0^t y(u)y(t-u)\,du.$$

Is it a linear equation?